UNIVERSITÀ DEGLI STUDI DI FIRENZE

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PhD Thesis

Planar Clusters

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Contents

Introduction

This thesis is focused on the study of minimal *planar soap bubble clusters*. The aim is to determine the configuration of N disjoint regions $E_1, \dots, E_N$ with fixed areas $a_1, \dots, a_N$, which minimizes the length of the interfaces: $\bigcup_{i=1}^N \partial E_i$. The $3D$ version of this problem (i.e regions enclosing prescribed volumes with minimal total surface area) has a simple physical model given by soap bubbles (see Figure 1).

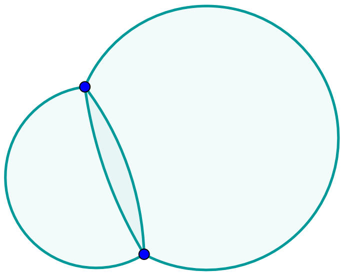

Figure 1: Double bubble in $\mathbb{R}^3$ provides the least-area way to enclose and separate the given volumes of air.

If we have a single bubble the problem becomes the classic isoperimetric problem already well known since the times of ancient Greeks, which knew the solution: the circle. But the first real proof is due to Steiner [14] in the nineteenth century. He proved that, if the solution exists then necessarily has to be a circle. Carathéodory [1] completed the proof showing the existence of the solution. We want to emphasize that the circle is connected, so intuitively, it would seem clear even in the presence of more than one region, that the best configuration is one in which each region is connected. This is called *the soap bubble conjecture*

Conjecture 0.1. *[12] All regions of a minimizing cluster are connected.*

In particular it is not known if the problem to determine the configuration of N disjoint connected regions $E_1, \ldots, E_N$ with fixed areas $a_1, \ldots, a_N$ has solution. As often happens, however, what is intuitive is particularly difficult to prove mathematically; this is the case, a method is not yet found to solve directly the conjecture but partial results are obtained a posteriori after finding an explicit solution of the problem. The problem can be presented in the more general case of clusters in $\mathbb{R}^n$; it is called *the generalized soap bubble problem*.

Almgren in 1976 [1] proved existence and regularity almost everywhere for $n \geq 3$ of a solution to the generalized soap bubble problem. Taylor improved this result for $n = 3$ in the same year (see [15]).In 1985, Bleicher [4] proved important properties of the solutions to the problem in 2 dimensions without giving a rigorous proof of their existence, while in 1992, Morgan [11] proved the existence of the solutions to the planar problem and properties of minimizers (the same is also proved by Maggi in [10]). Some years later, in $1994 - 1995$ Cox, Harrison, Hutchings, Kim, Light, Mauer and Tilton [6] proved that to each region E_i of the minimum one can associate a real number p_i (said pressure) so that each edge between regions E_i and E_j has curvature $p_i - p_j$. In 1996, Bleicher [5] proved a useful property that, in a minimizing bubble, any 2 components may meet at most once. This reduces many combinatorial possibilities for candidate bubble clusters.

Once the existence and local structure of minimizers have been completely established, the problem stated in Conjecture 0.1 remains the most

[1] Blaschke [3] credits Edler, Carathéodory and Study with existence results. Bandle [2] claims Carathéodory was first. Schmidt and Weierstrass completed the three dimensional analogue.

important open question. In 1994 the planar double bubble problem (i.e cluster of two bubbles)

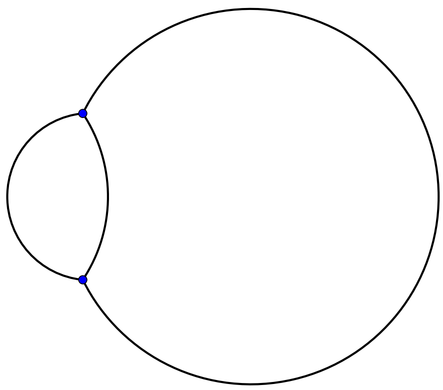

Figure 2: A standard double bubble.

was solved by Foisy, Alfaro, Brock, Hodges and Zimba [7], a group of students of Morgan: all minimizers are as in Figure 2.

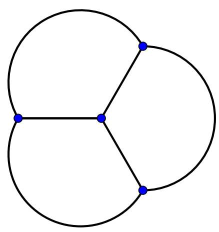

Figure 3: A standard triple bubble with the same area.

The case $N = 3$ (Figure 3) was fully proved in 2002 by Wichiramala in his PhD thesis [17], exploiting the PhD thesis of Vaughn [16] of 1998. Vaughn proved that any minimizing triple bubble with equal pressures and connected exterior is composed by connected regions. Other significant result is the proof of the classical honeycomb conjecture: any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling. It is proved in 1999 by Hales [8].

The purpose of this thesis is to solve Conjecture 0.1 in the case $N = 4$ and all regions with the same area.

We conclude this introduction with a short summary of each chapter of the thesis.

- **Chapter 1.** In this chapter we briefly recall previous results on the soap bubble problem and we give important definitions that are used throughout this dissertation. In particular we introduce the notion of weak minimizers, which is a minimizing cluster with areas $|\, E_i \,| \geq a_i$ (instead of $|\, E_i \,| = a_i$). We focus on progress in the planar case.

  The first section is devoted to the existence and regularity of soap bubbles.

  In the second section we introduce a significant concept: the pressure. Here the most important result is Corollary 1.47, that links the perimeter of a bubble with the pressures and areas of each region.

  In the last section we show that, under some suitable conditions, weak minimizers are minimizers, and if weak minimizers are standard (i.e each region is connected), then minimizers are standard (see Theorem 1.50).

- **Chapter 2.** Here, following also the PhD thesis of Wichiramala [17], we discuss geometric properties of planar soap bubbles.

  In the first section we introduce Möbius transformations, that are maps with particular properties; they transform straight lines and circles into straight lines and circles and they preserve angles between curves and orientation, as shown in Theorem 2.6 and Remark 2.9 respectively.

  In the second section, we determine some conditions under which some components are vertically symmetric, as shown in Corollary 2.16. Furthermore Lemma 2.18 is very interesting, since it describes the situation when there is a sequence of four-sided components.

Finally, in the last section, we conclude with Lemma 2.22, where we show how to simplify clusters by reducing one component with three edges.

- **Chapter 3.** In this chapter we introduce some new tools which are used in the following. In this first section, we present the key theorem of the thesis; it is Theorem 3.5 and it gives some necessary conditions on the quantity of area that different components of the same region must have. In particular, under suitable conditions, we are able to identify a big component and the small components in each region.

  In the second section we introduce three particular variations of a cluster in Lemma 3.11, Lemma 3.12 and Lemma 3.14. In the first we find the minimum quantity of area that a component of a disconnected region must have. In the second the goal is to remove a small component in favor of the big component. This will give an important estimate for the pressure of a region. Finally in the last we determine a simple estimate for the length of all edges of a weakly minimizing planar cluster.

  In the third section we conclude with an interesting lemma (Lemma 3.16) where we determine a significant estimate for the pressures of a standard double bubble.

- **Chapter 4.** In this chapter we show how the tools developed in Chapter 3 can be successfully used to prove, in an alternative way respect to the already presented solutions, that the double and triple bubbles with all regions of equal area are connected.

  In the first section we deal with the double bubble. The result is a direct consequence of Remark 4.7 and Corollary 4.8, that give an upper and lower estimate on the area of a small component respectively.

  In the second section we deal with the triple bubble. Remark 4.12 and Corollary 4.15 are the keys, because again they give an upper and lower estimate on the area of a small component respectively. Finally we also underline Lemma 4.21, that describes a component of a disconnected region and a component of a connected region.

- **Chapter 5.** This chapter contains the new result of this thesis. We consider a cluster of four regions of equal area and we prove that each region is connected (i.e. Conjecture 0.1 is true in the case in which each region has the same area). The chapter is divided in four sections.

  In the first section the most important results are Theorem 5.6 and Corollary 5.10, which give estimates on the area of disconnected regions.

  In the second section, exploiting Remark 5.7 and Corollary 5.10, we list all possible cases of disconnected planar weakly minimizing 4-cluster (see Remark 5.12), which remain to be excluded.

  Theorems 5.15, 5.23, 5.39 and 5.55, contained in the second, third and fourth section respectively, are the crucial theorems to prove the conjecture. Indeed they eliminate all possibilities of disconnected planar weakly minimizing 4-cluster, listed in Remark 5.12.

Chapter 1

Preliminary results

In this chapter, we summarize known results on the soap bubble problem and we give important definitions that are used throughout this dissertation. We focus on progress in the planar case.

In particular the first section is devoted to the existence and the regularity of soap bubbles.

In the second section we introduce a significant concept: the pressure. Here the most important result is Corollary 1.47, that links the perimeter of a bubble with the pressures and areas of each region.

In the last section we show a new approach in order to prove the planar soap bubbles conjecture. It is summarized in Theorem 1.50, which, under suitable conditions, shows when a weakly planar minimizing cluster is a planar minimizing cluster and underline that *the soap bubble conjecture* holds if every weak minimizer is standard.

1.1 Existence and regularity of soap bubbles

The core of this first section is the existence and the regularity of soap bubbles. In particular we are especially interested in the planar case.

We start with some definitions.

Definition 1.1. [10] A N**-cluster** $\mathbf{E}$ is an N-uple of sets, $\mathbf{E} := (E_1, E_2, \dots, E_N)$ with these properties

a) E_i is a subset of $\mathbb{R}^n$, $\mathcal{L}^n$-measurable for all $i = 1, \dots, N$;

b) $0 < |\, E_i\,| < +\infty$ for all $i = 1, \dots, N$;

c) $|\, E_i \cap E_j\,| = 0$ for all $i \neq j$;

d) $\mathrm{P}(E_i) = \mathcal{H}^{n-1}(\partial^* E_i) < +\infty$ for all $i = 1, \dots, N$.

Furthermore, given an N-cluster $\mathbf{E}$, we define

e) $E_0 := \mathbb{R}^n \setminus \bigcup_{i=1}^{N} E_i$, called exterior region;

f) $\mathrm{P}(\mathbf{E}) := \dfrac{1}{2}\sum_{j=0}^{N} \mathrm{P}(E_j)$.

We introduce the following notation to denote the set of N-clusters E of $\mathbb{R}^n$

$$\mathcal{E}_{n,N} := \{\mathbf{E} \,|\, \mathbf{E}\ \ N - \text{cluster of } \mathbb{R}^n\}.$$

We denote with $\mathrm{P(B)}$, $|\,\mathrm{B}\,|$, $\partial^*\mathrm{B}$ and $\mathcal{H}^{n-1}(\mathrm{B})$ respectively the perimeter, the volume, the reduced boundary and the $n-1$-dimensional Hausdorff measure of any subsets B of $\mathbb{R}^n$. We use the vector notation to denote a vector of given volumes $\mathbf{a} = (a_1, \dots, a_N)$ with $a_i > 0$ for all $i = 1, \dots, N$ and $m(\mathbf{E}) = (|\, E_i\,|, \dots, |\, E_N\,|)$.

The **soap bubble problem** is the following:

$$\min\left\{\mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{n,N}, m(\mathbf{E}) = \mathbf{a}\right\}, \tag{1.1}$$

namely it consist in the search for the least surface area way to enclose and separate N regions E_i of given volumes a_i. If $n = 2$, we call the problem the **planar soap bubble problem.**

We formulate the corresponding weak version of the problem (1.1):

$$\min\left\{\mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{n,N}, m(\mathbf{E}) \geq \mathbf{a}\right\}, \tag{1.2}$$

where $m(\mathbf{E}) \geq \mathbf{a}$ is $|\, E_i\,| \geq a_i$, for all $i = 1, \dots, N$.

We call this problem **weak soap bubble problem**.

Definition 1.2. We denote with

$$p_{n,N}(\mathbf{a}) = \min\left\{\mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{n,N}, m(\mathbf{E}) = \mathbf{a}\right\},$$

$$p^*_{n,N}(\mathbf{a}) = \min\left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{n,N}, m(\mathbf{E}) \geq \mathbf{a} \right\}.$$

Definition 1.3. Let $\mathbf{E}$ be a N-cluster. If $m(\mathbf{E}) = \mathbf{a}$, then we say that $\mathbf{E}$ is a **competitor** for the problem (1.1). Then we denote with

$$\mathcal{C}_{n,N}(\mathbf{a}) := \{\mathbf{E} \in \mathcal{E}_{n,N} \,|\, m(\mathbf{E}) = \mathbf{a}\},$$

the set of all competitors.

In the same way, given a N-cluster $\mathbf{E}$, we say that it is a **weak competitor** for the problem (1.2) if $m(\mathbf{E}) \geq \mathbf{a}$. Then we denote by

$$\mathcal{C}^*_{n,N}(\mathbf{a}) := \{\mathbf{E} \in \mathcal{E}_{n,N} \,|\, m(\mathbf{E}) \geq \mathbf{a}\},$$

the set of all weak competitors.

Remark 1.4. By Definition 1.2, it is clear that

$$\begin{aligned} p_{n,N}(\mathbf{a}) &= \inf\left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{C}_{n,N}(\mathbf{a}) \right\}, \\ p^*_{n,N}(\mathbf{a}) &= \inf\left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{C}^*_{n,N}(\mathbf{a}) \right\}. \end{aligned} \tag{1.3}$$

Since $\mathcal{C}_{n,N}(\mathbf{a}) \subset \mathcal{C}^*_{n,N}(\mathbf{a})$, then $p^*_{n,N}(\mathbf{a}) \leq p_{n,N}(\mathbf{a})$.

Remark 1.5. By (1.3) of previous remark, we have that

$$p^*_{n,N}(\mathbf{a}) = \inf_{\mathbf{b} \geq \mathbf{a}} p_{n,N}(\mathbf{b}).$$

where $\mathbf{b} \geq \mathbf{a}$ is $b_i \geq a_i$, for all $i = 1, \ldots, N$.

Definition 1.6. A N-cluster $\mathbf{E}$ is a **minimum** for the problem (1.1) if

1) $\mathbf{E} \in \mathcal{C}_{n,N}(\mathbf{a})$;

2) $P(\mathbf{E}) \leq P(\mathbf{E}')$ for all $\mathbf{E}' \in \mathcal{C}_{n,N}(\mathbf{a})$.

We denote with $\mathcal{M}_{n,N}(\mathbf{a})$ the set of minimizers, namely

$$\mathcal{M}_{n,N}(\mathbf{a}) := \{\mathbf{E} \in \mathcal{C}_{n,N}(\mathbf{a}) \,|\, P(\mathbf{E}) \leq P(\mathbf{E}'), \forall \mathbf{E}' \in \mathcal{C}_{n,N}(\mathbf{a})\}.$$

Similarly for the problem (1.2), given a N-cluster $\mathbf{E}$, we say that it is a **weak minimum** if

1) $\mathbf{E} \in \mathcal{C}^*_{n,N}(\mathbf{a})$;

2) $P(\mathbf{E}) \le P(\mathbf{E}')$ for all $\mathbf{E}' \in \mathcal{C}^*_{n,N}(\mathbf{a})$.

We denote by $\mathcal{M}^*_{n,N}(\mathbf{a})$ the set of weak minimizers, namely

$$\mathcal{M}^*_{n,N}(\mathbf{a}) := \{\mathbf{E} \in \mathcal{C}^*_{n,N}(\mathbf{a}) \,|\, P(\mathbf{E}) \le P(\mathbf{E}'), \forall \mathbf{E}' \in \mathcal{C}^*_{n,N}(\mathbf{a})\}.$$

Remark 1.7. If $\mathbf{E} \in \mathcal{M}^*_{n,N}(\mathbf{a})$, then $\mathbf{E} \in \mathcal{M}_{n,N}(m(\mathbf{E}))$. Otherwise there would exist $\mathbf{E}'$ such that $m(\mathbf{E}') = m(\mathbf{E})$ and $P(\mathbf{E}') < P(\mathbf{E}) \le P(\mathbf{E}'')$ for all $\mathbf{E}'' \in \mathcal{C}^*_{n,N}(\mathbf{a})$. Since $m(\mathbf{E}') = m(\mathbf{E}) \ge \mathbf{a}$, $\mathbf{E}' \in \mathcal{C}^*_{n,N}(\mathbf{a})$, then $P(\mathbf{E}) \le P(\mathbf{E}')$, so we get a contradiction.

Remark 1.8. Let $\mathcal{M}^*_{n,N}(\mathbf{a}) \neq \emptyset$. We note that when for all $\mathbf{E} \in \mathcal{M}^*_{n,N}(\mathbf{a})$ is true that $m(\mathbf{E}) = \mathbf{a}$, then weak minimizers and minimizers are the same. Indeed let be $\mathbf{E} \in \mathcal{M}^*_{n,N}(\mathbf{a})$, since $m(\mathbf{E}) = \mathbf{a}$, then $P(\mathbf{E}) \le P(\mathbf{E}'')$ for all $\mathbf{E}'' \in \mathcal{C}_{n,N}(\mathbf{a})$. Therefore $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{a})$, namely $\mathbf{E}$ is a minimizer.

On the other hand, since $m(\mathbf{E}) = \mathbf{a}$, $\mathbf{E}$ is a competitor for the problem (1.1), therefore, taken a minimizer $\mathbf{E}'$, then $P(\mathbf{E}) \ge P(\mathbf{E}')$. Furthermore, since $m(\mathbf{E}') = \mathbf{a} \ge \mathbf{a}$, $\mathbf{E}'$ is a weak competitor for the problem (1.2), therefore $P(\mathbf{E}) \le P(\mathbf{E}')$. So $\mathbf{E}'$ is a weak minimizer.

Remark 1.9. Furthermore, by the definitions of $p_{n,N}(\mathbf{a})$ and $\mathcal{M}_{n,N}(\mathbf{a})$ seen in Remark 1.4 and in Definition 1.6 respectively, we have that

$$\mathcal{M}_{n,N}(\mathbf{a}) = \left\{ E \in \mathcal{C}_{n,N}(\mathbf{a}) \,\middle|\, P(\mathbf{E}) = p_{n,N}(\mathbf{a}) \right\},$$

for any vector $\mathbf{a}$ of positive components.

The existence and basic regularity almost everywhere of the solutions to the soap bubble problem in $\mathbb{R}^n$ for $n \ge 3$ was proved by Almgrem in 1976 [1], while in 1994 Morgan proved the existence and regularity of solutions to the planar soap bubble problem (the same is also proved by Maggi in [10]).

Theorem 1.10. *[11][4][10] For all* $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N_+$ *there exists* $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$. *Every* $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$ *must satisfy these conditions:*

1. *it is composed by a finite number of arcs of circle (segments are considered edges with zero curvature);*
2. *each vertex is meeting point of exactly three edges that make angles of* $\dfrac{2\pi}{3}$*;*
3. *edges that separate the same pair of regions have the same curvature;*
4. *in each vertex* ***the cocycle condition*** *holds, namely the sum of the signed curvatures is zero.*

Remark 1.11. By Almgrem [1] and by Theorem 1.10 we have that the set $\mathcal{M}_{n,N}(\mathbf{a}) \neq \emptyset$ for any vector $\mathbf{a}$ of positive components. So by Remark 1.9, there exists $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{a})$, such that $P(\mathbf{E}) = p_{n,N}(\mathbf{a})$.

We recall the isoperimetric inequality

$$P(E) \geq n(\omega_n)^{\frac{1}{n}} |\, E \,|^{\frac{n-1}{n}}, \forall E \subset \mathbb{R}^n, E\ \mathcal{L}^n - \text{measurable}, |\, E \,| < +\infty,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We denote by $C_n := n(\omega_n)^{\frac{1}{n}}$, thus if $n = 2$, $C_2 = 2\sqrt{\pi}$.

We set $D_{\mathbf{a}} := [a_1, +\infty[\times \ldots \times [a_N, +\infty[$, where $a_i > 0$ for all i.

Here, we show a preliminary lemma for the proof of the existence of the minimum for the the soap bubble weak problem (1.2).

Lemma 1.12. *Let* $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{R}^N_+$ *and* $p_{n,N}$ *be the following function,*

$$p_{n,N} : D_{\mathbf{a}} \to \mathbb{R},\ \mathbf{b} = (b_1, \ldots, b_N) \mapsto p_{n,N}(\mathbf{b}) = \min \Big\{ P(\mathbf{E}) \,\big|\, \mathbf{E} \in \mathcal{C}_{n,N}(\mathbf{b}) \Big\}, \tag{1.4}$$

then:

1) $p_{n,N}(\mathbf{b}) \geq p_{n,j}(\mathbf{b}_j)$*, where* $\mathbf{b}_j$ *is a vector of* j*-components of the vector* $\mathbf{b}$*;*

2) $\lim\limits_{|\mathbf{b}| \to +\infty} p_{n,N}(\mathbf{b}) = +\infty$ *(*$|\mathbf{b}|$ *denotes the Euclidian norm of* $\mathbf{b} \in D_{\mathbf{a}}$*);*

3) $p_{n,N}$ *is continuous.*

Proof. First of all we note that the minimum of the set $\{P(\mathbf{E}) \,|\, \mathbf{E} \in \mathcal{C}_{n,N}(\mathbf{b})\}$ exists and there is $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{b})$, such that $P(\mathbf{E}) = p_{n,N}(\mathbf{b})$, as we have seen in Remark 1.11.

We show 1). Let $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{b})$ and j with $1 \leq j \leq N$, we consider $\mathbf{E}^j$, a vector of j-components of $\mathbf{E}$ with E_0^j, the complementary set of the union of j-components of $\mathbf{E}$, then $\mathbf{E}^j$ is a j-cluster and $\mathbf{E} \in \mathcal{C}_{n,j}(\mathbf{b}_j)$(see the Definition 1.1) and one has

$$p_{n,N}(\mathbf{b}) = P(\mathbf{E}) \geq P(\mathbf{E}^j) \geq p_{n,j}(\mathbf{b}_j).$$

This finishes 1).

The property 2) is a directed consequence of 1); in fact for all j fixed, such that $1 \leq j \leq N$, we obtain that

$$p_{n,N}(\mathbf{b}) \geq p_{n,j}(b_j) = C_n \cdot b_j^{\frac{n-1}{n}}.$$

From which, choosing j such that $b_j \to +\infty$, the claim follows.

Finally we prove 3). The idea is to show that $p_{n,N}$ is, fixed $\mathbf{b} \in D_{\mathbf{a}}$, upper and lower semicontinuous in $\mathbf{b}$. We check that $p_{n,N}$ is upper semicontinuous in $\mathbf{b}$: let $\mathbf{x} \in D_{\mathbf{a}}$ with $\mathbf{x} \geq \mathbf{b}$ (i.e $x_i \geq b_i$ for all $i = 1, \ldots, N$), then taken $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{b})$, we consider $\mathbf{E}' = \mathbf{E} \cup (B_1, \ldots, B_N) = (E_1 \cup B_1, \ldots, E_n \cup B_N)$, where B_i are wirwise disjoint balls, each ball B_i is disjoint from $\mathbf{E}$ and $\mid B_i \mid = \mathrm{x}_i - b_i$. $\mathbf{E}'$ is in $\mathcal{C}_{n,N}(\mathbf{x})$, thus we get

$$\begin{aligned} p_{n,N}(\mathbf{x}) \leq P(\mathbf{E}') = P(\mathbf{E}) + \sum_{i=1}^{n} P(B_i) = p_{n,N}(\mathbf{b}) + C_n \sum_{i=1}^{N} (x_i - b_i)^{\frac{n-1}{n}} \\ = p_{n,N}(\mathbf{b}) + C_n \, N \, || \, \mathbf{x} - \mathbf{b} \, ||^{\frac{n-1}{n}}. \end{aligned} \tag{1.5}$$

From (1.5), for all $\varepsilon > 0$, chosen δ go that

$$0 < \delta < \left(\frac{\varepsilon}{C_n \, N}\right)^{\frac{n}{n-1}}, \tag{1.6}$$

we have that for all $\mathbf{x}$, $\mathbf{x} \geq \mathbf{b}$ and $||\mathbf{x} - \mathbf{b}|| < \delta$ one has $p_{n,N}(\mathbf{x}) \leq p_{n,N}(\mathbf{b}) + \varepsilon$.

We must also consider the case where $\mathbf{x} \in D_{\mathbf{a}}$, $\mathbf{x} \to \mathbf{b}$ and $\mathbf{x}$ has at least one component x_i such that $x_i < b_i$. Without loss of generality we can suppose that the components of $\mathbf{x}$, which are smaller than the corresponding components of $\mathbf{b}$, are the first components of $\mathbf{x}$. So we use the following notations: $\mathbf{x} = (\mathbf{x}_k, \mathbf{x}_{N-k})$, with $\mathbf{x}_k < \mathbf{b}_k$, $\mathbf{x}_{N-k} \geq \mathbf{b}_{N-k}$, where we let $\mathbf{b} = (\mathbf{b}_k, \mathbf{b}_{N-k})$, $\mathbf{x}_k = (x_1, \ldots, x_k)$, $\mathbf{x}_{N-k} = (x_{k+1}, \ldots, x_N)$ and the same for $\mathbf{b}_k$ and $\mathbf{b}_{N_k}$. Let $\mathbf{x}$ be as above and $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{x})$, then we let $\lambda = \left(\prod_{i=1}^{k} \frac{b_i}{x_i}\right)^{\frac{1}{n}}$ and $\mathbf{E}' := \lambda \cdot \mathbf{E}$. We will prove the following facts:

a) $\mathbf{x}' \geq \mathbf{b}$, where $\mathbf{x}' = \lambda^n \cdot \mathbf{x}$;

b) if $\mathbf{x} \to \mathbf{b}$ then $\mathbf{x}' \to \mathbf{b}$;

c) $P(\mathbf{E}') = p_{n,N}(\mathbf{x}') \geq p_{n,N}(\mathbf{x})$.

We have that $\mathbf{x}' = m(\mathbf{E}') = \lambda^n \cdot m(\mathbf{E}) = \lambda^n \cdot \mathbf{x}$, with

$$x'_j = |\, E'_j \,| = \lambda^n \cdot x_j = \begin{cases} b_j \left(\prod_{i \neq j} \frac{b_i}{x_i}\right), & \text{if } 1 \leq j \leq k, \\ \lambda^n \cdot x_j, & \text{if } k+1 \leq j \leq N, \end{cases}$$

therefore, since $\mathbf{b}_k > \mathbf{x}_k$ and $\mathbf{x}_{N-k} \geq \mathbf{b}_{N-k}$, $a)$ follows.

We see $b)$; we have

$$\begin{aligned} ||\mathbf{x}' - \mathbf{b}|| &= ||\lambda^n \cdot \mathbf{x} - \mathbf{b}|| \\ &= \sqrt{\sum_{j=1}^{N} \left(\lambda^n \cdot x_j - b_j\right)^2}. \end{aligned}$$

If $1 \leq j \leq k$ one has

$$\begin{aligned} \left|\lambda^n \cdot x_j - b_j\right| &= \left|b_j \left(\prod_{i \neq j} \frac{b_i}{x_i}\right) - b_j\right| \\ &= \frac{b_j \left|\prod_{i \neq j} b_i - \prod_{i \neq j} x_i\right|}{\prod_{i \neq j} x_i}, \end{aligned}$$

while, if $k+1 \leq j \leq N$ we have

$$\left|\lambda^n \cdot x_j - b_j\right| = \frac{\left|\left(\prod_{i=1}^{k} b_i\right) x_j - \left(\prod_{i=1}^{k} x_i\right) b_j\right|}{\prod_{i=1}^{k} x_i}$$

$$= \frac{\left| \left(\prod_{i=1}^{k} b_i - \prod_{i=1}^{k} x_i \right) x_j - \left(\prod_{i=1}^{k} x_i \right) (b_j - x_j) \right|}{\prod_{i=1}^{k} x_i}$$

$$\leq \frac{\left| \prod_{i=1}^{k} b_i - \prod_{i=1}^{k} x_i \right| |x_j| + |b_j - x_j| \left| \prod_{i=1}^{k} x_i \right|}{\prod_{i=1}^{k} x_i}.$$

Thus, by previous estimates, it is clear that, if $\mathbf{x} \to \mathbf{b}$, then $\mathbf{x}' \to \mathbf{b}$. This proves b).

Finally we show c); we argue by contradiction, so we suppose there exists $\mathbf{E}'' \in \mathcal{C}_{n,N}(\mathbf{x}')$ such that $P(\mathbf{E}'') < P(\mathbf{E}')$. We consider $\mathbf{E}''' := \frac{\mathbf{E}''}{\lambda}$, then $m(\mathbf{E}''') = \frac{m(\mathbf{E}'')}{\lambda^n} = \frac{\mathbf{x}'}{\lambda^n} = \mathbf{x}$, hence $\mathbf{E}''' \in \mathcal{C}_{n,N}(\mathbf{x})$. So it follows

$$P(\mathbf{E}) = p_{n,N}(\mathbf{x}) \leq P(\mathbf{E}''') = \frac{P(\mathbf{E}'')}{\lambda^{n-1}} < \frac{P(\mathbf{E}')}{\lambda^{n-1}} = P(\mathbf{E});$$

this is a contradiction, thus c) is true. Since $\displaystyle\prod_{i=1}^{k} \frac{b_i}{x_i} \geq 1$, it follows that $p_{n,N}(\mathbf{x}') \geq p_{n,N}(\mathbf{x})$; therefore, from a), b) and c), eventually taking a smaller δ in (1.6), we have that

$$p_{n,N}(\mathbf{x}) \leq p_{n,N}(\mathbf{x}') \leq p_{n,N}(\mathbf{b}) + \varepsilon.$$

So the upper semicontinuity of $p_{n,N}$ is proved.

Now we will prove the lower semicontinuity of $p_{n,N}$; the idea is the same that we used for the upper semicontinuity. Let $\mathbf{x} \in D_{\mathbf{a}}$ with $\mathbf{x} \leq \mathbf{b}$ and $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{x})$, then we consider $\mathbf{E}' = \mathbf{E} \cup (B_1, \ldots, B_N) = (E_1 \cup B_1, \ldots, E_n \cup B_N)$, where B_i are pairwise disjoint balls, each ball is disjoint from $\mathbf{E}$ and $|\, B_i \,| = b_i - x_i$. $\mathbf{E}'$ is in $\mathcal{C}_{n,N}(\mathbf{b})$, thus we obtain

$$\begin{aligned} p_{n,N}(\mathbf{b}) \leq P(\mathbf{E}') = P(\mathbf{E}) + \sum_{i=1}^{n} P(B_i) = p_{n,N}(\mathbf{x}) + C_n \sum_{i=1}^{N} (x_i - b_i)^{\frac{n-1}{n}} \\ = p_{n,N}(\mathbf{x}) + C_n \, N \, (||\, \mathbf{x} - \mathbf{b} \,||)^{\frac{n-1}{n}}. \end{aligned} \tag{1.7}$$

From (1.7) for all $\varepsilon > 0$, chosen $0 < \delta < \left(\frac{\varepsilon}{C_n N}\right)^{\frac{n}{n-1}}$, we have that for all $\mathbf{x} \in D_{\mathbf{a}}$, $\mathbf{x} \leq \mathbf{b}$ and $||\mathbf{x} - \mathbf{b}|| < \delta$ it holds $p_{n,N}(\mathbf{b}) \leq p_{n,N}(\mathbf{x}) + \varepsilon$.

From here, as in the case of the upper semicontinuity, we show that it is always possible to be in the situation, up to a rescaling of $\mathbf{x}$, where $\mathbf{x} \leq \mathbf{b}$. So we must also consider the case where $\mathbf{x} \in D_{\mathbf{a}}$, $\mathbf{x} \to \mathbf{b}$ and $\mathbf{x}$ has at least one component x_i such that $x_i > b_i$. Without loss of generality we must think that the components of $\mathbf{x}$, which are greater than the corresponding components of $\mathbf{b}$, are the first components of $\mathbf{x}$. So we use the following notations: $\mathbf{x} = (\mathbf{x}_k, \mathbf{x}_{N-k})$, with $\mathbf{x}_k > \mathbf{b}_k$, $\mathbf{x}_{N-k} \leq b_{N-k}$ where $\mathbf{b} = (\mathbf{b}_k, \mathbf{b}_{N-k})$, $\mathbf{x}_k = (x_1, \dots, x_k)$, $\mathbf{x}_{N_k} = (x_{k+1}, \dots, x_N)$ and the same for $\mathbf{b}_k$ and $\mathbf{b}_{N_k}$. Let $\mathbf{x}$ be as above, $\lambda = \left(\prod_{i=1}^{k} \frac{b_i}{x_i}\right)^{\frac{1}{n}}$ and $\mathbf{E}' := \lambda \cdot \mathbf{E}$, where $\mathbf{E} \in \mathcal{M}_{n,N}(\mathbf{x})$. Therefore the following facts hold:

d) $\mathbf{x}' \leq \mathbf{b}$, where $\mathbf{x}' = \lambda^n \cdot \mathbf{x}$;

e) if $\mathbf{x} \to \mathbf{b}$ then $\mathbf{x}' \to \mathbf{b}$;

f) $P(\mathbf{E}') = p_{n,N}(\mathbf{x}') \leq p_{n,N}(\mathbf{x})$.

The proof of $d)$, $e)$ and $f)$ is the same as the one we have already presented in the case of the upper semicontinuity; then from this also the lower semicontinuity of $p_{n,N}$ follows; together the upper and the lower semicontinuity give the continuity of $p_{n,N}$.

□

Corollary 1.13. *The problem* (1.2) *admits minimum.*

Proof. By Remark 1.5 it suffices to prove the existence of the minimum of the problem

$$\inf \left\{ p_{n,N}(\mathbf{b}) \,\middle|\, \mathbf{b} \geq \mathbf{a} \right\}, \tag{1.8}$$

so the proof is finished. The existence of the minimum for the problem (1.8) is a direct consequence of 2) and 3) of Lemma 1.12. □

At the end of this section we introduce other important definitions, that are used throughout this dissertation.

Definition 1.14. Let $\mathbf{E}$ be a N-cluster in $\mathbb{R}^n$, we say that E_i is a **connected region** of $\mathbf{E}$ if for any subset E_i^1, E_i^2 of E_i such that

i) $E_i = E_i^1 \cup E_i^2$;

ii) E_i^j is $\mathcal{L}^n$-measurable for any j ;

iii) $|\, E_i^1 \cap E_i^2 \,| = 0$;

iv) $P(E_i^j) < +\infty$ for any j;

v) $P(E_i) = P(E_i^1) + P(E_i^2)$,

then either $|E_i^1| = 0$ or $|\, E_i^2 \,| = 0$.

Definition 1.15. Let $\mathbf{E}$ be a N-cluster in $\mathbb{R}^n$, we say that C is a **component** of some region E_i of $\mathbf{E}$ if

i) C is a connected subset of E_i

ii) $|\, C \,| > 0$;

iii) $P(C) < +\infty$

iv) $P(E_i) = P(C) + P(E_i \setminus C)$.

In particular a bounded component of E_0 is specifically called an **empty chamber** as it does not contribute area to any of N bounded regions E_i ($i \neq 0$).

Definition 1.16. Let $\mathbf{E} \in \mathcal{E}_{n,N}$ and C be a component of a some region E_i of $\mathbf{E}$, we say that C is **inner** if $\mathcal{H}^{n-1}(\partial^* C \cap \partial^* E_0) = 0$. While we say that C is **external** if $\mathcal{H}^{n-1}(\partial^* C \cap \partial^* E_0) > 0$.

Definition 1.17. Let $\mathbf{E} \in \mathcal{E}_{n,N}$ and C_i, C_j be two components of the regions E_i and E_j of $\mathbf{E}$ respectively, we say that C_i and C_j are **disjoint** if $C_i \cap C_j = \emptyset$. While we say that C_i and C_j are **adjacent** if $\mathcal{H}^{n-1}(\partial^* C_i \cap \partial^* C_j) > 0$.

Definition 1.18. Let $\mathbf{E}$ be a N-cluster in $\mathbb{R}^n$ and C be a component of a region E_i of $\mathbf{E}$, we say that

i) C is small if $|\, C \,| < \frac{|\, E_i \,|}{2}$;

ii) C is big if $|\,C\,| > \frac{|\,E_i\,|}{2}$.

Remark 1.19. Let $\mathbf{E}$ be a (weak) minimizing N-cluster. We note that each disconnected region E_i can be seen as finite disjoint union of its components by Almgrem [1] and Theorem 1.10.

.

Definition 1.20. Let $\mathbf{E}$ be a N-cluster in $\mathbb{R}^n$, we say that $\mathbf{E}$ is **standard** if each region E_i is connected.

Definition 1.21. Let $\mathbf{E} \in \mathcal{E}_{n,N}$, we define the vector, called **connection type**, $I_{\mathbf{E}} := (M(1), \ldots, M(N))$, where $M(i)$ denotes the number of small components for any region E_i of $\mathbf{E}$.

In particular we explicitly note that if $\mathbf{E}$ is a minimizer, by Almgrem [1] and Theorem 1.10, $M(i)$ is finite for all $i = \ldots, N$.

Remark 1.22. It is clear that any connected region E_i of $\mathbf{E} \in \mathcal{E}_{n,N}$ is a big component. We set the number of small components, $M(i)$, equal to zero for any connected region.

The **soap bubble conjecture** can be phrased as follows: every minimizing cluster is standard.

1.2 The concepts of pressure and results on minimizing clusters

Here we introduce an important definition: the definition of pressure of a region and furthermore we conclude with some significant results for a minimizing cluster, that link together the concepts of perimeter, area and pressure. From now on we will focus on the planar soap bubble problem

$$\min \left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{2,N},\, m(\mathbf{E}) = \mathbf{a} \right\}, \tag{1.9}$$

and on the corresponding weak problem

$$\min \left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{2,N},\, m(\mathbf{E}) \geq \mathbf{a} \right\}, \tag{1.10}$$

unless otherwise noted.

Definition 1.23. Let $\mathbf{E}$ be a N-cluster of $\mathbb{R}^2$, we say that $p_1 \ldots, p_N \in \mathbb{R}$ are the pressures of $E_1, \ldots, E_N$ respectively, if each edge between E_i and E_j has curvature $|\, p_i - p_j \,|$ and curves into the lower pressure region with the convention that the pressure of the exterior region is zero.

Remark 1.24. Note that a priori regions may have negative pressures.

Existence of pressures implies the cocycle condition at all vertices and further implies that all edges separating a specific pair of regions have the same curvature.

Later, following the work of Cox, Harrison, Hutchings, Kim, Light, Mauer and Tilton [6], we show that for a minimizing planar N-cluster it is possible to define the pressure for each region.

Definition 1.25. The **sign** of the curvature of a directed edge is considered positive (negative) if edge is turning left (right).

Remark 1.26. When considering a component C, we implicitly direct its edges counter-clockwise with respect to C thus to the left on each edge. Hence the signed curvature of an edge of a component is well-defined.

Remark 1.27. Another convention for the **oriented** curvature that we use is the one represented in Figure 1.1.

Figure 1.1: *Sign conventions for the* ***oriented*** *curvature of an edge crossed by a path.*

Remark 1.28. Let e be an edge of a component C. We say that e is convex (concave) if its signed curvature is non negative (non positive). If all edges of a component C are convex (concave), we say that C is convex (concave).

Definition 1.29. An edge of a cluster is **redundant** if it separates a region from itself.

Remark 1.30. Clearly a cluster with redundant edges is not minimizing.

Definition 1.31. Let $\mathbf{E}$ be a planar N-cluster, we say that $\mathbf{E}$ is **regular** if $\mathbf{E}$ satisfies the properties of Theorem 1.10, it has pressures for its regions and it has not redundant edges. We also call a regular N-cluster a **N-bubble**.

Proposition 1.32. *[4] A minimizing planar N-cluster $\mathbf{E}$ is path connected and each component is simply connected.*

Proof. We argue by contradiction and we suppose that $\mathbf{E}$ is not path connected, thus by sliding two pieces of $\mathbf{E}$ until they touch, we create a cluster of the same perimeter and areas but with an invalid meeting between edges. This contradicts 2. of Theorem 1.10. Hence $\mathbf{E}$ is path connected.

Arguing in the same way seen previously, we obtain that each component is simply connected.

□

Proposition 1.33. *[5] For a minimizing planar N-cluster any two components may meet at most once along a single edge.*

An edge e is said to be incident to a vertex v if v is an endpoint of e.

Definition 1.34. An **incident** edge of a component C is an incident edge at a vertex of C that is not an edge on the boundary of C.

Corollary 1.35. *[7] A minimizing planar N-cluster has no two sided components if $N \geq 3$.*

Proof. If there is a two sided component, then the two components surrounding the two sided component meet twice unless the two incident edges on the two sided component are the same edge. By the previous proposition the latter case is true; therefore the two sided component with this third edge form a standard double bubble. But $N \geq 3$, then there is another bounded component not attached to this part, contradicting Proposition 1.32. □

Definition 1.36. The **turning angle** of an edge of a component is the product of its signed curvature and its length.

Remark 1.37. The product of the length and the **absolute curvature** of an edge L is also the central angle subtended by L, but at the same time it is also the double of the angle between the tangent to L in B (or the tangent to L in A) and the edge L (see Figure 1.2).

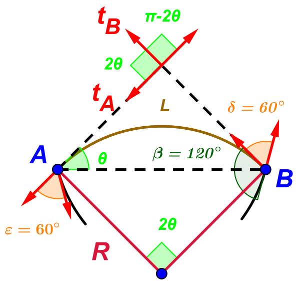

Figure 1.2: *It is clear that* $L \cdot \frac{1}{R} = L \cdot k = 2\theta$, *where* R *is the radius of curvature of* L *and* k *is its curvature.*

Lemma 1.38. *[17] For an* n*-sided component of a bubble, the sum of all edges' turning angles is* $\left(\frac{6-n}{3}\right)\pi$ *if the component is bounded and is* $\left(\frac{-6-n}{3}\right)\pi$ *if the component is unbounded.*

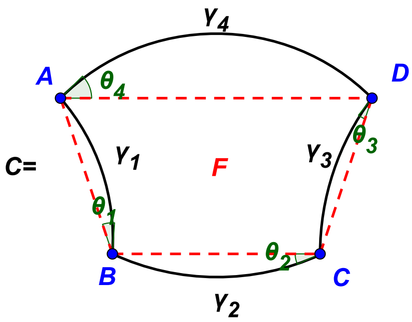

Figure 1.3: *A bounded component* C *with four edges and the corresponding polygon* F.

Proof. We show in detail the case of bounded component C. We consider a component C with n edges $\gamma_1, \ldots, \gamma_n$ with the convention for the orientation view in Remark 1.26. We build the polygon F determined by the vertices of C. Since C has n edges, then S has n sides (see Figure 1.3). Let θ_i be the angle between γ_i and the corresponding side of F and we denote with L_j the length of γ_j. We call $S_1 = \{j \in \{1, \ldots, n\} \,|\, \widetilde{k}_j > 0\}$ and $S_2 = \{j \in \{1, \ldots, n\} \,|\, \widetilde{k}_j < 0\}$, where $\widetilde{k}_j$ is the signed curvature of γ_j. The sum of the inner angles of F is $n\pi - 2\pi$, but at the same time it is equal to $n \cdot \frac{2\pi}{3} - 2\sum_{i \in S_1} \theta_i + 2\sum_{i \in S_2} \theta_i$, because C is a component of a bubble, then its inner angles are $\frac{2\pi}{3}$. So we have

$$n\pi - 2\pi = n \cdot \frac{2\pi}{3} - \sum_{i \in S_1} L_i\, \widetilde{k}_i - \sum_{i \in S_2} L_i\, \widetilde{k}_i,$$

namely the statement

$$\sum_{i \in S_1} L_i\, \widetilde{k}_i + \sum_{i \in S_2} L_i\, \widetilde{k}_i = 2\pi - n\pi + n \cdot \frac{2\pi}{3} = \left(\frac{6-n}{3}\right)\pi.$$

In the case of unbounded component C, the argument is the same, but in this situation we must consider the external angles of F (again, see Figure 1.3, where the component C is unbounded). The inner angles of C are always $\frac{2\pi}{3}$ (because C is a component of a bubble), while the sum of the external angles of F is equal to $2n\pi - (n\pi - 2\pi) = n\pi + 2\pi$. The proof is then concluded as before.

$\square$

Now we follow [6] in defining variations.

Definition 1.39. Let $\mathbf{E}$ be a regular N-cluster of $\mathbb{R}^2$, a variation of $\mathbf{E}$ is a $\mathcal{C}^1$ family of clusters $(\mathbf{E}_t)_{|t|<\varepsilon}$, where $\mathbf{E}_t = \mathbf{E}(t, x) \colon [-\varepsilon, \varepsilon] \times \mathbb{R}^2 \to \mathbb{R}^2$ is such that

1) $\mathbf{E}(0, x) = \mathbf{E}$,

2) $\mathbf{E}_t$ is injective for all t fixed in $[-\varepsilon, \varepsilon]$,

with $\varepsilon > 0$.

Following [9] and [13] we find a formula for the first derivative of the perimeter of a bubble.

Consider a planar N-cluster $\mathbf{E}$ with smooth **interfaces** E_{ij} (precisely E_{ij} represents the union of the edges between E_i and E_j) between E_i and E_j. Let N_{ij} be the unit normal vector on E_{ij} from E_j into E_i (see Figure 1.4).

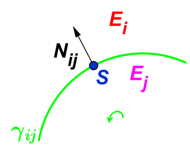

Figure 1.4: *An example of an edge between E_i and E_j.*

Consider a continuous **variation** $V = \{\mathbf{E}_t = \mathbf{E} \to \mathbb{R}^2\}_{|t|<\varepsilon}$ of $\mathbf{E}$, that is smooth on each E_{ij} up to the boundary. The **associated initial velocity** is $X := \frac{\mathrm{d}\mathbf{E}_t}{\mathrm{d}t}\Big|_{t=0}$. The **scalar normal component** of X from E_j to E_i is $u_{ij} := X \cdot N_{ij}$. Let k_{ij} [1] be the oriented curvature of E_{ij}; this is nonnegative if E_i has higher-pressure. It is clear that N_{ij}, u_{ij} and k_{ij} are skew-symmetric in their indices. Let N, u and k be the disjoint union functions $\coprod_{i<j} N_{ij}$, $\coprod_{i<j} u_{ij}$ and $\coprod_{i<j} k_{ij}$. The **normal component** of X is uN, where uN denotes the pointwise product of the functions u and N. Given a scalar or vector valued function $f = \coprod f_{ij}$ defined on the interfaces of $\mathbf{E}$, we define a function $Y(f)$ on the vertices of $\mathbf{E}$ by $Y(f)(p) = f_{ij}(p) + f_{jh}(p) + f_{hi}(p)$ if E_i, E_j and E_h meet at p (in that order counterclockwise). If $Y(f)(p) = 0$, we say that f or f_{ij} **agree** in p. For a bubble, since N_{ij} agree at p for any X and the associated normal component u, $Y(u)(p) = X \cdot Y(N)(p) = 0$. Hence u_{ij} agree at p. Initially the area of E_i changes at the rate $\sum_{j\neq i} \int_{E_{ij}} u_{ij} = \frac{\mathrm{d}a_i}{\mathrm{d}t}\Big|_{t=0}$, precisely in each interface E_{ij} there is a change of $\int_{E_{ij}} u_{ij}$, thus in totaly we must sum each previous contribution.

We will calculate the first variation formula of the perimeter for a planar N-cluster. We let $T(p)$ be the sum of the unit tangent vectors to the edges

[1] Let γ_{ij} be the parameterization of one edge of E_{ij} with respect to the arc-length s, then we have that $\gamma''_{ij}(s) = k_{ij}(s)\, N_{ij}(s)$, namely $k_{ij}(s) = \gamma''_{ij} \cdot N_{ij}(s)$.

meeting at p. Note that $T(p) = *Y(N)(p)$, where $*$ denotes the rotation of $90°$ clockwise.

Lemma 1.40. *[9] Let $\mathbf{E}$ be a planar N-cluster and $V = (\mathbf{E}_t)_{|t|<\varepsilon}$ a variation of $\mathbf{E}$ with associated initial velocity X, normal scalar component u_{ij} on E_{ij}, interfaces $\mathcal{C}^\infty$, then the first derivative of the perimeter at the initial time is:*

$$\begin{aligned}\left.\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{dt}}\right|_{t=0} &= -\sum_{j>i}\int_{E_{ij}} k_{ij}u_{ij} - \sum_{\textit{vertex } p} X(p)\cdot T(p)\\ &= -\int_{\mathbf{E}} k\cdot u - \sum_{\textit{vertex } p} X(p)\cdot T(p).\end{aligned} \tag{1.11}$$

In particular, if $\mathbf{E}$ is a bubble then

$$\left.\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{dt}}\right|_{t=0} = -\sum_{j>i}\int_{E_{ij}} k_{ij}u_{ij} = -\int_{\mathbf{E}} k\cdot u. \tag{1.12}$$

Proof. Let $V = (\mathbf{E}_t)_{|t|<\varepsilon}$ be a variation of a planar N-cluster $\mathbf{E}$ with associated initial velocity X. In order to determine the statement we find the first variation of the length of each edge of $\mathbf{E}$. Let e be an edge of $\mathbf{E}$ of length l_0 and signed curvature k; we denote with $\gamma_0\colon [0, l_0] \to \mathbb{R}^2$ its parameterization with respect to the arc-length s (note that $|\gamma_0'| = 1$). For all $t \in [-\varepsilon, \varepsilon]$, $\gamma_t\colon [0, l_0] \to \mathbb{R}^2$ is a parameterization of the deformed edge e_t of e at the time t according to the variation V (see Figure 1.5).

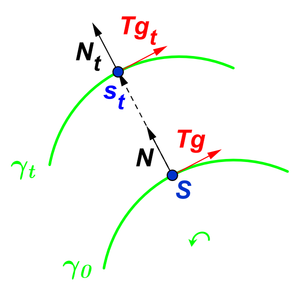

Figure 1.5: *The edge e with its deformed edge e_t.*

Observe that this parameterization is not necessarily respect to the arc-length.We call Tg and N the unit tangent vector and the unit normal vector on γ_0 respectively, where N is obtained by Tg with a counterclockwise rotation of 90° degrees. We denote with $\dot{f} = \frac{\partial f}{\partial t}$ and $f' = \frac{\partial f}{\partial s}$, where t and s are the temporal and spatial variable respectively. In this way we have $\gamma_0^{'} = Tg$, $\gamma_0^{''} = k\ N = Tg'$, and $\frac{\mathrm{d}\gamma_t}{\mathrm{dt}}\Big|_{t=0} = (\dot{\gamma}_t)\Big|_{t=0} = X$. Expanding in series of Taylor the function γ_t for $t = 0$ we obtain that $\gamma_t(s) = \gamma_0(s) + t\,X(s) + O(t^2)$, thus $\gamma_t^{'}(s) = \gamma_0^{'}(s) + t\,X'(s) + O(t^2)$. Then we have that (note that $|\,\gamma_0^{'}\,| = 1$)

$$\begin{aligned} |\,\gamma_t^{'}(s)\,|^2 &= |\,\gamma_0^{'}(s)\,|^2 + 2\Big(\gamma_0^{'}(s)\cdot X'(s)\Big)t + O(t^2) \\ &= 1 + 2\Big(\gamma_0^{'}(s)\cdot X'(s)\Big)t + O(t^2). \end{aligned} \tag{1.13}$$

Furthermore by Taylor expansion, we know that

$$\sqrt{1+h} = 1 + \frac{h}{2} + O(h^2). \tag{1.14}$$

Now let l_t be the length of e_t, then we get:

$$\begin{aligned} l_t &= \int_0^{l_0} |\gamma_t^{'}(s)|\,\mathrm{d}s \overset{1.13}{=} \int_0^{l_0} \sqrt{1 + 2\Big(\gamma_0^{'}(s)\cdot X'(s)\Big)t + O(t^2)}\,\mathrm{d}s \\ &\overset{1.14}{=} \int_0^{l_0} \left(1 + \Big(\gamma_0^{'}(s)\cdot X'(s)\Big)t + O(t^2)\right)\mathrm{d}s \\ &= l_0 + t\int_0^{l_0} \gamma_0^{'}(s)\cdot X'(s)\,\mathrm{d}s + O(t^2). \end{aligned}$$

Thus, integrating by parts, since the interfaces E_{ij} are smooth (so γ_0 is smooth), we obtain that

$$\begin{aligned} \frac{\mathrm{d}l_t}{\mathrm{dt}}\bigg|_{t=0} &= \int_0^{l_0} \gamma_0^{'}(s)\cdot X'(s)\,\mathrm{d}s = [\gamma_0^{'}(s)\cdot X(s)]_{s=0}^{s=l_0} - \int_0^{l_0} \gamma_0^{''}(s)\cdot X(s)\,\mathrm{d}s \\ &= [Tg(s)\cdot X(s)]_{s=0}^{s=l_0} - \int_0^{l_0} k(s)\,N(s)\cdot X(s)\,\mathrm{d}s \\ &= [Tg(s)\cdot X(s)]_{s=0}^{s=l_0} - \int_0^{l_0} k(s)\,u(s)\,\mathrm{d}s \\ &= -\int_0^{l_0} k(s)\,u(s)\,\mathrm{d}s + \big(Tg(l_0)\cdot X(l_0) - Tg(0)\cdot X(0)\big). \end{aligned}$$

Then, when we sum the contribution of each edge of $\mathbf{E}$, we have that

$$\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{dt}}\bigg|_{t=0} = -\sum_{j>i}\int_{E_{ij}} k_{ij}(s)\,u_{ij}(s)\,\mathrm{d}s - \sum_{\text{vertex } p} X(p)\cdot T(p). \tag{1.15}$$

In particular, if $\mathbf{E}$ is a bubble, (1.12) holds, because

$$\sum_{p \text{ vertex}} X(p) \cdot T(p) = 0.$$

Indeed each vertex is the meeting point of exactly three edges that make angles of $\frac{2\pi}{3}$ (see Figure 1.6), thus, for all vertex p, we get that

$$X(p) \cdot T(p) = |\, X(p) \,| \left(\cos \alpha + \cos \left(\frac{2\pi}{3} - \alpha \right) + \cos \left(\frac{2\pi}{3} + \alpha \right) \right) = 0,$$

since $\cos \alpha + \cos\left(\frac{2\pi}{3} - \alpha\right) + \cos\left(\frac{2\pi}{3} + \alpha\right) = 0$ for all α.

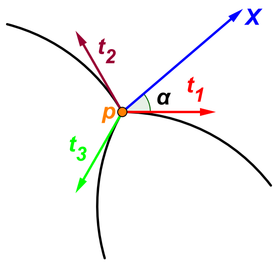

Figure 1.6: *In a vertex p three edges meet, whose tangents t_1, t_2 and t_3 define angles of $\frac{2\pi}{3}$.*

□

Remark 1.41. We explicitly note that the identity (1.12) also holds if $\mathbf{E}$ is a minimizing planar N-cluster. Indeed the proof of (1.12) is based, as shown in previous lemma, on each vertex is a meeting point of exactly three edges that define angles of $\frac{2\pi}{3}$. By Theorem 1.10, this property is still true in the case that $\mathbf{E}$ is a minimizing planar N-cluster.

Proposition 1.42. *[6][5] For any closed path that crosses only edges of a minimizing planar N-cluster $\mathbf{E}$, the sum of the oriented curvatures of the crossed edges is zero.*

Proof. We consider a closed path γ (see Figure 1.7) that crosses a minimizing planar N-cluster $\mathbf{E}$. Let v_i be the vertices of $\mathbf{E}$ inside γ and γ_i be the directed curves (oriented as γ) around each v_i such that each γ_i crosses $\mathbf{E}$ only in the three incident edges of v_i and nowhere else.

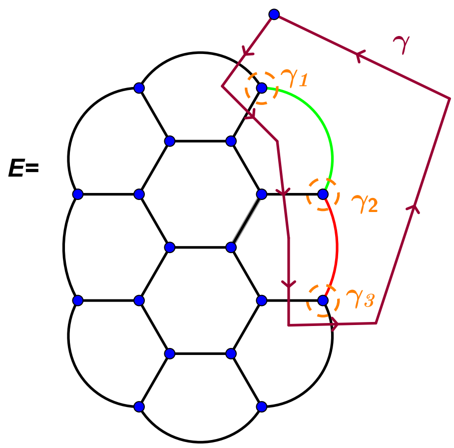

Figure 1.7: *A closed path γ that crosses $\mathbf{E}$ with the directed curves γ_i around the vertices v_i, that have the same orientation of γ.*

By 4. of Theorem 1.10, the sum of the oriented curvatures of the three edges on each v_i is zero. Then the sum of the oriented curvatures of the edges crossed be γ is equal to the sum over v_i of the sum of the oriented curvatures of the three edges crossed by γ_i, that is a sum of zero and hence is equal to zero as claimed (note that the contribution of the oriented curvatures of edges in common between two γ_i is null, because these arcs are crossed in two directions, one the opposite of the other; for example in Figure 1.7 the green edge is considered by γ_1 and γ_2 and the red edge is counted by γ_2 and γ_3 in a opposite direction respectively). □

Remark 1.43. The previous Proposition is also true if $\mathbf{E}$ is a planar regular N-cluster. Indeed, the previous proof is based on the cocycle condition, which is also true in all vertices of a regular planar N-cluster $\mathbf{E}$ (see Definition 1.31).

Proposition 1.44. *[6] A minimizing planar N-cluster $\mathbf{E}$ is regular.*

Proof. Let $\mathbf{E}$ be a minimizing planar N-cluster. By Theorem 1.10 and by Definition 1.31, we have only to show that each region of $\mathbf{E}$ has a pressure. The proof is divided in three parts:

1) first of all we define pressure for a component C of each region;

2) then we prove that the previous definition is well posed;

3) finally we prove that different components of the same region have the same pressure; so the pressure of a region is the pressure of any of its component.

We show 1). Fixed a region E_i we consider one of its components C and let γ be an external path to $\mathbf{E}$, such that γ is not closed, does not pass through the vertices of $\mathbf{E}$ and it arrives inside C (see Figure 1.8). Then we define the pressure of C as the sum of the signed curvatures of the edges crossed by γ. In formula

$$p_\gamma(C) := \sum_\gamma k_\gamma, \tag{1.16}$$

where k_γ represents a signed curvature of an edge crossed by γ.

We prove 2). We take another path γ_1 with the same characteristics of γ, then we must see that the definition (1.16) is independent from the choice of the path, namely $p_\gamma(C) = p_{\gamma_1}(C)$.

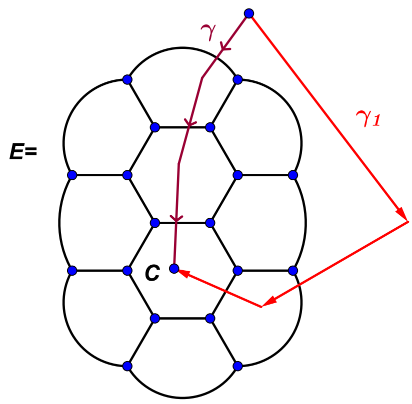

Figure 1.8: *Two path γ and γ_1 external to $\mathbf{E}$ and that arrive inside C. In order to prove the independence of the definition of pressure of a component C we consider the path $\gamma + (-\gamma_1)$.*

Hence we link γ with γ_1, considering a new path $\gamma + (-\gamma_1)$ (see Figure 1.8), then by Proposition 1.42 $\sum_{\gamma+(-\gamma_1)} k_{\gamma+(-\gamma_1)} = 0$, thus

$$0 = \sum_{\gamma+(-\gamma_1)} k_{\gamma+(-\gamma_1)} = \sum_{\gamma} k_\gamma + \sum_{-\gamma_1} k_{(-\gamma_1)}$$
$$= \sum_{\gamma} k_\gamma - \sum_{\gamma_1} k_{\gamma_1} = p_\gamma(C) - p_{\gamma_1}(C);$$

this completes 2).

Finally we see 3). Now in (1.16), by 2) we can write $p(C) = \sum_{\gamma} k_\gamma$. We consider two different components C_1 and C_2 of the same region E_i and we prove that $p(C_1) = p(C_2)$. If $p(C_1) \neq p(C_2)$, then without loss of generality we can assume that $p(C_1) < p(C_2)$ and so we may create a variation $V = (\mathbf{E}_t)_{|t|<\varepsilon}$, that moves some area from C_2 to C_1 (see Figure 1.9) in order to conserve all areas of the regions at the initial time.

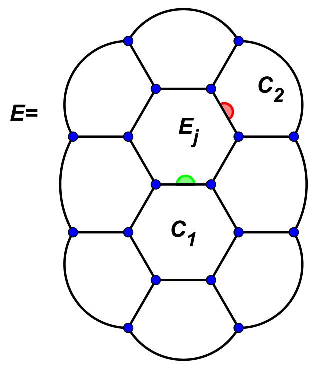

Figure 1.9: *The variation V removes the red area in C_2 and gives the green area (which is the same of the red area) to C_1; so the component of E_j adjacent to C_1 and C_2 does not change its area as also the region E_i.*

In particular for the region E_i we have $\frac{\mathrm{d}a_i(t)}{\mathrm{dt}}\big|_{t=0} = 0$, because $\frac{\mathrm{d}a_{C_2}(t)}{\mathrm{dt}}\big|_{t=0} = -\frac{\mathrm{d}a_{C_1}(t)}{\mathrm{dt}}\big|_{t=0}$ (note that C_1 takes area from C_2, thus $\frac{\mathrm{d}a_{C_1}(t)}{\mathrm{dt}}\big|_{t=0} > 0$). By (1.12) of Lemma 1.40 (see also Remark 1.41) we get that (recall that initially the area of E_i changes of $\int_{E_{ij}} u_{ij}$ in the interface E_{ij})

$$\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{dt}}\Big|_{t=0} = \Big(p(C_1) - p(C_2)\Big)\frac{\mathrm{d}a_{C_1}(t)}{\mathrm{dt}}\Big|_{t=0} < 0.$$

This contradicts the minimality of $\mathbf{E}$ and concludes the proof.

□

Remark 1.45. By Remark 1.7 and Proposition 1.44, we explicitly note that a planar weak minimizer $\mathbf{E}$ is a regular cluster. Hence it is possible to define the pressure of for each region.

Proposition 1.46. *[6] In a planar regular N-cluster $\mathbf{E}$ with areas $a_1, \ldots, a_N$ and pressures $p_1, \ldots, p_N$, and any variation $(\mathbf{E}_t)_{|t|<\varepsilon}$, we have*

$$\left.\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{d}t}\right|_{t=0} = \sum_{i=1}^{N} p_i \left.\frac{\mathrm{d}a_i(t)}{\mathrm{d}t}\right|_{t=0}, \tag{1.17}$$

where $a_i(t) = |\,E_i(t)\,|$ denotes the area of the i^{th} bounded region of $\mathbf{E}_t$.

Proof. Let $V = (\mathbf{E}_t)_{|t|<\varepsilon}$ be a variation of a planar regular N-cluster $\mathbf{E}$ with associated initial velocity X and let u_{ij} be the scalar normal component of X on E_{ij} (recall that E_{ij} is the union of edges between E_i and E_j). By (1.12) of Lemma 1.40 we know that $\frac{\mathrm{d}P(\mathbf{E}_t)}{\mathrm{d}t}\big|_{t=0} = -\sum_{i<j}\int_{E_{ij}} k_{ij}\, u_{ij} = -\sum_{i<j}\int_{E_{ij}}(p_i - p_j)\, u_{ij}$. Furthermore the total area lost by E_i in favor of E_j is $-\sum_{j\neq i}\int_{E_{ij}} u_{ij} = \frac{\mathrm{d}a_i(t)}{\mathrm{d}t}\big|_{t=0}$, since $\int_{E_{ij}} u_{ij}$ is the initial rate of decrease in the area of E_i taken by E_j. Note that $u_{ij} = -u_{ji}$ for all index i and j, thus called $a_{ij} := -\int_{E_{ij}} u_{ij}$, we have that $a_{ij} = -a_{ji}$. Observing that $p_0 = 0$, we get that

$$\begin{aligned}
\sum_{i<j} k_{ij}\, a_{ij} &= \sum_{i<j}(p_i - p_j)\, a_{ij} = \sum_{i<j} p_i\, a_{ij} - \sum_{i<j} p_j\, a_{ij} \\
&= \sum_{i<j} p_i\, a_{ij} - \sum_{j<i} p_i\, a_{ji} = \sum_{i>0}\sum_{i<j} p_i\, a_{ij} - \sum_{j\geq 0}\sum_{j<i} p_i\, a_{ji} \\
&= \sum_{i>0}\sum_{i<j} p_i\, a_{ij} - \sum_{i>0}\sum_{j<i} p_i\, a_{ji} = \sum_{i>0}\left(\sum_{j>i} p_i\, a_{ij} - \sum_{j<i} p_i\, a_{ji}\right) \\
&= \sum_{i>0}\left(\sum_{j>i} p_i\, a_{ij} + \sum_{j<i} p_i\, a_{ij}\right) = \sum_{i>0} p_i \sum_{j\neq i} a_{ij} = \sum_{i>0} p_i \frac{\mathrm{d}a_i(t)}{\mathrm{d}t}\Big|_{t=0}.
\end{aligned}$$

□

We present an important corollary of the previous proposition, that contains a formula, which links the perimeter of a bubble with the pressures and areas of each region.

Corollary 1.47. *Let B be a N-bubble of areas $a_1 \ldots, a_N$ and pressures $p_1, \ldots, p_N$, then*

$$P(B) = 2\sum_{i=1}^{N} p_i\, a_i. \tag{1.18}$$

Proof. We consider the following variation $B_t = B(t,x)\colon [-\varepsilon, \varepsilon] \times \mathbb{R}^2 \to \mathbb{R}^2$, with $B_t(x) = (t+1)x$ and $\varepsilon > 0$. With this choice $P(B_t) = (t+1)P(B)$ and $a_i(t) = (t+1)^2\, a_i$ for any i. Thus $\frac{\mathrm{d}P(B_t)}{\mathrm{dt}}\big|_{t=0} = P(B)$ and $\frac{\mathrm{d}a_i(t)}{\mathrm{dt}}\big|_{t=0} = 2a_i$, then by (1.17) we obtain the claim.

□

Remark 1.48. In a N-bubble B of areas $a_1, \ldots, a_N$ and pressures $p_1, \ldots, p_N$, from (1.18) the highest pressure must be positive, because $P(B) > 0$.

1.3 The weak approach

In this section we show a new approach, in order to prove the planar soap bubbles conjecture, that allows to consider the exterior region connected and to take as competitors any clusters $\mathbf{E}$ with $m(\mathbf{E}) \geq \mathbf{a}$.

We present two significant statements, one proposition and one theorem. In the first we show that for a weak minimizer the pressures are non negative and the exterior region is connected and in the second we prove that, under suitable conditions, weak minimizers are minimizer and the soap bubble conjecture applies if every weak minimizer is standard.

We recall that, by Remark 1.7, each weak minimizer is a minimizer, then by Proposition 1.44, each weak minimizer is a regular cluster. Therefore if $\mathbf{E}$ is a weak minimizer, the pressures are defined for all its region.

Proposition 1.49. *[17] The exterior region of a weak minimizer for area $a_1, \ldots, a_N$ is connected. Its pressures $p_1, \ldots, p_N$ are non negative. Furthermore if a region has area greater than a_i then $p_i = 0$.*

Proof. Let $\mathbf{E}$ be a weak minimizer. If the exterior region E_0 is not connected then at least one empty chamber of E_0 exists; thus we can reassign C to be part of one of the neighboring components and then we remove the redundant edge. Then we have a shorter cluster that satisfies the condition of $m(\mathbf{E}) \geq \mathbf{a}$, contradicting the minimality of $\mathbf{E}$.

If there exists an index i such that p_i is negative, we can define a variation $V = (\mathbf{E}_t)_{|t|<\varepsilon}$ such that the area of E_i increase and the other remain the same; so the length decreases by Proposition 1.46, giving again a contradiction.

If the area of E_i is greater than a_i and $p_i > 0$, we can define, as before, a variation $V = (\mathbf{E}_t)_{|t|<\varepsilon}$ such that the area of E_i decrease making sure that the area of each region is at least a_i. By Proposition 1.46 the length decreases again. Hence we get a shorter cluster that still contains areas bigger than $a_1, \ldots, a_N$; this is a contradiction.

□

Theorem 1.50. *Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$ and N_C be the total number of bounded components of $\mathbf{E}$. If $N_C \leq 6$, then $m(\mathbf{E}) = \mathbf{a}$ and $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$.*

In particular, if $N \leq 6$, the soap bubble conjecture *holds if every weak minimizer is standard, where N is the number of the regions of the problem* (1.10).

Proof. It is clear that N is not larger than N_C. We explicitly note that, by Proposition 1.49, E_0 is connected, thus there are no empty chambers. Therefore the only bounded components are components of some regions E_i with $i \neq 0$. We suppose by contradiction that the statement is false. Therefore an index $i \in \{1, \ldots, N\}$ exists such that $|\, E_i \,| > a_i$. By Proposition 1.49, its pressure $p_i = 0$ and it is the lowest pressure region. Hence the turning angle of any edge of E_i is non positive, then the turning angle of any its component is non positive. Therefore, by Lemma 1.38, any component of E_i has at least six edges. By Proposition 1.33 and by the fact that $N_C \leq 6$, we have that any component of E_i has exactly six edges. So the turning angle of any its component is zero. Since the turning angle of any its edges is non positive, by Lemma 1.38, we get that any component of E_i is an hexagon, namely its edges are a straight lines with zero curvature. Thus $N_C = 6$ and the pressure of any region is zero, in particular the pressure of the highest pressure region. By Remark 1.48 this is a contradiction; by Definition 1.6 we have that $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$.

If every $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$ is standard, then $N_C = N$, therefore if $N \leq 6$

we have that $N_C \leq 6$. Hence $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$, thus, taken $\mathbf{E}' \in \mathcal{M}_{2,N}(\mathbf{a})$, $P(\mathbf{E}) \geq P(\mathbf{E}')$ (note that if $\mathbf{E} \in \mathcal{M}_{2,N}(\mathbf{a})$, then $\mathbf{E} \in \mathcal{C}_{2,N}(\mathbf{a})$). At the same time $m(\mathbf{E}') = \mathbf{a} \geq \mathbf{a}$ (i.e $\mathbf{E}' \in \mathcal{C}^*_{2,N}(\mathbf{a})$), thus $P(\mathbf{E}') \geq P(\mathbf{E})$. Since $P(\mathbf{E}) = P(\mathbf{E}')$, then $\mathbf{E}' \in \mathcal{M}^*_{2,N}(\mathbf{a})$ and so, by assumption $\mathbf{E}'$ is standard.

$\square$

Remark 1.51. Theorem 1.50 shows that to prove the planar soap bubble conjecture for $N \leq 6$ it suffices to consider nonstandard clusters with exterior region connected and with non negative pressures and to prove they are not weakly minimizing.

Chapter 2

The geometry of planar soap bubbles

In this chapter, following also the PhD thesis of Wichiramala [17], we discuss geometric properties of planar soap bubbles.

In the first section we introduce Möbius transformations, that are maps with particular properties, namely they transform straight lines and circles into straight lines and circles and they preserve angles between curves and orientation as shown in Theorem 2.6 and Remark 2.5.

In the second section we determine some conditions under which some components are vertically symmetric, as shown in Corollary 2.16. Furthermore Lemma 2.18 is very interesting, since it describes the situation when there is a sequence of four-sided components.

Finally, in the last section, we conclude with Lemma 2.22, where we show how to simplify clusters by reducing one component with three edges.

2.1 Möbius transformations

We introduce an important class of functions which have some nice properties: **Möbius transformations**. We place $\Sigma = \mathbb{C} \cup \{\infty\}$ and we define the set of Möbius transformations

$$\mathcal{M} = \left\{ F : \Sigma \to \Sigma \,\middle|\, F(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\},$$

where

$$F\left(-\frac{d}{c} \right) = \infty, \quad \text{if } c \neq 0,$$

and

$$F(\infty) = \begin{cases} \infty, & \text{if } c = 0 \\ \frac{a}{c}, & \text{if } c \neq 0. \end{cases}$$

Remark 2.1. We observe that $F \in \mathcal{M}$ is an homeomorphism on Σ, because F is continuous, and its inverse transformation is $F^{-1}(z) = \frac{dz-b}{-cz+a} \in \mathcal{M}$. The following maps are particular and important elements of $\mathcal{M}$:

i) the inversion $F(z) = \frac{1}{z}$;

ii) the translation $F(z) = z + a$ with $a \in \mathbb{C}$;

iii) the similarity $F(z) = az$ with $a \in \mathbb{C} \setminus \{0\}$.

These transformations are called **elementary transformations.**

It is easy to see that each $F \in \mathcal{M}$ is a composition of elementary transformations. Indeed if $c = 0$, then (note that by the condition $ad - bc \neq 0$, since $c = 0$, a and d are different from zero)

$$F(z) = \frac{az}{d} + \frac{b}{d} = (f_2 \circ f_1)(z),$$

where $f_1(z) = \frac{az}{d}$ and $f_2(z) = z + \frac{b}{d}$.

While if $c \neq 0$, then

$$F(z) = \frac{az+b}{cz+d} + b = (f_4 \circ f_3 \circ f_2 \circ f_1)(z),$$

where $f_1(z) = z + \frac{d}{c}$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{bc-ad}{c^2} \cdot z$ and $f_4(z) = z + \frac{a}{c}$.

Definition 2.2. Let Ω be an open set of $\mathbb{R}^2$ and $f:\Omega\to\mathbb{R}^2$ a function. We say that f is a **conformal map** in $p\in\Omega$ if it preserves the amplitude of angles between curves through p.

In particular if f also preserves the orientation we say that f is a **direct conformal map** in p.

Finally we say that f is a (direct) conformal map in Ω if f is a (direct) conformal map in all $p\in\Omega$.

We recall that the angle between two curves is the angle between the two tangent lines to the curves in their common point.

Remark 2.3. If f is a differentiable function in Ω, the condition to preserve the orientation is expressed by $\det Df(x)>0$ for all $x\in\Omega$, where Df is the Jacobian matrix of f.

Proposition 2.4. *Let Ω be a subset of $\mathbb{C}$ and $f:\Omega\to\mathbb{C}$ a holomorphic function in Ω with $f'(z)=\frac{\mathrm{d}f(z)}{\mathrm{dz}}\neq 0$ for all $z\in\Omega$, then f is a direct conformal map in Ω.*

Proof. We denote $z=x+\mathrm{i}y$ for all $z\in\mathbb{C}$ and $f(z)=P(z)+\mathrm{i}Q(z)$, then $f(x,y)=(P(x,y),Q(x,y))$. Since f is holomorphic in Ω and $f'(z)\neq 0$ for all $z\in\Omega$, then[1] $\det Df(x,y)=|\,f'(z)\,|^2>0$. Now we prove that f preserves angles. Let $\gamma:(-1,1)\to\mathbb{R}^2$, $\gamma=(\gamma_1,\gamma_2)$ be a curve such that $\gamma(0)=z=x+\mathrm{i}y=(x,y)$ and $v=\gamma'(0)\neq 0$. Thus, in view of Cauchy-Riemann equation, the tangent vector $\widetilde{v}$ to $f\circ\gamma$ is given by

$$\widetilde{v}=\left.\frac{\mathrm{d}(f\circ\gamma)(t)}{\mathrm{dt}}\right|_{t=0}=Df(x,y)\cdot\gamma'(0)$$

[1]We recall that if $f:\Omega\subseteq\mathbb{C}\to\mathbb{C}$ is holomorphic in Ω, then we have that

$$\begin{cases} f'(z)=\frac{\partial f(x,y)}{\partial x}=\frac{\partial P(x,y)}{\partial x}+\mathrm{i}\frac{\partial Q(x,y)}{\partial x}\\ \mathrm{i}\frac{\partial f(x,y)}{\partial x}=\frac{\partial f(x,y)}{\partial y},\end{cases}$$

where $\mathrm{i}\frac{\partial f(x,y)}{\partial x}=\frac{\partial f(x,y)}{\partial y}$ is the Cauchy-Riemann equation, that is

$$\begin{cases}\frac{\partial P(x,y)}{\partial y}=-\frac{\partial Q(x,y)}{\partial x}\\ \frac{\partial Q(x,y)}{\partial y}=\frac{\partial P(x,y)}{\partial x}.\end{cases}$$

Therefore we obtain that

$$\det Df(x,y)=\det\begin{pmatrix}\frac{\partial P(x,y)}{\partial x} & \frac{\partial P(x,y)}{\partial y}\\ \frac{\partial Q(x,y)}{\partial x} & \frac{\partial Q(x,y)}{\partial y}\end{pmatrix}=\det\begin{pmatrix}\frac{\partial P(x,y)}{\partial x} & -\frac{\partial Q(x,y)}{\partial x}\\ \frac{\partial Q(x,y)}{\partial x} & \frac{\partial P(x,y)}{\partial x}\end{pmatrix}=|f'(z)|^2.$$

$$
\begin{aligned}
&= \begin{pmatrix} \frac{\partial P(x,y)}{\partial x} & -\frac{\partial Q(x,y)}{\partial x} \\ \frac{\partial Q(x,y)}{\partial x} & \frac{\partial P(x,y)}{\partial x} \end{pmatrix} \cdot \gamma'(0) \\
&= \left(\frac{\partial P(x,y)}{\partial x} \cdot \gamma_1'(0) - \frac{\partial Q(x,y)}{\partial x} \cdot \gamma_2'(0), \quad \frac{\partial Q(x,y)}{\partial x} \cdot \gamma_1'(0) + \frac{\partial P(x,y)}{\partial x} \cdot \gamma_2'(0) \right) \\
&= f'(z) \cdot v,
\end{aligned}
$$

where the last $\cdot$ denotes the complex multiplication. By assumption $f'(z)$ is different from zero for all $z \in \Omega$, thus we have that

$$
\begin{aligned}
&f'(z) = |f'(z)| \mathrm{e}^{\mathrm{i}\theta(z)}, \\
&v = |\, v \,| \mathrm{e}^{\mathrm{i}\theta}.
\end{aligned}
$$

So it follows that

$$\widetilde{v} = |f'(z)| \cdot |\, v \,| \mathrm{e}^{\mathrm{i}(\theta(z)+\theta)},$$

namely $\widetilde{v}$ is obtained expanding or contracting v of a factor $|f'(z)|$ and finally turning it of an angle $\theta(z)$. The key observation is that the angle $\theta(z)$ depends only on z and not on the curve γ through z. Therefore, given two curves γ_1 and γ_2 through z with tangent vectors v_1 and v_2 respectively, the corresponding tangent vectors $\widetilde{v_1}$ and $\widetilde{v_2}$ to $f \circ \gamma_1$ and $f \circ \gamma_2$ differ from v_1 and v_2 of the same angle $\theta(z)$. It follows that if v_1 and v_2 define an angle α, then $\widetilde{v_1}$ and $\widetilde{v_2}$ define the same angle.

□

Remark 2.5. By the previous proposition it is clear that

a) $F \in \mathcal{M}$ is a direct conformal map in $\mathbb{C}$ if $c = 0$,

b) $F \in \mathcal{M}$ is a direct conformal map in $\mathbb{C} \setminus \{-\frac{d}{c}\}$ if $c \neq 0$.

Indeed if $c = 0$, $F'(z) = \frac{a}{d} \neq 0$ (in this case, since $ad - bc \neq 0$, $ad \neq 0$, therefore a and d are different from zero), while if $c \neq 0$, $F'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$, thanks to the condition $ad - bc \neq 0$.

We note that if $c = 0$, $F \in \mathcal{M}$ is holomorphic in $\mathbb{C}$, because it is a composition of one translation and one similarity (see Remark 2.1). At the same time if $c \neq 0$, by Remark 2.1, $F \in \mathcal{M}$ is a composition of elementary transformations (translation, similarity and inversion), therefore it is holomorphic in $\mathbb{C} \setminus \{-\frac{d}{c}\}$ (note that the inversion is holomorphic in $\mathbb{C} \setminus \{0\}$).

We conclude this section with two significant results about Möbius transformations. The first shows that Möbius maps transform straight lines and circles into straight lines and circles. The second proves that Möbius transformations are direct conformal maps in $\mathbb{C}$.

Theorem 2.6. *The elementary transformations, the translation and the similarity,transform straight lines (circles) into straight lines (circles).*

The inversion map $F(z) = \frac{1}{z}$ transforms straight lines R and circles C into straight lines and circles in this way:

1) *if C is a circle with center c and radius r, which does not pass through the origin $O = (0,0)$, $F(C)$ is a circle with center $\frac{\bar{c}}{|c|^2-r^2}$ and radius $\frac{r}{||c|^2-r^2|}$;*

2) *if C is a circle with center c and radius r, which passes through the origin $O = (0,0)$ and c is not on the x-axis, $F(C)$ is a straight line with slope $\frac{\mathbb{R}\mathrm{e}\,(c)}{\mathrm{Im}(c)}$ and y-intercept $q = -\frac{1}{2\mathrm{Im}(c)}$;*

3) *if C is a circle with center c and radius r, which passes through the origin $O = (0,0)$ and c is on the x-axis (but it is not the origin), $F(C)$ is the vertical line with equation $z + \bar{z} = \frac{1}{\mathbb{R}\mathrm{e}\,(c)}$;*

4) *if R is a straight line with slope m and y-intercept $q \neq 0$, $F(C)$ is the circle with center $c = -\frac{m}{2q} - \frac{\mathrm{i}}{2q}$ and radius $r = \frac{\sqrt{m^2+1}}{2|q|}$;*

5) *if R is a straight line with slope m and y-intercept $q = 0$, $F(C)$ is the straight line with slope $-m$ and y-intercept $q = 0$;*

6) *if R is a vertical line with equation $z + \bar{z} = 2k$ and $k \neq 0$, $F(C)$ is the circle with center $c = \frac{1}{2k}$ and radius $r = \frac{1}{2|k|}$;*

7) *if R has equation $x = 0$, then it is kept.*

In particular any $F \in \mathcal{M}$ transforms straight lines and circles into straight lines and circles.

Proof. We recall that the equation of a circle with center $c \in \mathbb{C}$ and radius r is

$$z \cdot \bar{z} - z \cdot \bar{c} - \bar{z} \cdot c + |c|^2 - r^2 = 0, \tag{2.1}$$

while the straight line $y = mx + q$ can be written as

$$z(1 - \mathrm{i}m) - \bar{z}(1 + \mathrm{i}m) - 2\mathrm{i}q = 0, \tag{2.2}$$

if the straight line is not vertical with slope m and y-intercept q, while if it is vertical and has equation $x = k$, the equation in z is

$$z + \bar{z} = 2k. \tag{2.3}$$

It is easy to verify the statement about to the translations and homotheties, therefore we only show the case of the inversion function $F(z) = \frac{1}{z}$.

We begin to prove the first three assertions. The circle of equation (2.1) becomes

$$(|c|^2 - r^2)z \cdot \bar{z} - z \cdot c - \bar{z} \cdot \bar{c} + 1 = 0. \tag{2.4}$$

If $|c|^2 - r^2 \neq 0$, we can divide by $|c|^2 - r^2$ obtaining

$$z \cdot \bar{z} - z \cdot \left(\frac{c}{|c|^2 - r^2}\right) - \bar{z} \cdot \left(\frac{\bar{c}}{|c|^2 - r^2}\right) + \frac{1}{|c|^2 - r^2} = 0.$$

By (2.1), this is the equation of a circle with center $\frac{\bar{c}}{|c|^2 - r^2}$ and radius $\frac{r}{|\,|c|^2 - r^2\,|}$, because the square of the radius is

$$\frac{\bar{c}}{|c|^2 - r^2} \cdot \frac{c}{|c|^2 - r^2} - \frac{1}{|c|^2 - r^2} = \left(\frac{r}{|c|^2 - r^2}\right)^2.$$

If $|c|^2 - r^2 = 0$, i.e. the circle C passes through the origin O, by equation (2.4), $F(C)$ is the following straight line:

$$z \cdot c + \bar{z} \cdot \bar{c} - 1 = 0. \tag{2.5}$$

By multiplying by i we have

$$z \cdot (\mathrm{i}c) + \bar{z} \cdot (\mathrm{i}\bar{c}) - \mathrm{i} = 0,$$

where $\mathrm{i}c = -(\mathrm{Im}(c) - \mathrm{i}\mathbb{R}\mathrm{e}\,(c))$ and $\mathrm{i}\bar{c} = (\mathrm{Im}(c) + \mathrm{i}\mathbb{R}\mathrm{e}\,(c))$. Thus if $\mathrm{Im}(c)$ is different from zero, the previous equation becomes

$$z \cdot \left(1 - \mathrm{i}\frac{\mathbb{R}\mathrm{e}\,(c)}{\mathrm{Im}(c)}\right) - \bar{z}\left(1 + \mathrm{i}\frac{\mathbb{R}\mathrm{e}\,(c)}{\mathrm{Im}(c)}\right) - 2\mathrm{i}\left(-\frac{1}{2\mathrm{Im}(c)}\right) = 0.$$

By (2.2), this is the equation of a straight line with slope $m = \frac{\mathbb{R}\mathrm{e}\,(c)}{\mathrm{Im}(c)}$ and y-intercept $q = -\frac{1}{2\mathrm{Im}(c)}$.

If $|c|^2 - r^2 = 0$ and $\mathrm{Im}(c) = 0$, i.e. the circle C passes through the origin O and the center is on the x-axis, by (2.5), $F(C)$ is

$$z \cdot \mathbb{R}\mathrm{e}\,(c) + \bar{z} \cdot \mathbb{R}\mathrm{e}\,(c) - 1 = 0. \tag{2.6}$$

Since r is positive and $|c|^2 - r^2 = 0$, then $|\,\mathbb{R}\mathrm{e}\,(c)\,| = r$, therefore $\mathbb{R}\mathrm{e}\,(c)$ is different from zero. Hence the equation (2.6) represents a vertical straight line of equation $z + \bar{z} = \frac{1}{\mathbb{R}\mathrm{e}\,(c)}$.

Now we prove $4)$ and $5)$. The straight line of equation (2.2) becomes

$$\bar{z} \cdot (1 - \mathrm{i}m) - z \cdot (1 + \mathrm{i}m) - 2\mathrm{i}q\bar{z} \cdot z = 0. \tag{2.7}$$

If $q \neq 0$, we can divide by $-2\mathrm{i}q$ obtaining

$$\bar{z} \cdot z - z \cdot \left(-\frac{m}{2q} + \frac{\mathrm{i}}{2q}\right) - \bar{z} \cdot \left(-\frac{m}{2q} - \frac{\mathrm{i}}{2q}\right) = 0.$$

By (2.1), it is an equation of a circle with center $-\frac{m}{2q} - \frac{\mathrm{i}}{2q}$ and it passes through the origin, therefore its radius is $|\,-\frac{m}{2q} - \frac{\mathrm{i}}{2q}\,| = \frac{\sqrt{m^2+1}}{2|q|}$.

If $q = 0$, i.e. the straight line R passes through the origin O, by (2.7), $F(R)$ is the following straight line

$$\bar{z} \cdot (1 - \mathrm{i}m) - z \cdot (1 + \mathrm{i}m) = 0. \tag{2.8}$$

By (2.2), it is an equation of a non vertical straight line passing through the origin (thus the y-intercept is $q = 0$) with slope $-m$.

Finally we prove $6)$ and $7)$. The straight line of equation (2.3) becomes

$$\bar{z} + z = 2k \cdot \bar{z} \cdot z. \tag{2.9}$$

If $k \neq 0$, we can divide by $2k$ obtaining

$$\bar{z} \cdot z - z \cdot \frac{1}{2k} - \bar{z} \cdot \frac{1}{2k} = 0.$$

By (2.1), it is an equation of a circle with center $\frac{1}{2k}$ and it passes through the origin, therefore its radius is $\frac{1}{2|k|}$.

If $k = 0$, i.e. the vertical straight line R is the y-axis, by (2.9), $F(R)$ is R because its equation is

$$\bar{z} + z = 0.$$

By Remark 2.1 any Möbius transformation is a composition of elementary transformations. Then it is obvious that any Möbius function transforms straight lines and circles into straight lines and circles.

□

Definition 2.7. Let r_1 and r_2 be two intersecting lines in a point p. We say that the angle to ∞ between r_1 and r_2 is the supplementary of the angle defined in p.

Proposition 2.8. *The inversion $F(z) = \frac{1}{z}$ is a direct conformal map on $\mathbb{C}$.*

Proof. By Remark 2.5 for all $z \in \mathbb{C} \setminus \{0\}$, F is a direct conformal map. We prove that F keeps the oriented angles also in $z = 0$.

We denote by $R_\theta(z) = \mathrm{e}^{i\theta} z$ a rotation of angle θ. We observe that $(F \circ R_\theta)(z) = (R_{-\theta} \circ F)(z)$ for all $z \in \mathbb{C} \setminus \{0\}$ (note that $F^{-1} = F$), thus we can show the statement up to rotations.

Furthermore let T be the tangent line to the curve γ in the origin, then $F(T)$ is parallel to the tangent line, T', to $F(\gamma)$. We parameterize γ with a cartesian parameterization

$$\Phi \colon I \to \mathbb{R}^2, \Phi(u) = (u, \varphi(u)), \tag{2.10}$$

where $0 \in I$, I is an open interval of R, $\varphi \in \mathcal{C}^2(I, \mathrm{R})$ and $\varphi(0) = 0$. We see the inversion F as a function of real variables, i.e. $F(x, y) = (\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2})$. Certainly $F \circ \Phi$ is a parameterization of $F(\gamma)$, where

$$(F \circ \Phi)(u) = \left(\frac{u}{u^2 + \varphi(u)^2}, -\frac{\varphi(u)}{u^2 + \varphi(u)^2} \right), \qquad u \in I.$$

In order to prove that $F(T)$ is parallel to T', it is sufficient to show that the slopes of $F(T)$ and T' are the same. By (2.10) the slope of T is $\varphi'(0)$, therefore, by Theorem 2.6, the slope of $F(T)$ is

$$-\varphi'(0). \tag{2.11}$$

The cartesian coordinate of $F(\gamma)$ is $U := \frac{u}{u^2+\varphi(u)^2}$. We note that $U \to +\infty$ if and only if $u \to 0$; furthermore

$$\frac{\mathrm{d}}{\mathrm{d}U} = \left(- \frac{(u^2+\varphi(u)^2)^2}{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)} \right) \frac{\mathrm{d}}{\mathrm{d}u}.$$

Thus the slope of T' is

$$\lim_{u\to 0} \left(- \frac{(u^2+\varphi(u)^2)^2}{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)} \right) \cdot \frac{\mathrm{d}}{\mathrm{d}u} \left(- \frac{\varphi(u)}{u^2+\varphi(u)^2} \right). \tag{2.12}$$

We observe that

$$-\frac{\varphi(u)}{u^2+\varphi(u)^2} = \frac{u}{u^2+\varphi(u)^2} \cdot \left(- \frac{\varphi(u)}{u} \right), \qquad u \neq 0. \tag{2.13}$$

By Taylor expansion we have that

$$-\frac{\varphi(u)}{u} = -\varphi'(0) + O(u), \qquad u \to 0.$$

Hence, by (2.13), we get that

$$-\frac{\varphi(u)}{u^2+\varphi(u)^2} = \frac{u}{u^2+\varphi(u)^2} \cdot \big(- \varphi'(0) + O(u) \big), \qquad u \to 0.$$

By simple calculations we find that

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{u}{u^2+\varphi(u)^2} \right) = - \frac{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)}{(u^2+\varphi(u)^2)^2},$$

and $\frac{\mathrm{d}}{\mathrm{d}u}(-\varphi'(0)+O(u))$ is bounded around $u=0$. Therefore we have that

$$\lim_{u\to 0} \left(- \frac{(u^2+\varphi(u)^2)^2 \cdot \big(- \varphi'(0) + O(u) \big)}{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)} \right) \cdot \frac{\mathrm{d}}{\mathrm{d}u} \left(- \frac{u}{u^2+\varphi(u)^2} \right) = -\varphi'(0),$$

$$\lim_{u\to 0} \left(- \frac{(u^2+\varphi(u)^2) \cdot u}{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)} \right) \cdot \frac{\mathrm{d}}{\mathrm{d}u} \left(- \varphi'(0) + O(u) \right) = 0,$$

because it holds that

$$\lim_{u\to 0} \left(\frac{(u^2+\varphi(u)^2) \cdot u}{u^2-\varphi(u)^2+2u\cdot\varphi(u)\cdot\varphi'(u)} \right) = 0.$$

Thus, by (2.12), we derive that the slope of T' is $-\varphi'(0)$. By (2.11), we see that $F(T)$ and T' have the same slope, therefore $F(T)$ and T' are parallel. Hence recalling that the angle between two curves is the angle between the two tangent lines to the curves in their common point, we need to prove that F keeps the oriented angles between two straight lines that pass through the origin. Up to rotations we can think that one straight line has equation $x = 0$ and the other has equation $y = mx$. By Theorem 2.6 the straight line of equation $x = 0$ is kept, while the straight line of equation $y = mx$ becomes the straight line of equation $y = -mx$. We establish to measure the angle between two straight lines counterclockwise. Thus, by Definition 2.7 we get the statement (see also Figure 2.1).

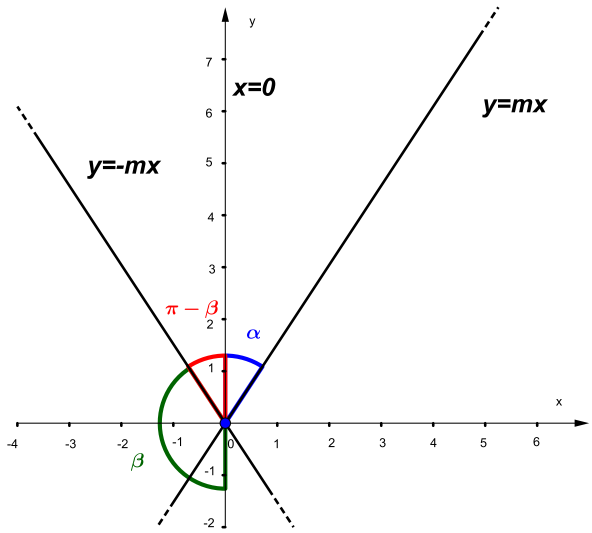

Figure 2.1: *In the origin $O = (0, 0)$ two straight lines of equations $x = 0$ and $y = mx$ pass creating an angle α. The inversion function fixes the straight line of equation $x = 0$ and transforms the straight line of equation $y = mx$ into the straight line of equation $y = -mx$ defining an angle to the infinity of $\pi - \beta = \alpha$.*

□

Remark 2.9. By Remark 2.5 and Proposition 2.8 it follows that any Möbius transformation is a direct conformal map on $\mathbb{C}$.

2.2 The geometry of planar bubbles

This section establishes important facts about basic planar geometry of lines and circles.

Lemma 2.10. *[17] At a vertex where three circular arcs meet at $120°$ angles, the cocycle condition is equivalent to requiring that the three circular edges leaving the vertex meet again in a single point and thus form a standard double bubble in the extended plane ($\mathbb{R}^2 \cup \infty$).*

Proof. Let three edges meet at $120°$ angles in a point p. First of all we prove that if the three edges satisfy the cocycle condition (the sum of the three oriented curvatures is zero), then the three edges meet again in other point q.

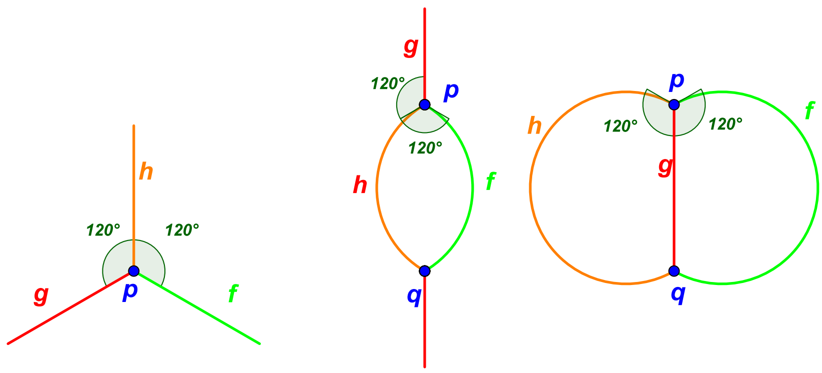

Figure 2.2: *The cases where at least one edge is straight; in the first picture there are three straight edges and in the other only one edge is straight.*

At the beginning we consider the situation where two edges are straight. Then by the cocycle condition, the third edge is also straight. Hence the three edges meet again at infinity at $120°$ (see Figure 2.2).

We suppose now that there is only one straight edge, therefore by the cocycle condition, the other two edges have the same absolute curvature. So the two edges define a component with two sides with two possibilities as in Figure 2.2.

Finally we take the case where each absolute curvature is positive; we suppose that two edges g and h meet in another point q (see Figure 2.3), then we must prove that the third edge, f, meets q. We consider the edge c of the circle C, that crosses p and q and which has the same direction of the edge f in p (i.e the two arcs have the same tangent in p and the same position respect to it), then we just show that $k(c) = k(f)$, where we denote by $k(l)$ the absolute curvature of any edge l (see Figure 2.3). In this way we have two arcs c and f with the same absolute curvature, that pass through p and have the same direction in p, then the two edges are the same and so f meets q.

By the assumptions, we know that (we use the convention seen in Remark 1.27)

$$k(g) - k(h) - k(f) = 0,$$

hence we want to prove that also

$$k(g) - k(h) - k(c) = 0;$$

so we get $k(c) = k(f)$.

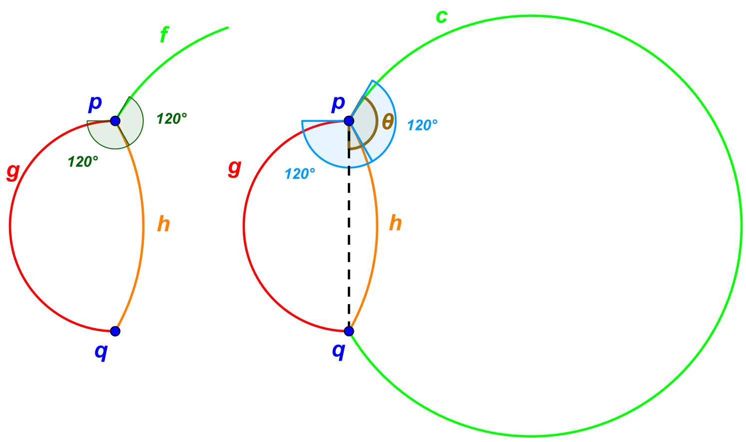

Figure 2.3: *The edges g and h meet again in q; we must see that the third edge, f, has the same absolute curvature of edge c of the circle, that crosses p and q and it has the same direction of edge f in p.*

Indeed, observing Figure 2.3, we have that (we denote by $d(p, q)$ the distance between two point, p and q)

$$\begin{aligned} k(c) &= \frac{2\sin\theta}{d(p,q)}, \\ k(g) &= \frac{2\sin\left(2\pi - \frac{2\pi}{3} - \theta\right)}{d(p,q)} = -\frac{2\sin\left(\frac{2\pi}{3}+\theta\right)}{d(p,q)}, \\ k(h) &= \frac{2\sin\left(\theta - \frac{2\pi}{3}\right)}{d(p,q)}. \end{aligned} \tag{2.14}$$

Using also the convention view in Remark 1.27, we get that

$$\begin{aligned} k(g) - k(h) - k(c) &= -\frac{2}{d(p,q)}\left(\sin\left(\frac{2\pi}{3}+\theta\right) + \sin\left(\theta - \frac{2\pi}{3}\right) + \sin\theta\right) \\ &= -\frac{2}{d(p,q)}\left(\frac{\sqrt{3}\cos\theta}{2} - \frac{\sin\theta}{2} - \frac{\sin\theta}{2} - \frac{\sqrt{3}\cos\theta}{2} + \sin\theta\right) \\ &= 0, \qquad \forall\theta. \end{aligned}$$

Finally we prove the viceversa, namely if the three edges meet again in q forming a standard double bubble, then the sum of the three oriented curvature in p is zero. The proof is based on Figure 2.3, on the formulas (2.14) and on the fact that $\sin\left(\frac{2\pi}{3}+\theta\right) + \sin\left(\theta - \frac{2\pi}{3}\right) + \sin\theta = 0$ for all θ.

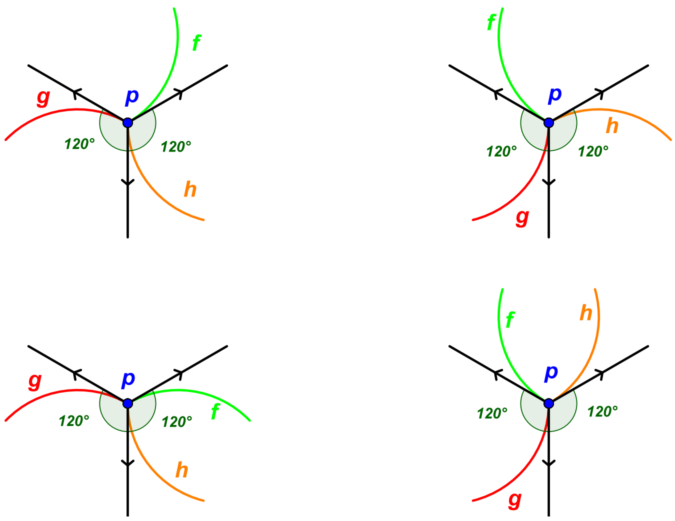

Figure 2.4: The four cases for three edges meet in a point p with positive absolute curvature.

Remark 2.11. We explicitly observe that the possibilities in Figure 2.4 are the only cases for three edges to meet in a point p with positive absolute curvature; at the top there are two possibilities where the cocycle condition is not satisfied, at the bottom other two possibilities where the cocycle condition is satisfied.

Corollary 2.12. *[17] The cocycle condition is invariant under a Möbius map.*

Proof. By Theorem 2.6, any Möbius map transforms straight lines and circles into straight lines and circles and, by Remark 2.9, it preserves the oriented angles, therefore any standard double bubble is also sent to another standard double bubble. Thus, by Lemma 2.10 we have the claim. □

In the next lemmas we discuss about geometry of consecutive edges of a component where it has internal angles of $120°$ (see Figure 2.5).

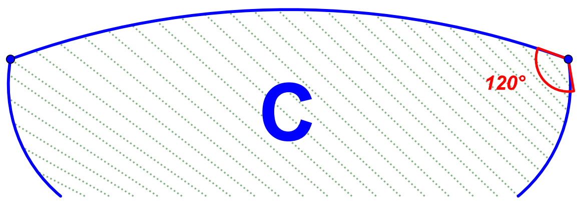

Figure 2.5: Internal angles of a component are $120°$. The shade denotes interior of a component.

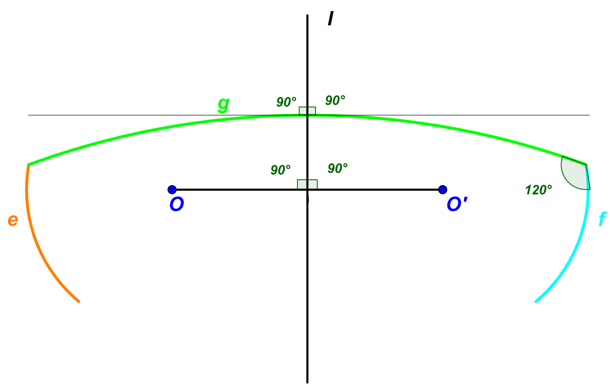

Figure 2.6: Three consecutive edges e, g and f with internal angles of $120°$ where e and f have the same signed curvature but different centers, O and O'.

Lemma 2.13. *[17] Let e, g and f be consecutive edges of a component C, such that e and f have the same signed curvature and different centers, O and O' respectively. Denote by l the axis of the segment $\overline{OO'}$, then l is the axis of g (see Figure 2.6).*

Proof. Let E and F be the circle to which e and f belong; having different centers, E and F are distinct.

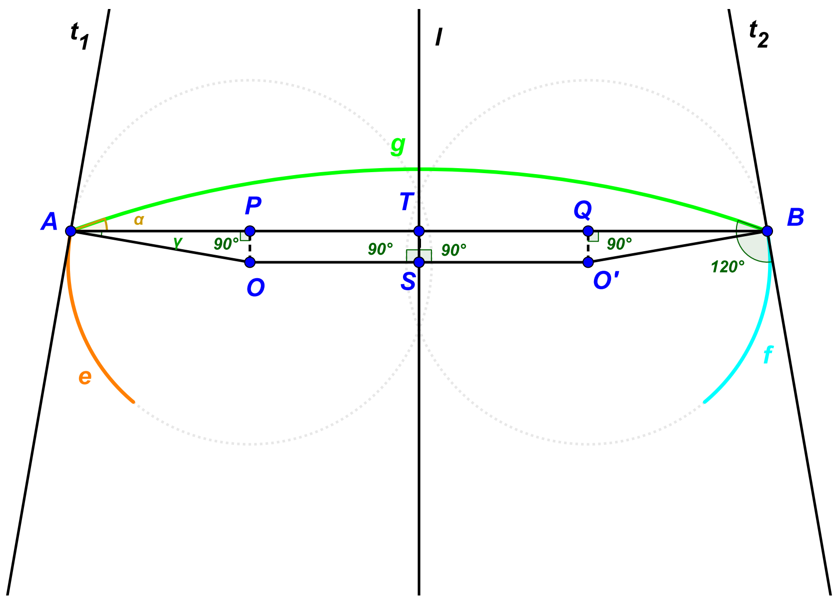

Figure 2.7: *Three consecutive edges e, g and f with internal angles of $120°$ where e and f have the same signed curvature but different centers. So the axis of the segment $\overline{OO'}$ is the axis of g.*

Looking at Figure 2.7 and denoted by α, the angle between the line through the vertices A and B of the edge g and the arc g, then the angle between the tangent t_1 to E in A and $\overline{AB}$ is $\frac{2\pi}{3} - \alpha$. The same is the angle between the tangent t_2 to F in B and $\overline{AB}$. Furthermore the angle between t_1 and the radius $\overline{OA}$ of E is $90°$, and at the same time, the angle between t_2 and the radius $\overline{O'B}$ of F. Therefore $O\hat{A}P = O'\hat{B}Q := \gamma$ and so

$$\overline{OP} = d(O, \overline{AB}) = R \sin\gamma = O'Q = d(O', \overline{AB}).$$

Thus $\overline{AB}$ and $\overline{OO'}$ are parallel; furthermore in this way l is perpendicular to $\overline{AB}$ and $\overline{AT} = \overline{TB}$, because $\overline{AP} = \overline{AO}\cos\gamma = \overline{O'B}\cos\gamma = \overline{QB}$ and $\overline{PT} = \overline{OS} = \overline{SO'} = \overline{TQ}$, since l is the axes of the segment $\overline{OO'}$. This completes the proof. $\square$

Remark 2.14. We explicitly note that in Lemma 2.13 it is very significant that the edges have the same signed curvature, indeed it allows us to prove that the straight lines $\overline{AB}$ and $\overline{OO'}$ are parallel.

Definition 2.15. Let e and f be two edges; we say that e and f are **cocircular** if e and f are in the same circle.

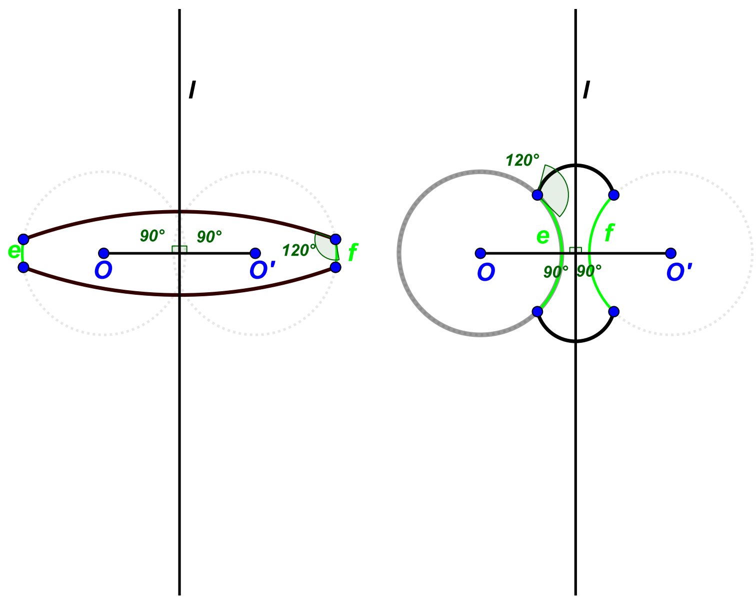

Figure 2.8: *Two examples of components C with four sides and vertically symmetric.*

Corollary 2.16. *Let C be a component with four edges, whose lateral sides have the same signed curvature and which are not cocircular, then C is vertically symmetric.*

Proof. Let e and f be the lateral sides of C, component with four edges; since e and f are not cocircular, then e and f have different centers, O and O' respectively, as shown in Figure 2.8. Furthermore by assumption e and

f have the same signed curvature, so by Lemma 2.6, if we consider the axis l of the segment joining the centers O and O', l is the axis of the top and bottom sides of C. Therefore l is the vertical axis of C. $\square$

Now we show two important lemmas. In the first one we determine the unique shape for a particular component with four edges and in the second we describe the situation where there is a sequence of four-sided components.

Lemma 2.17. *Let C be a four-sided component with inner angles of $\frac{2\pi}{3}$ and two opposite concave edges (concave edge according to Definition 1.28), then, if the concave sides are cocircular, the top and the bottom edge of C are strictly convex and strictly concave respectively. In particular the shape of C, up to translations, rotations and homoteties, is uniquely determined and it is represented in Figure 2.9.*

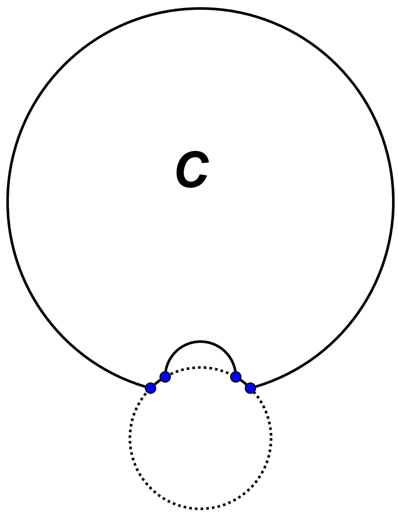

Figure 2.9: *The unique shape of a four-sided component C with two opposite concave circular edges and internal angles of $\frac{2\pi}{3}$.*

Proof. Since the two opposite edges are concave and cocircular, then they are external to the circle. Furthermore C has four edges, therefore its turning angle is $\frac{2\pi}{3}$, thus at least one of the other sides must be strictly convex (i.e its signed curvature must be positive). It is clear that the bottom edge of C must be strictly concave because, as shown in Figure 2.10, otherwise the condition of inner angles of $\frac{2\pi}{3}$ is contradicted. This concludes the proof.

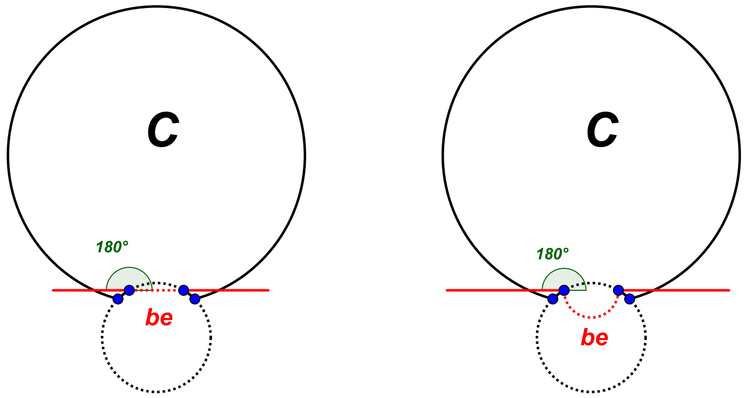

Figure 2.10: *In the first and second picture the bottom edge is straight and strictly convex respectively. In both, we note that there are two inner angles greater than* π.

□

Lemma 2.18. *[17] In a sequence of four-sided components of a bubble* $\mathbf{E}$ *(see Figure 2.11), if the sides* u_i *and* v_i *are cocircular, then any edges* u_j *and* v_j *are cocircular.*

Proof. By assumption there exists an index i such that the edges u_i and v_i are cocircular. We note that, since this sequence of four-sided components is in a bubble $\mathbf{E}$, in each vertex the cocycle condition holds. Therefore all edges u_j and v_j have the same absolute curvature respectively. We just show that u_j and v_j also have the same center. This is a consequence of Lemma 2.10. Indeed we consider a couple of edges u_j and v_j, respectively consecutive to u_i and v_i, as shown in Figure 2.11.

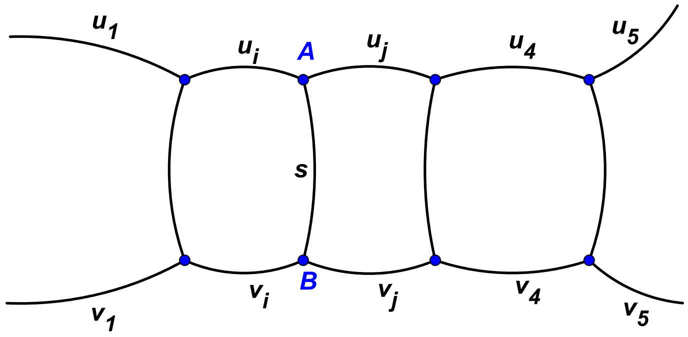

Figure 2.11: *A sequence of four-sided components of a bubble* **E**.

Since in the vertices A and B it is true the cocycle condition and the edges u_i and v_i are cocircular meeting the edge s in A and B, then, by Lemma 2.10, both edges u_j and v_j belong to the same arc, that passes to A and B with the same direction of u_j in A and the same direction of v_j in B. Therefore u_j and v_j also have the same center. We repeat the same argument for another couple of edges, so the lemma statement is clear.

$\square$

Now we present a lemma, that describes the curvature of a inner and external edge to a circle.

Lemma 2.19. *Let $\mathcal{C}$ be a circle of radius R and let L be an arc of a circle joining two points P and Q of $\mathcal{C}$. If L meets inside $\mathcal{C}$ at inner angles $\frac{2\pi}{3}$ as in Figure 2.12, then its curvature is given by*

$$k_L^i(\theta) := \frac{1}{R} \cdot \frac{\sin\left(\frac{\pi}{6} - \theta\right)}{\cos\theta}, \quad \theta \in \left] -\frac{\pi}{2}, \frac{\pi}{2}\right[. \tag{2.15}$$

Instead if L meets outside $\mathcal{C}$ at inner angles $\frac{2\pi}{3}$ as in Figure 2.13, then its curvature is given by

$$k_L^e(\theta) := \frac{1}{R} \cdot \frac{\sin\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}, \quad \theta \in \left] -\frac{\pi}{2}, \frac{\pi}{2}\right[. \tag{2.16}$$

In particular the functions k_L^i and k_L^e are bijective and k_L^i, k_L^e are strictly decreasing and increasing respectively.

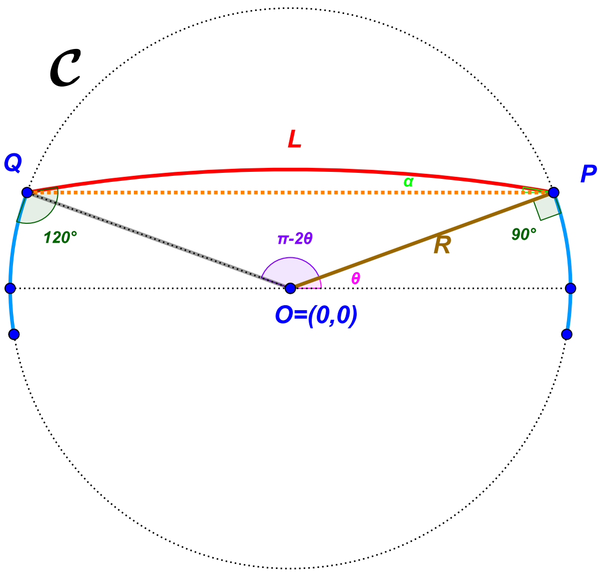

Figure 2.12: *An edge L meets inside the circle $\mathcal{C}$ at inner angle $\frac{2\pi}{3}$.*

Proof. We can assume that the circle $\mathcal{C}$ is centered in the origin of the plane $O = (0,0)$, otherwise we translate it. We call P and Q the meeting point between L and $\mathcal{C}$ and we respectively denote by α and θ, the angle between L and the joining line of its vertices P and Q and the angle determined by P on the circle. We denote by $d(A,B)$ the distance between two point in the plane. Initially we prove (2.15); observing Figure 2.12 we have that $\frac{\pi}{2} + \theta + \alpha = \frac{2\pi}{3}$, thus

$$\alpha = \frac{\pi}{6} - \theta. \tag{2.17}$$

Now $P = R(\cos\theta, \sin\theta)$ and $Q = R(-\cos\theta, \sin\theta)$. By the formulas in Proposition 5.4 we get that (note that $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$)

$$k_L^i(\alpha) = \frac{2\sin\alpha}{d(P,Q)} \overset{(2.17)}{=} \frac{2\sin\left(\frac{\pi}{6} - \theta\right)}{2R|\cos\theta|} = \frac{1}{R} \cdot \frac{\sin\left(\frac{\pi}{6} - \theta\right)}{\cos\theta}.$$

This is (2.15).

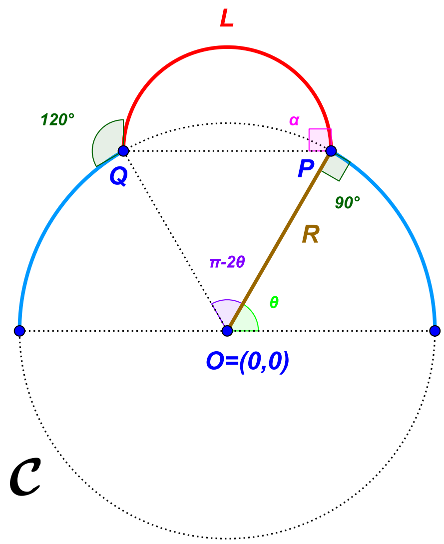

Figure 2.13: *An edge L meets outside the circumference $\mathcal{C}$ at inner angle $\frac{2\pi}{3}$.*

Finally we prove (2.16). In this case we are in the situation illustrated in Figure 2.13. By assumption L meets $\mathcal{C}$ at inner angles $\frac{2\pi}{3}$, then the external angles between L and $\mathcal{C}$ are $\frac{4\pi}{3}$. Therefore we have that $\frac{\pi}{2}+\theta+\alpha = \frac{4\pi}{3}$, thus

$$\alpha = \frac{5\pi}{6} - \theta. \tag{2.18}$$

We also obtain that $P = R(\cos\theta, \sin\theta)$ and $Q = R(-\cos\theta, \sin\theta)$. By the formulas in Proposition 5.4 we get that (note that $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$)

$$k_L^e(\alpha) = \frac{2\sin\alpha}{d(P,Q)} \overset{(2.18)}{=} \frac{2\sin\left(\frac{5\pi}{6}-\theta\right)}{2R|\cos\theta|} = \frac{1}{R}\cdot\frac{\sin\left(\frac{5\pi}{6}-\theta\right)}{\cos\theta}.$$

This is (2.16).

We see that the functions k_L^i and k_L^e are bijective. Their first derivative are

$$\frac{\mathrm{d}\,k_L^i(\theta)}{\mathrm{d}\theta} = -\frac{\sqrt{3}}{2}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{R},$$
$$\frac{\mathrm{d}\,k_L^e(\theta)}{\mathrm{d}\theta} = \frac{\sqrt{3}}{2}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{R}.$$

Thus the first derivatives are negative and positive respectively, therefore k_L^i and k_L^e are strictly decreasing and increasing respectively. Then k_L^i and k_L^e are injective. It is very simple to see that

$$\lim_{\theta \to -\frac{\pi}{2}^+} k_L^i(\theta) = +\infty,$$
$$\lim_{\theta \to \frac{\pi}{2}^-} k_L^i(\theta) = -\infty,$$

$$\lim_{\theta \to -\frac{\pi}{2}^+} k_L^e(\theta) = -\infty,$$
$$\lim_{\theta \to \frac{\pi}{2}^-} k_L^e(\theta) = +\infty.$$

Hence k_L^i and k_L^e are surjective (note that k_L^i and k_L^e are continuous), then the proof is concluded.

□

Remark 2.20. We note that in Lemma 2.19, the monotonicity and the bijectivity of the functions k_L^i and k_L^e do not depend by the radius R of the circle.

2.3 Reduction of a bubble with three edges

In this section we show how to simplify clusters by reducing any component with three edges. We call this method **reduction** of a three-sided component.

We begin with a lemma, that gives the uniqueness of a three-sided component with internal angles of $\frac{2\pi}{3}$ on an equilateral triangle.

Lemma 2.21. *Let T be an equilateral triangle, then the three-sided component C with internal angles of $\frac{2\pi}{3}$ and the same vertices of T is unique and it is as in Figure 2.14, namely the angles between the edges of C and the corresponding sides of T (i.e the chord line of the edge of C) are $30°$. In particular each edge of C has the same curvature.*

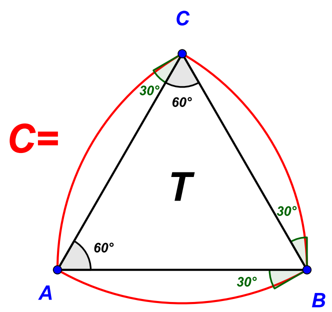

Figure 2.14: *The only three-sided component C with internal angles of $\frac{2\pi}{3}$ and the same vertices of an equilateral triangle T.*

Proof. Fixed an equilateral triangle T, let A, B and C be its vertices and l be the length of its side. It is clear that we can make a three-sided component with internal angles of $\frac{2\pi}{3}$ and the same vertices of T. Indeed we fix a vertex of T and the corresponding opposite side, then we consider the circle with center in this vertex and radius l. Since T is equilateral, this circle passes through the vertices of the fixed side, creating an angle of $\frac{\pi}{6}$ between the side of T and the arc of the circle for the vertices of the side (see Figure 2.15).

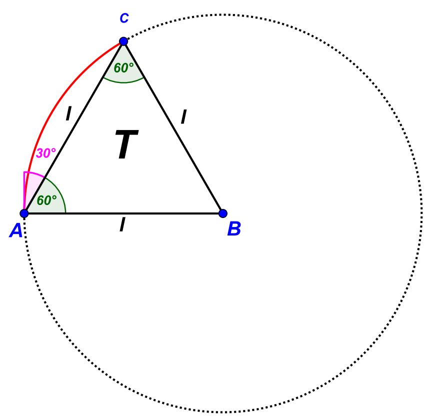

Figure 2.15: *The sides $\overline{AB}$ and $\overline{CB}$ are equal, because T is equilateral, therefore if we consider the circle with center in B and radius l, then it passes from A and C, creating an angle of $\frac{\pi}{6}$ between itself and the side of T, that crosses the vertices A and C.*

We repeat this construction for the other vertices and the corresponding opposite sides and so we obtain the component as illustrated in Figure 2.14.

Now we must see that, given T, we cannot realize any other three sided component C with internal angles of $\frac{2\pi}{3}$ and the same vertices of T. We call θ_1, θ_2, θ_3 the half of the turning angle of the edges L_1, L_2 and L_3 of C respectively. We note that, since the vertices of T must be the vertices of C, then each side of T represents the chord line of an arc of C. By Lemma 1.38, we know that $\sum_{i=1}^{3} \theta_i = \frac{\pi}{2}$, thus there exists an index $i \in \{1, 2, 3\}$ such that $0 < \theta_i < \pi$ (i.e there exists at least one edge of C with positive signed curvature[2]). Without loss of generality we can assume $i = 1$ and we consider positive the angles that are exterior to the triangle. The edges L_2 and L_3 cannot be straight, because if L_2 is straight (the argument is the same if L_3 is straight), then $\theta_1 = \frac{\pi}{3}$, because the internal angles of C must be of $\frac{2\pi}{3}$ and T is equilateral. Therefore L_3 is also straight, then C would have an internal angle of $\frac{\pi}{3}$ as illustrated in Figure 2.16, but this is a contradiction.

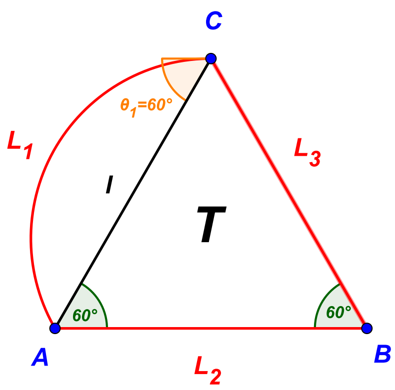

Figure 2.16: *If L_2 is straight, since the internal angles of C must be of $\frac{2\pi}{3}$ and T is equilateral, then L_3 must be straight and so C would have an internal angle of $\frac{\pi}{3}$.*

[2]We recall that the curvature k of an edge through the points P and Q is $k = \frac{2 \sin \theta}{d(P,Q)}$, where θ is the angle between the edge and the segment.

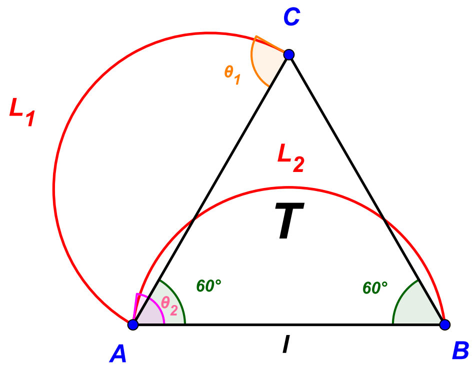

Figure 2.17: *The situation in the case if* $\min\limits_{i=2,3} \theta_i = \theta_2 \leq -\dfrac{\pi}{3}$.

Therefore we have that $\min(|\theta_2|, |\theta_3|) > 0$. Furthermore we prove that, if $\min(\theta_2, \theta_3) < 0$, then

$$\min(\theta_2, \theta_3) > -\frac{\pi}{3}, \tag{2.19}$$

namely, the arcs of C are external or inner to T.

For example we suppose that $\min(\theta_2, \theta_3) = \theta_2$; if $\min(\theta_2, \theta_3) = \theta_3$ the argument is the same. We proceed by contradiction, then we are in the situation described in Figure 2.17. We recall that $0 < \theta_1 < \pi$ and the internal angles of C must be of $\frac{2\pi}{3}$, but in the angle between L_1 and L_2 we have that

$$\frac{2\pi}{3} = \theta_1 + \theta_2 < \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

This is a contradiction, thus $\min(\theta_2, \theta_3) > -\frac{\pi}{3}$.

We note that we have decided to consider positive the angles that are exterior to the triangle and by (2.19) the angles θ_1, θ_2, θ_3 are external or internal to T (θ_1 is always external). Since the internal angles of C must be of $\frac{2\pi}{3}$, θ_1, θ_2, θ_3 must satisfy the following linear system (in Figure 2.18 the

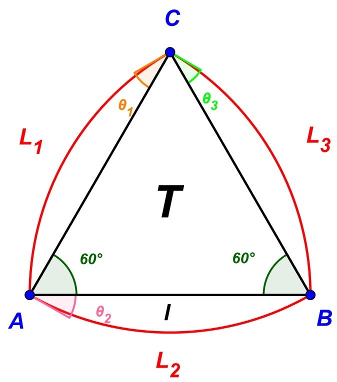

Figure 2.18: *This is the situation when* $\min\limits_{i=1,2,3} \theta_i > 0$.

case when $\min\limits_{i=1,2,3} \theta_i > 0$ is represented)

$$\begin{cases} \theta_1 + \frac{\pi}{3} + \theta_2 = \frac{2\pi}{3} \\ \theta_2 + \frac{\pi}{3} + \theta_3 = \frac{2\pi}{3} \\ \theta_3 + \frac{\pi}{3} + \theta_1 = \frac{2\pi}{3}. \end{cases}$$

The only solution of this system is $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{6}$. Therefore C is as in Figure 2.14.

□

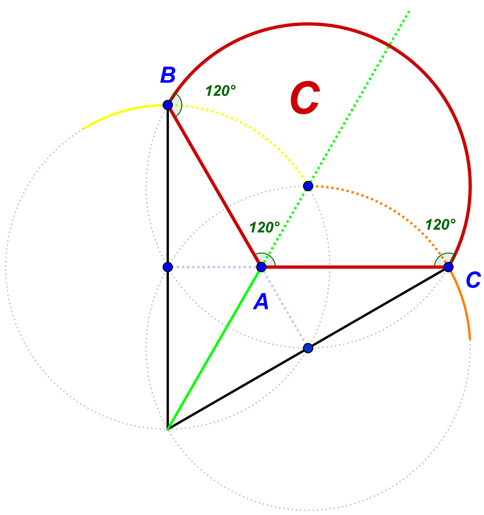

Figure 2.19: *A component C with three edges, whose incident edges are prolonged.*

Lemma 2.22. *[17] Let C be a component with three edges of a planar regular N-cluster $\mathbf{E}$, then, extending its three incident edges (incident edge according to the Definition 1.34) into the component, they meet in a point satisfying the cocycle condition (see Figure 2.19).*

Proof. By Definition 1.31 we know that in each vertex of a planar regular N-cluster $\mathbf{E}$, the cocycle condition is satisfied. By Lemma 2.10 this means that the three edges leaving the vertex meet again and thus form a standard double bubble. Thus we can extend in this way the three incident edges of a three-sided component C of $\mathbf{E}$. We recall that the orientation of any component is fixed as in Remark 1.26. We choose F, a Möbius map, such that the three vertices of C go into the vertices of an equilateral triangle as in Figure 2.20.

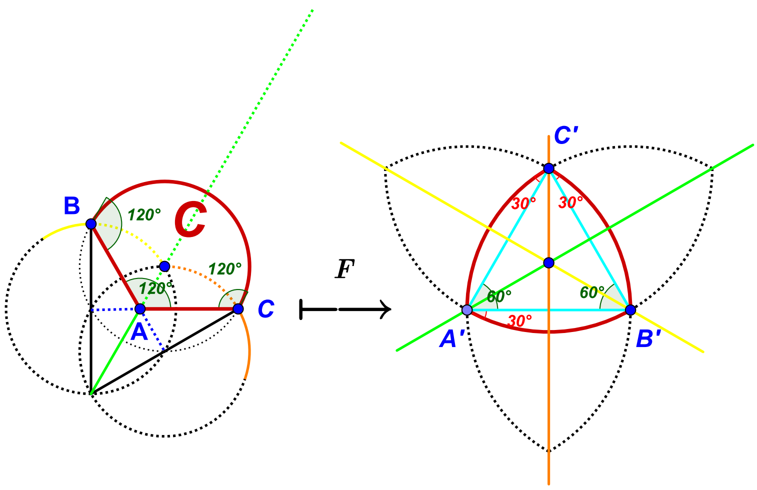

Figure 2.20: *A component C with three edges and its extended incident edges are mapped by a Möbius map into an equal-sided three component.*

We explicitly note that, by Remark 2.9, any Möbius transformation is a direct conformal map on $\mathbb{C}$. Hence C and $F(C)$ have the same orientation and so in this way the interior of the image of C is the image of the interior of C. Since the vertices A', B' and C' of $F(C)$ are vertices of an equilateral triangle, by Lemma 2.21, $F(C)$ is a three component with equal edges and each arc has the same curvature. Furthermore in the vertices A', B' and C' of $F(C)$ the cocycle condition applies because it is true in the vertices A, B and C and F is a Möbius map (see Corollary 2.12). Thus the three incident edges of C are mapped by F in three straight lines, which are the axis of symmetry of $F(C)$ (see Figure 2.20). In the inner point of $F(C)$, where the image of extended incident edges of C meet, the angles are $120°$ creating a standard double bubble at the infinity; so by Lemma 2.10 here it is satisfied the cocycle condition. Now we can return back with F^{-1} (since F is a Möbius map then F^{-1} is a Möbius map) and so we obtain the statement. $\square$

Remark 2.23. The Möbius map of the previous lemma, up to translations and homotheties, that sends the three vertices A, B and C of a three-sided component of a regular planar N-cluster $\mathbf{E}$ in the three vertices A', B' and C' of an equilateral triangle, is

$$F(z) = \frac{z \cdot \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}(C - B)}{z(C - B \cdot \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}) - CB(1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{3}})}. \tag{2.20}$$

Indeed, without loss of generality, we can assume that $A = (0, 0)$ (otherwise we translate the three-sided component; in complex notation $A = 0$) and $B = (B, 0)$ (otherwise we rotate the three-sided component; in complex notation the imaginary part of B is null and so B $\in \mathbb{R}$). We show that a Möbius transformation exists, that transforms A, B and C in the following equilateral triangle $A' = (0, 0)$, $B' = (1, 0)$ and $C' = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ (in complex notation they are 0, 1 and $\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ respectively). If we prove this, then three vertices A, B, C can be transformed in any equilateral triangle composing F with suitable translations or homotheties.

In order to prove that A, B and C can be mapped in A', B' and C', we must imposed the following relations (we recall that $F(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$)

$$\begin{cases} b = 0 \\ aB = cB + d \\ aC = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}(c \cdot C + d). \end{cases}$$

Since three vertices A, B and C are distinct we have that (we can divide for B and C that are non zero)

$$\begin{cases} b = 0 \\ a = \frac{cB+d}{B} \\ a = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}(c \cdot C+d)}{C}. \end{cases}$$

From the second and third equation we get that

$$c \cdot CB(1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}) + d(C - B \cdot \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}) = 0.$$

If $C - B\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} = 0$, then we have that the starting triangle is equilateral, therefore we can choose $F(z) = z$. Thus $C - B\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} \neq 0$ and so we find that

$$d = -\frac{c \cdot CB(1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{3}})}{C - B \cdot \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}. \tag{2.21}$$

Since we have that $b = 0$, then $ad - bc = ad \neq 0$, thus a and d must be different from zero. By (2.21), we obtain that $c \neq 0$. Therefore with simple algebraic computations we have (2.20).

Chapter 3

Conditions on area, variations and estimates of bubbles

This chapter is divided in three section.

In this first section, we present the key theorem of the thesis; it is Theorem 3.5 and it gives some necessary conditions on the quantity of area that different components of the same region must have.

In the second section we introduce three particular variations in Lemma 3.11, Lemma 3.12 and Lemma 3.14. In the first we find the minimum quantity of area that a component of a disconnected region must have. In the second the goal is to promote the external components of a region respect to its inner components; this will give an important estimate for the pressure of a region. In particular it is very significant in the case when a big component of a region is external.

Finally in the last Lemma we determine a simple estimate for all edges of a weakly minimizing N-cluster for the problem (1.10).

In the third section we conclude with an interesting lemma, Lemma 3.16, where we determine a significant estimate for the pressures of a standard double bubble.

3.1 Necessary conditions on area

Lemma 3.1. *Let C and D be real constants such that D is non negative and $\sqrt{D} \leq C \leq \sqrt{2D}$, then the solution of the following inequality*

$$\sqrt{x} + \sqrt{D-x} \leq C, \quad 0 \leq x \leq D, \tag{3.1}$$

is

$$0 \leq x \leq \frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} \quad or \quad \frac{D + \sqrt{C^2 \cdot (2D - C^2)}}{2} \leq x \leq D. \tag{3.2}$$

Proof. With easy algebraic steps and by assumption on the constants C and D, we obtain that the solution of (3.1) is the same of the following inequality

$$4x^2 - 4x \cdot D + (C^2 - D)^2 \geq 0,$$

that is just (3.2).

□

Remark 3.2. Let C and D be real constants such that D is positive and $\sqrt{D} < C < \sqrt{2D}$, then

$$0 < \frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{D}{2}, \quad \frac{D}{2} < \frac{D + \sqrt{C^2 \cdot (2D - C^2)}}{2} < D. \tag{3.3}$$

First of all we note that $\frac{D+\sqrt{C^2 \cdot (2D-C^2)}}{2} = D - \frac{D-\sqrt{C^2 \cdot (2D-C^2)}}{2}$. Therefore we just show that $0 < \frac{D-\sqrt{C^2 \cdot (2D-C^2)}}{2} < \frac{D}{2}$. By assumption it is immediately clear that $\frac{D-\sqrt{C^2 \cdot (2D-C^2)}}{2} < \frac{D}{2}$, while

$$\frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} > 0$$

is equivalent to

$$C^2 \cdot (2D - C^2) < D^2$$

and it is the same as

$$C^4 - 2D \cdot C^2 + D^2 = (C^2 - D)^2 > 0.$$

By assumption $C^2 > D$, thus the last inequality is true.

This proves (3.3).

Remark 3.3. Let C and D be real constants such that D is positive and $\sqrt{D} < C < \sqrt{D\left(1 + \frac{2\sqrt{2}}{3}\right)}$, then

$$0 < \frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{D}{3}, \quad \frac{2D}{3} < \frac{D + \sqrt{C^2 \cdot (2D - C^2)}}{2} < D. \tag{3.4}$$

Since $1 + \frac{2\sqrt{2}}{3} < 2$, then $\sqrt{D} < C < \sqrt{2D}$, thus, by Remark 3.2,

$$0 < \frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{D}{2}, \quad \frac{D}{2} < \frac{D + \sqrt{C^2 \cdot (2D - C^2)}}{2} < D.$$

Therefore to prove (3.4), it is enought to show that $\frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{D}{3}$ when $D > 0$ and $\sqrt{D} < C < \sqrt{D\left(1 + \frac{2\sqrt{2}}{3}\right)}$. Indeed we have that

$$\frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{D}{3}$$

is equivalent to

$$\sqrt{C^2 \cdot (2D - C^2)} > \frac{D}{3},$$

that is same as

$$C^4 + C^2(-2D) + \left(\frac{D}{3}\right)^2 < 0.$$

The solution of the previous inequality is

$$D \cdot \left(1 - \frac{2\sqrt{2}}{3}\right) < C^2 < D \cdot \left(1 + \frac{2\sqrt{2}}{3}\right). \tag{3.5}$$

By assumption $C > \sqrt{D}$, thus $C^2 > D > D \cdot \left(1 - \frac{2\sqrt{2}}{3}\right)$. Thus (3.5) is equivalent to

$$-\sqrt{D\cdot\left(1+\frac{2\sqrt{2}}{3}\right)}<C<\sqrt{D\cdot\left(1+\frac{2\sqrt{2}}{3}\right)}.$$

Also by assumption $C > \sqrt{D}$ and $D > 0$, thus $C > 0$, so the previous relation is equivalent to

$$C<\sqrt{D\cdot\left(1+\frac{2\sqrt{2}}{3}\right)}.$$

This completes the proof.

Remark 3.4. We explicitly note the following estimate holds on $p^*_{2,N}(\mathbf{a})$

$$p^*_{2,N}(\mathbf{a}) > \sqrt{\pi}\left(\sum_{i=1}^{N}\sqrt{a_i}+\sqrt{a_0}\right), \tag{3.6}$$

where $a_0 := \sum_{i=1}^{N} a_i$.

In particular

$$\frac{p^*_{2,N}(\mathbf{a})-\sqrt{\pi}\left(\sum_{j\neq i}\sqrt{a_j}+\sqrt{a_0}\right)}{\sqrt{\pi}} > \sqrt{a_i}, \tag{3.7}$$

for all $i = 1, \dots, N$.

Indeed we consider $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$, then, by the isoperimetric inequality, we get that (note that, by Remark 1.7, $\mathbf{E} \in \mathcal{M}_{2,N}(m(\mathbf{E}))$, thus, by Theorem 1.10, E_i can not be a circle)

$$P(\mathbf{E}) = \frac{1}{2}\left(\sum_{i=1}^{N} P(E_i) + P(E_0)\right) > \sqrt{\pi}\left(\sum_{i=1}^{N}\sqrt{a_i}+\sqrt{a_0}\right).$$

Therefore, we have (3.6).

Given $\mathbf{a} \in \mathbb{R}^N_+$, $\mathbf{a} = (a_1, \dots, a_N)$, we set $a_0 := \sum_{i=1}^{N} a_i$ and let i be an index in $\{1, \dots, N\}$, then we place

$$\Phi_{i,\mathbf{a}}(p) :\ = \frac{p-\sqrt{\pi}\left(\sqrt{a_0}+\sum_{j\neq i}\sqrt{a_j}\right)}{\sqrt{\pi}},$$

$$x_{i,\mathbf{a}}(p) : = \frac{a_i - \sqrt{\Phi^2_{i,\mathbf{a}}(p) \cdot \left(2a_i - \Phi^2_{i,\mathbf{a}}(p)\right)}}{2}, \quad \forall p \in \{p \,|\, \Phi_{i,\mathbf{a}}(p) \le \sqrt{2a_i}\}, \tag{3.8}$$

where p is the value of the perimeter of $\mathbf{E} \in \mathcal{C}^*_{2,N}(\mathbf{a})$.

Theorem 3.5. *Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$ and p be the perimeter of a weak competitor such that $\Phi_{i,\mathbf{a}}(p) < \sqrt{2a_i}$. If $\mathbf{E}$ is not standard with disconnected region E_i, then each disjoint union U of components of E_i, satisfies that $0 < |\,U\,| \le x_{i,\mathbf{a}}(p)$ or $|\,U\,| \ge a_i - x_{i,\mathbf{a}}(p)$.*

Proof. By Remark 1.19 we can see E_i as finite disjoint union of its components. We suppose by contradiction that there exists one disjoint union U of components of the region E_i such that $x_{i,\mathbf{a}}(p) < |\,U\,| < a_i - x_{i,\mathbf{a}}(p)$. By (3.7) of Remark 3.4 and by assumption on p we have that

$$\sqrt{a_i} < \Phi_{i,\mathbf{a}}(p) < \sqrt{2a_i}. \tag{3.9}$$

We set $D = a_i$ and $C = \Phi_{i,\mathbf{a}}(p)$, then, by (3.9) and by Remark 3.2, we get that

$$0 < x_{i,\mathbf{a}}(p) = \frac{D - \sqrt{C^2 \cdot (2D - C^2)}}{2} < \frac{a_i}{2}. \tag{3.10}$$

Thus it follows that, by the minimality of $\mathbf{E}$ and by the isoperimetric inequality, (note that, by (3.10), $|\,E_i \setminus U\,| = |\,E_i\,| - |\,U\,| \ge a_i - |\,U\,| > x_{i,\mathbf{a}}(p) > 0$)

$$\begin{aligned} p \ge P(\mathbf{E}) &\ge \frac{1}{2}\left(P(U) + P(E_i \setminus U) + P(E_0) + \sum_{j \ne i} P(E_j)\right) \\ &\ge \sqrt{\pi}\left(\sqrt{|\,U\,|} + \sqrt{a_i - |\,U\,|} + \sqrt{a_0} + \sum_{j \ne i} \sqrt{a_j}\right). \end{aligned} \tag{3.11}$$

Let $x = |\,U\,|$, therefore we have the following inequality

$$\sqrt{x} + \sqrt{a_i - x} \le \frac{p - \sqrt{\pi}\left(\sum_{j \ne i} \sqrt{a_j} + \sqrt{a_0}\right)}{\sqrt{\pi}} = \Phi_{i,\mathbf{a}}(p),$$

for $x_{i,\mathbf{a}}(p) < x < a_i - x_{i,\mathbf{a}}(p)$. It contradicts Lemma 3.1, because by (3.9), it follows that $\sqrt{D} < C < \sqrt{2D}$.

□

In the following remark we show that, in Theorem 3.5, it is better to choose a value p of a perimeter of $\mathbf{E} \in \mathcal{C}^*_{2,N}(\mathbf{a})$ "near" to $p^*_{2,N}(\mathbf{a})$ and such that $\Phi_{i,\mathbf{a}}(p) < \sqrt{2a_i}$.

Remark 3.6. Let q be the value of the perimeter of $\mathbf{E} \in \mathcal{C}^*_{2,N}(\mathbf{a})$ such that $\Phi_{i,\mathbf{a}}(q) < \sqrt{2a_i}$, then the function (the index i is fixed in $\{1, \dots, N\}$)

$$x_{i,\mathbf{a}} : [p^*_{2,N}(\mathbf{a}), q] \to \mathbb{R}, \qquad x_{i,\mathbf{a}}(p) = \frac{a_i - \sqrt{\Phi^2_{i,\mathbf{a}}(p) \cdot \left(2a_i - \Phi^2_{i,\mathbf{a}}(p)\right)}}{2}, \tag{3.12}$$

is strictly increasing. In particular $x_{i,\mathbf{a}}(p^*_{2,N}(\mathbf{a})) \leq x_{i,\mathbf{a}}(q)$.

We set $I = [p^*_{2,N}(\mathbf{a}), q]$; the first derivative of $\Phi_{i,\mathbf{a}}$ is $\Phi^{'}_{i,\mathbf{a}}(p) = \frac{1}{\sqrt{\pi}}$, therefore $\Phi_{i,\mathbf{a}}$ is strictly increasing on I. Thus, by (3.7) of Remark 3.4 and by assumption on q, we have that

$$\sqrt{a_i} < \Phi_{i,\mathbf{a}}\big(p^*_{2,N}(\mathbf{a})\big) \leq \Phi_{i,\mathbf{a}}(p) \leq \Phi_{i,\mathbf{a}}(q) < \sqrt{2a_i}, \quad \forall p \in I. \tag{3.13}$$

Then the function $x_{i,\mathbf{a}}$ is well defined on I and $x_{i,\mathbf{a}}(p) = (F \circ \Phi_{i,\mathbf{a}})(p)$, where $F(x) := \frac{a_i - \sqrt{x^2 \cdot (2a_i - x^2)}}{2}$, with $x \in [\Phi_{i,\mathbf{a}}\big(p^*_{2,N}(\mathbf{a})\big), \Phi_{i,\mathbf{a}}(q)]$. By algebraic calculations the first derivative of F is (note that $x \geq \Phi_{i,\mathbf{a}}\big(p^*_{2,N}(\mathbf{a})\big) > \sqrt{a_i} > 0$)

$$F'(x) = \frac{x^2 - a_i}{\sqrt{2a_i - x^2}}. \tag{3.14}$$

Then the first derivative of $x_{i,\mathbf{a}}$ is

$$x^{'}_{i,\mathbf{a}}(p) = \frac{F'\big(\Phi_{i,\mathbf{a}}(p)\big)}{\sqrt{\pi}} = \frac{\Phi^2_{i,\mathbf{a}}(p) - a_i}{\sqrt{\pi \cdot \left(2a_i - \Phi^2_{i,\mathbf{a}}(p)\right)}}. \tag{3.15}$$

From (3.13) the claim follows.

Remark 3.7. By Remark 1.19, given $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$, each disconnected region E_i of $\mathbf{E}$ can be seen as disjoint union of its components. In the following remark we will show, as under specific conditions, there is a particular decomposition for E_i.

Remark 3.8. Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$ and p be the perimeter of a weak competitor such that $\Phi_{i,\mathbf{a}}(p) < \sqrt{a_i \cdot \left(1 + \frac{2\sqrt{2}}{3}\right)}$. We show that if the region E_i of $\mathbf{E}$ is not connected then

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \dots E_i^{M(i)}, \tag{3.16}$$

with

a) $|\, E_i^0 \,| \geq |\, E_i \,| - x_{i,\mathbf{a}}(p) \geq a_i - x_{i,\mathbf{a}}(p)$;

b) $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq x_{i,\mathbf{a}}(p)$

where E_i^j is a component of E_i for any $j = 0, \dots, M(i)$ (note that $M(i)$ is finite by Theorem 1.10 and $M(i) > 1$, because E_i is disconnected). Furthermore it holds that $0 < x_{i,\mathbf{a}}(p) < \frac{a_i}{3}$, therefore any E_i^j ($j = 1, \dots, M(i)$) is a small component and E_i^0 is the big component by Definition 1.18 (note that $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$, thus $|\, E_i \,| \geq a_i$).

By (3.7) of Remark 3.4 and by assumption on p we have that

$$\sqrt{a_i} < \Phi_{i,\mathbf{a}}(p) < \sqrt{a_i \cdot \left(1 + \frac{2\sqrt{2}}{3}\right)}. \tag{3.17}$$

We set $D = a_i$ and $C = \Phi_{i,\mathbf{a}}(p)$, then, by (3.17) and by Remark 3.3, we get that

$$0 < x_{i,\mathbf{a}}(p) < \frac{a_i}{3}. \tag{3.18}$$

Therefore if we prove (3.16) with the properties $a)$ and $b)$, by Definition 1.18, any E_i^j is a small component,while E_i^0 is the big component.

By assumption on p and by Theorem 3.5 (note that $1 + \frac{2\sqrt{2}}{3} < 2$), we know that $0 < |\, U \,| \leq x_{i,\mathbf{a}}(p)$ or $|\, U \,| \geq a_i - x_{i,\mathbf{a}}(p)$ for any disjoint union U of components of E_i (since E_i is not connected, then $\mathbf{E}$ is not standard). In order to prove (3.16), we just show that E_i has one and only one component C with $|\, C \,| \geq a_i - x_{i,\mathbf{a}}(p)$. Indeed if it is true, by Theorem 3.5, it follows that $|\, E_i \setminus C \,| < x_{i,\mathbf{a}}(p)$, thus $|\, C \,| = |\, E_i \,| - |\, E_i \setminus C \,| \geq |\, E_i \,| - x_{i,\mathbf{a}}(p)$.

We explicitly note that the condition

$$\Phi_{i,\mathbf{a}}(p) < \sqrt{a_i \cdot \left(1 + \frac{2\sqrt{2}}{3}\right)}$$

is equivalent to

$$\sqrt{\pi}\left(\sqrt{a_0}+\sum_{j\neq i}\sqrt{a_j}+\sqrt{a_i\left(1+\frac{2\sqrt{2}}{3}\right)}\right)-p>0. \tag{3.19}$$

We underline that

$$2\cdot\sqrt{\frac{2}{3}}>\sqrt{1+\frac{2\sqrt{2}}{3}}. \tag{3.20}$$

Therefore, if there are at least two components C_i^1 and C_i^2 of E_i with their area greater or equal to $a_i - x_{i,\mathbf{a}}(p)$, by the isoperimetric inequality and by the minimality of $\mathbf{E}$, we get that (note that $P(E_i)\geq P(C_i^1)+P(C_i^2)$ and, by (3.18), $a_i - x_{i,\mathbf{a}}(p) > \frac{2a_i}{3}$)

$$\begin{aligned} p \geq P(\mathbf{E}) &\geq \frac{1}{2}\left(P(E_0)+\sum_{j\neq i}P(E_j)+P(C_i^1)+P(C_i^2)\right) \\ &\geq \sqrt{\pi}\left(\sqrt{a_0}+\sum_{j\neq i}\sqrt{a_j}+2\sqrt{\frac{2a_i}{3}}\right). \end{aligned}$$

Hence, by (3.19) and (3.20), we obtain that

$$\begin{aligned} 0 &\geq \sqrt{\pi}\left(\sqrt{a_0}+\sum_{j\neq i}\sqrt{a_j}+2\sqrt{\frac{2a_i}{3}}\right)-p \\ &\geq \sqrt{\pi}\left(\sqrt{a_0}+\sum_{j\neq i}\sqrt{a_j}+\sqrt{a_i\left(1+\frac{2\sqrt{2}}{3}\right)}\right)-p>0. \end{aligned}$$

This is clearly a contradiction.

Now we prove the existence of a component C, which satisfies that $|\,C\,| \geq a_i - x_{i,\mathbf{a}}(p)$. We show that if $E_i = \bigsqcup_{j=1}^{M(i)} C_j$, where $0<|\,C_j\,|\leq x_{i,\mathbf{a}}(p)$ for all $j=1,\dots,M(i)$, then there exists $s\in\{2,\dots,M(i)-1\}$ such that $x_{i,\mathbf{a}}(p) < \sum_{k=1}^{s}|\,C_k\,| < a_i - x_{i,\mathbf{a}}(p)$. Therefore, if we call $E_i^1 := \bigsqcup_{j=1}^{s} C_j$, $\mathbf{E}$ would have a component with area which contradicts Theorem 3.5, given the condition on p and by the minimality of $\mathbf{E}$. We know that

1) $\sum_{j=1}^{M(i)}|\,C_j\,|\geq a_i$,

2) $0 < x_{i,\mathbf{a}}(p) < \frac{a_i}{3}$,

3) $0 < |\, C_j \,| \leq x_{i,\mathbf{a}}(p), \qquad \forall j = 1 \dots, M(i)$.

At the beginning we prove the existence of $s \in \{2, \dots, M(i) - 1\}$ such that $\sum_{j=1}^{s} |\, C_j \,| > x_{i,\mathbf{a}}(p)$. We proceed by contradiction, therefore it follows that $a_i \overset{1)}{\leq} \sum_{j=1}^{M(i)-1} |\, C_j \,| + |\, C_{M(i)} \,| \overset{3)}{\leq} 2x_{i,\mathbf{a}}(p)$, finding $x_{i,\mathbf{a}}(p) \geq \frac{a_i}{2}$, but this contradicts 2).

We consider

$$s = \min \left\{ t \in \{2, \dots, M(i) - 1\} \,\middle|\, \sum_{j=1}^{t} |\, C_j \,| > x_{i,\mathbf{a}}(p) \right\} \tag{3.21}$$

then $\sum_{j=1}^{s} |\, C_j \,| < a_i - x_{i,\mathbf{a}}(p)$. We suppose it is not true, thus we have that

$$a_i - x_{i,\mathbf{a}}(p) \leq \sum_{j=1}^{s} |\, C_j \,| = \sum_{j=1}^{s-1} |\, C_j \,| + |\, C_s \,| \overset{\text{(3.21) and 3)}}{\leq} x_{i,\mathbf{a}}(p) + x_{i,\mathbf{a}}(p) = 2x_{i,\mathbf{a}}(p)$$

getting $x_{i,\mathbf{a}}(p) \geq \frac{a_i}{3}$, that is an absurd, because it contradicts 2). This concludes the proof.

3.2 Variations of bubbles

Lemma 3.9. *Let A be a positive constant and $f(x) = x \cdot \left(\sqrt{1 + \frac{2A}{x}} - 1 \right)$, $x > 0$. Then f is strictly increasing.*

Proof. The first derivative of f is

$$\frac{x + A - \sqrt{x(x + 2A)}}{\sqrt{x(x + 2A)}}.$$

Now the first derivative is positive because

$$\sqrt{x(x + 2A)} < x + A,$$

is equivalent to

$$x(x + 2A) < (x + A)^2,$$

that is the same as

$$0 < A^2.$$

$\square$

Lemma 3.10. *Let A be a positive constant and $f(x) = x - A \cdot \left(\sqrt{1 + \frac{2x}{A}} - 1 \right)$, $x \geq 0$. Then f is strictly increasing.*

Proof. The proof is a direct consequence of the first derivative of f, which is

$$f'(x) = \frac{\sqrt{A + 2x} - \sqrt{A}}{\sqrt{A + 2x}}.$$

$\square$

Lemma 3.11. *Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$. If $\mathbf{E}$ is not standard, then the following inequalities hold:*

1) $S \leq \frac{|S_i|}{2(|E_i| - |S_i|)} \cdot P(\mathbf{E})$,

2) $|S_i| \geq \frac{16\pi}{N_r^2} \cdot \left(\frac{|E_i| - |S_i|}{P(\mathbf{E})} \right)^2$,

3) $|S_i| \geq |E_i| - \frac{(P(\mathbf{E}) \cdot N_r)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot |E_i|}{(P(\mathbf{E}) \cdot N_r)^2}} - 1 \right)$,

4) $|S_i| \geq a_i - \frac{(P(\mathbf{E}) \cdot N_r)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot a_i}{(P(\mathbf{E}) \cdot N_r)^2}} - 1 \right)$,

where S_i is a component of some disconnected region E_i, S is the maximum sum of the lengths of the edges of S_i adjacent to the same region and N_r is the number of adjacent regions to S_i.

In particular, for any p, perimeter of a weak competitor, it follows that

$$|S_i| \geq a_i - \frac{(p \cdot N_r)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot a_i}{(p \cdot N_r)^2}} - 1 \right), \tag{3.22}$$

and finally

$$|S_i| \geq a_i - \frac{(p \cdot N)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot a_i}{(p \cdot N)^2}} - 1 \right). \tag{3.23}$$

Proof. Wherever S_i is (in Figure 3.1, $i = 4$), we consider a new cluster $\mathbf{E}'$

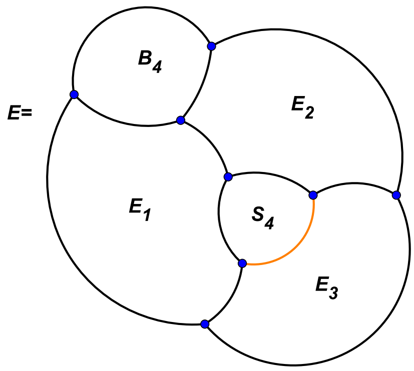

Figure 3.1: *Cluster* $\mathbf{E}$ *where* E_4 *is the disjoint union between its components* B_4 *and* S_4*. We remove the orange edge of* S_4 *for* E_3*.*

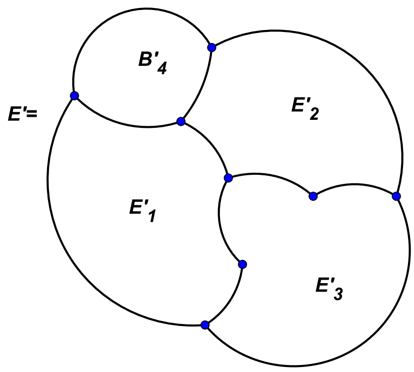

Figure 3.2: *The new cluster* $\mathbf{E}'$*.*

as follows: remove S, the maximum sum of the lengths of the edges of S_i adjacent to the same region E_j, and give the area of S_i to the region E_j obtaining that $\mathbf{E}'$:=$(E'_1, \dots, E'_i, E'_j, \dots, E'_N)$ with $|\,E'_k\,| = |\,E_k\,|$ for all $k \neq i, j$ and $|\,E'_j\,| = |\,E_j\,| + |\,S_i\,|$, $|\,E'_i\,| = |\,E_i\,| - |\,S_i\,|$ (see Figure 3.2). From $\mathbf{E}'$ we make a weak competitor for the problem (1.10); in order to do this we need that $|E_i'| \geq |E_i| \geq a_i$ (in fact it would be enough $|E_i'| \geq a_i$), thus we dilate by a factor λ the cluster $\mathbf{E}'$ obtaining a new cluster $\mathbf{E}''$ such that $\mathbf{E}'' = \lambda\mathbf{E}'$, where we impose that $|E_i''| = |E_i|$ (in Figure 3.3 $\mathbf{E}''$ is the dashed cluster).

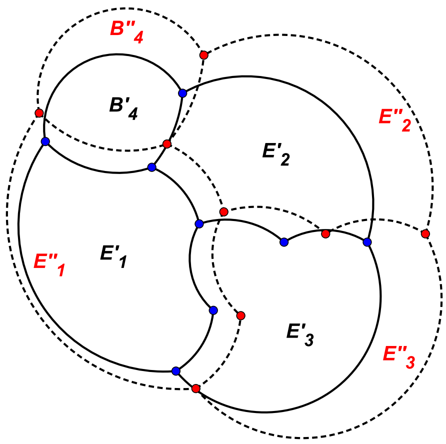

Figure 3.3: *The weak competitor* $\mathbf{E}''$.

Thus $|E_i| = |E_i''| = \lambda^2 |E_i'| = \lambda^2 \cdot (|E_i| - |S_i|)$, which shows that $\lambda^2 = \frac{|E_i|}{|E_i|-|S_i|} = 1 + \frac{|S_i|}{|E_i|-|S_i|}$. Since $\lambda > 1$, then $\mathbf{E}''$ is a weak competitor for the problem (1.10). Let $f(x) = \sqrt{1+x} - 1 - \frac{x}{2}$ for $x \geq 0$. We have that $f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{1+x}} - 1\right) \leq 0$, namely the function f is decreasing. Then $f(x) \leq f(0) = 0$, so that $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for any $x \geq 0$. Therefore it holds that (note that $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$)

$$\begin{aligned} P(\mathbf{E}) &\leq P(\mathbf{E}'') = P(\lambda\mathbf{E}') = \lambda P(\mathbf{E}') \\ &= \lambda(P(\mathbf{E}) - S) = \sqrt{1 + \frac{|\mathrm{S}_i|}{|E_i| - |S_i|}} \cdot (P(\mathbf{E}) - S) \\ &\leq \left(1 + \frac{|S_i|}{2(|E_i| - |S_i|)}\right) \cdot (P(\mathbf{E}) - S), \end{aligned}$$

obtaining that

$$S \leq S \cdot \left(1 + \frac{|S_i|}{2(|E_i| - |S_i|)}\right) \leq \frac{|S_i|}{2(|E_i| - |S_i|)} \cdot P(\mathbf{E}), \tag{3.24}$$

that is 1) of the statement.

Let N_r be the number of adjacent regions to S_i, then $S \geq \frac{P(S_i)}{N_r}$. By (3.24) and using the isoperimetric inequality, we get that

$$\frac{2\sqrt{\pi}\sqrt{|\,S_i\,|}}{N_r} \leq \frac{P(S_i)}{N_r} \leq \frac{|\,S_i\,|}{2(|\,E_i\,| - |\,S_i\,|)} \cdot P(\mathbf{E}).$$

Now dividing by $\sqrt{|\,S_i\,|}$ and squaring, we obtain

$$|\,S_i\,| \geq \frac{16\pi}{N_r^2} \cdot \left(\frac{|\,E_i\,| - |\,S_i\,|}{P(\mathbf{E})}\right)^2, \tag{3.25}$$

that is 2) of the statement.

We set $C = \frac{N_r^2}{16\pi}$ in (3.25), and we underline that the previous inequality is equivalent to the following inequality in the variable $|\,S_i\,|$

$$|\,S_i\,|^2 + |\,S_i\,| \cdot (-2|\,E_i\,| - C \cdot P(\mathbf{E})^2) + |\,E_i\,|^2 \leq 0. \tag{3.26}$$

Placing

$$s_1 := \frac{(2|\,E_i\,| + C \cdot P(\mathbf{E})^2) - C \cdot P(\mathbf{E})^2 \cdot \sqrt{1 + \frac{4|\,E_i\,|}{C \cdot P(\mathbf{E})^2}}}{2},$$
$$s_2 := \frac{(2|\,E_i\,| + C \cdot P(\mathbf{E})^2) + C \cdot P(\mathbf{E})^2 \cdot \sqrt{1 + \frac{4|\,E_i\,|}{C \cdot P(\mathbf{E})^2}}}{2},$$

the solution of (3.26) is

$$s_1 \leq |\,S_i\,| \leq s_2. \tag{3.27}$$

If we replace $C = \frac{N_r^2}{16\pi}$, of course we obtain 3)

$$|\,S_i\,| \geq |\,E_i\,| - \frac{(P(\mathbf{E}) \cdot N_r)^2}{32\pi}\left(\sqrt{1 + \frac{64\pi \cdot |\,E_i\,|}{(P(\mathbf{E}) \cdot N_r)^2}} - 1\right). \tag{3.28}$$

If we set $a = \frac{(N_r \cdot P(\mathbf{E}))^2}{32\pi}$ and $b = \frac{64\pi}{(N_r \cdot P(\mathbf{E}))^2}$ in (3.28), then $a \cdot b = 2$ and

$$|\,S_i\,| \geq |\,E_i\,| - a\left(\sqrt{1 + \frac{2|\,E_i\,|}{a}} - 1\right). \tag{3.29}$$

We call

$$f(x) = x - a\left(\sqrt{1+\frac{2x}{a}} - 1\right),$$

with $x = |\, E_i \,| \geq a_i$. By Lemma 3.10, we know that f is strictly increasing, therefore

$$f(x) \geq f(a_i) = a_i - a\left(\sqrt{1+\frac{2a_i}{a}} - 1\right).$$

By (3.29) and if we replace $a = \frac{(N_r \cdot P(\mathbf{E}))^2}{32\pi}$, of course we obtain $4)$

$$|\, S_i \,| \geq a_i - \frac{(P(\mathbf{E}) \cdot N_r)^2}{32\pi}\left(\sqrt{1+\frac{64\pi \cdot a_i}{(P(\mathbf{E}) \cdot N_r)^2}} - 1\right).$$

Let p be the perimeter of a weak competitor, then, by Remark 3.4,

$$\sqrt{\pi}\left(\sum_{i=1}^{N} \sqrt{a_i} + \sqrt{a_0}\right) < P(\mathbf{E}) \leq p,$$

where $a_0 = \sum_{i=1}^{N} a_i$. We set $a = \frac{N_r^2}{32\pi}$ and $b = \frac{64\pi \cdot a_i}{N_r^2}$ in $4)$, then $a \cdot b = 2a_i$ and

$$|\, S_i \,| \geq a_i - a \cdot P(\mathbf{E})^2\left(\sqrt{1+\frac{2a_i}{a \cdot P(\mathbf{E})^2}} - 1\right). \tag{3.30}$$

We call

$$f(x) = x \cdot \left(\sqrt{1+\frac{2a_i}{x}} - 1\right),$$

with $a \cdot \pi \cdot \big(\sum_{i=1}^{N} \sqrt{a_i} + \sqrt{a_0}\big)^2 < x = a \cdot P(\mathbf{E})^2 \leq a \cdot p^2$. By Lemma 3.9, we know that f is strictly increasing, therefore

$$f(x) \leq f(a \cdot p^2) = \frac{(N_r \cdot p)^2}{32\pi} \cdot \left(\sqrt{1+\frac{64\pi \cdot a_i}{(N_r \cdot p)^2}} - 1\right).$$

By (3.30), we have (3.22)

$$|\, S_i \,| \geq a_i - \frac{(p \cdot N_r)^2}{32\pi}\left(\sqrt{1+\frac{64\pi \cdot a_i}{(p \cdot N_r)^2}} - 1\right).$$

Since $\mathbf{E}$ is a weak minimum, then there are not redundant edges, therefore $2 \leq N_r \leq N$, where N is the number of the regions of the problem (1.10). We set $a = \frac{p^2}{32\pi}$ and $b = \frac{64\pi \cdot a_i}{p^2}$ in (3.22), then $a \cdot b = 2a_i$ and

$$|\, S_i \,| \geq a_i - a \cdot N_r^2 \left(\sqrt{1 + \frac{2a_i}{a \cdot N_r^2}} - 1 \right). \tag{3.31}$$

We call

$$f(x) = x \cdot \left(\sqrt{1 + \frac{2a_i}{x}} - 1 \right),$$

with $4a \leq x = a \cdot N_r^2 \leq a \cdot N^2$. By Lemma 3.9, we know that f is strictly increasing, therefore

$$f(x) \leq f(a \cdot N^2) = \frac{(N \cdot p)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot a_i}{(N \cdot p)^2}} - 1 \right).$$

By (3.31), we have (3.23)

$$|\, S_i \,| \geq a_i - \frac{(N \cdot p)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi \cdot a_i}{(N \cdot p)^2}} - 1 \right).$$

□

Lemma 3.12. *Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$. If $\mathbf{E}$ is not standard, then for all external and disconnected regions E_i (i.e. E_i is adjacent to the exterior region E_0) we have the following inequalities:*

a) $\alpha \geq \frac{1}{N_{C^i}} \cdot \sqrt{\frac{\pi}{|\, C^i \,|}} \cdot L_e^i - 1$;

b) $k_e^i \geq \frac{2}{N_{C^i}} \cdot \sqrt{\frac{\pi}{|\, C^i \,|}} - \frac{2}{L_e^i}$,

where L_e^i represents the external edge of one external component C_e^i of E_i, k_e^i is the curvature of L_e^i (since L_e^i is external, k_e^i corresponds with the pressure of E_i), α is the angle between L_e^i and the segment that links the vertices of L_e^i, C^i is another component of E_i and N_{C^i} represents the number of regions adjacent to C^i.

Proof. We create a weak competitor for the problem (1.10) thanks to the variation illustrated in Figure 3.4.

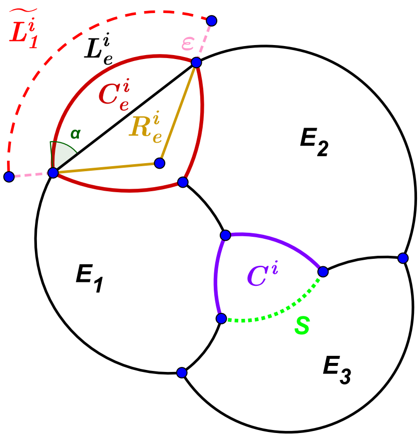

Figure 3.4: *The variation gives a weak competitor for the problem* (1.10) *because the area of E_i remains the same, while the area of the region, that has in common with C^i the maximum sum of the lengths of the edges, increases of area of C^i.*

We want to eliminate the component C^i in favor of C_e^i acting in the following way: we delete the maximum sum of the lengths of the edges of C^i adjacent to the same region and we recover the lost area of C^i extending outside the radius of curvature, R_e^i, of L_e^i of a quantity ε and finally by closing all in the most natural way. Called N_{C^i} the number of regions adjacent to C^i, we have that

$$P(C^i) \leq N_{C^i} \cdot S, \tag{3.32}$$

where S denotes the maximum sum of the lengths of the edges of C^i adjacent to the same region. Furthermore the following identities hold:

i) $L_e^i = 2R_e^i\,\alpha$,

ii) $\widetilde{L_e^i} = 2(R_e^i + \varepsilon)\,\alpha$,

iii) $A_e^i = {R_e^i}^2\,\alpha$,

iv) $\widetilde{A_e^i} = (R_e^i + \varepsilon)^2\,\alpha$,

v) $|\,C^i\,| = \widetilde{A_e^i} - A_e^i = \alpha\,\varepsilon(\varepsilon + 2R_e^i)$, where A_e^i and $\widetilde{A_e^i}$ respectively are the area of circular sector of radius R_e^i and amplitude 2α and the area of circular sector of radius $R_e^i + \varepsilon$ and amplitude 2α respectively.

From $v)$, ε satisfies the equation:

$$\alpha\varepsilon^2 + (2R_e^i\,\alpha)\varepsilon - |\,C^i\,| = 0,$$

and so, since ε is positive, we get that

$$\varepsilon = \frac{-R_e^i\,\alpha + \sqrt{(R_e^i\,\alpha)^2 + \alpha\,|\,C^i\,|}}{\alpha} \overset{i)}{=} R_e^i\left(\sqrt{1 + \frac{2|\,C^i\,|}{L_e^i\,R_e^i}} - 1\right).$$

Since $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for all $x \geq 0$, we have that

$$\varepsilon \leq \frac{|\,C^i\,|}{L_e^i}. \tag{3.33}$$

Since $\mathbf{E}$ is a weak minimum, then the performed variation gives a non negative variation of perimeter, that is, by $i)$, $ii)$ and (3.32),

$$\begin{aligned} 0 \leq \Delta P = \widetilde{L_e^i} + 2\varepsilon - L_e^i - S &\leq 2\varepsilon(1+\alpha) - \frac{P(C^i)}{N_{C^i}} \\ &\overset{(3.33)}{\leq} 2(1+\alpha)\frac{|\,C^i\,|}{L_e^i} - \frac{P(C^i)}{N_{C^i}}; \end{aligned}$$

from which and by the isoperimetric inequality, it is clear that

$$\frac{2\sqrt{\pi\,|\,C^i\,|}}{N_{C^i}} \leq \frac{P(C^i)}{N_{C^i}} \leq 2(1+\alpha)\frac{|\,C^i\,|}{L_e^i}.$$

Thus, simplifying, we obtain $a)$, namely

$$\alpha \geq \frac{1}{N_{C^i}} \cdot \sqrt{\frac{\pi}{|\,C^i\,|}} \cdot L_e^i - 1.$$

Now, using $a)$ and $k_e^i = \frac{2\alpha}{L_e^i}$, we have $b)$, namely

$$k_e^i \geq \frac{2}{N_{C^i}} \cdot \sqrt{\frac{\pi}{|\,C^i\,|}} - \frac{2}{L_e^i}.$$

□

Remark 3.13. We clearly note that the number of adjacent regions to C^i, N_{C^i}, is less or equal to N, the number of region of problem (1.10).

Lemma 3.14. *Let $\mathbf{E} \in \mathcal{M}^*_{2,N}(\mathbf{a})$, then*

$$L_C \leq 2\sqrt{\pi \,|\, C \,|}, \tag{3.34}$$

for any edge L_C of a component C of a region E_i.

Proof. We consider some component C of a region E_i of a weakly minimizing N-cluster $\mathbf{E}$. We take one of its edges L_C, then we make a variation, that produces a weak competitor $\mathbf{E}'$ for the problem (1.10). This variation consists in eliminating the edge L_C in order to donate the area of C to the region that has L_C in common with C and finally recovering the lost area of C with an external circle, with the same area of C. Hence $P(\mathbf{E}') = P(\mathbf{E}) - L_C + 2\sqrt{\pi | \, C \, |}$. Since by assumption $\mathbf{E}$ is a weak minimum, the variation of perimeter must be non negative and so we obtain

$$0 \leq \Delta P = 2\sqrt{\pi | \, C \, |} - L_C.$$

$\square$

3.3 Estimates on pressure in a standard double bubble

Finally we present an interesting and simple lemma, where we determine an important estimate for the pressures of a standard double bubble. Before that, we premise a preliminary lemma where we show significant properties of some functions used in the following.

Lemma 3.15. *Consider the following functions*

$$A(x) = \frac{x - \sin x \cos x}{4 \sin^2(x)}, \quad x \in]-\pi, \pi[$$

$$f(x) = 2 \sin\left(\frac{2\pi}{3} + x\right) \sqrt{A\left(\frac{2\pi}{3} + x\right) - A(x)}, \quad x \in \left] -\frac{\pi}{3}, \frac{\pi}{3}\right[.$$

The functions A and f are strictly increasing.

Proof. We consider the function A; first of all we note that $A(-x) = -A(x)$ for any $x \in]-\pi, \pi[$. Therefore we can restrict x on interval $[0, \pi[$. By direct computations, we have that

$$A'(x) = \frac{\sin x - x\cos x}{2\sin^3(x)}.$$

In order to prove that A is strictly increasing, it is sufficient to show that $A'(x)$ is positive for all $x \in [0,\pi[$. First of all $\lim_{x\to 0} A'(x) = \frac{1}{6}$, thus, since $\sin x > 0$ for all $x \in]0,\pi[$, it just show that $l(x) := \sin x - x\cos x > 0$ for any $x \in]0,\pi[$. For $x = \frac{\pi}{2}$, $l(\frac{\pi}{2}) = 1$, while for $x \neq \frac{\pi}{2}$, $l(x) = \cos x(\tan x - x)$; therefore for $0 < x < \frac{\pi}{2}$ the functions $\cos x$ and $\tan x - x$ are both positive, while for $\frac{\pi}{2} < x < \pi$ are both negative. Thus we have the first claim.

Now we prove the monotonicity of the function f. By the monotonicity of A we get that $A(\frac{2\pi}{3}+x)-A(x) > 0$ for all $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$. Furthermore, since $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$, $\frac{\pi}{3} < \frac{2\pi}{3}+x < \pi$; thus the function f is positive for $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$. Since f is positive,we just show that f^2 is strictly increasing. With simple calculations we get that

$$f^2(x) = \frac{2\pi}{3} + x + \frac{\sin\left(\frac{\pi}{3}+2x\right)}{2} - \frac{\cos\left(\frac{\pi}{3}+2x\right)+1}{1-\cos(2x)} \cdot \frac{2x-\sin(2x)}{2},$$

$$\Big(f^2(x)\Big)' = \frac{2x-\sin(2x)}{2\sin x^3} \cdot \left(\cos\left(\frac{\pi}{3}+x\right)+\cos x\right).$$

First of all $\lim_{x\to 0}\Big(f^2(x)\Big)' = \frac{3}{2}\cdot\left(\lim_{x\to 0}\frac{2x-\sin(2x)}{2\sin x^3}\right) = \frac{3}{2}\cdot\frac{4}{3} = 2$. We call $k(x) := 2x - \sin(2x)$ and $k_1(x) := \cos\left(\frac{\pi}{3}+x\right) + \cos x$. It is clear that $\Big(f^2(x)\Big)' = \frac{k(x)\cdot k_1(x)}{2\sin x^3}$. The first derivatives of k and k_1 are $k'(x) = 4\sin x^2$, $k_1(x)' = -\frac{\sqrt{3}}{2}(\sqrt{3}\sin x + \cos x)$ respectively. Therefore it follows that k is strictly increasing for $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$ and k is positive for $x \in]0,\frac{\pi}{3}[$ and negative for $x \in]-\frac{\pi}{3},0[$. While the function k_1 is strictly increasing in $]-\frac{\pi}{3},-\frac{\pi}{6}[$ and strictly decreasing in $[-\frac{\pi}{6},\frac{\pi}{3}[$, then $k_1(x) > \min\left(k_1(-\frac{\pi}{3}),k_1(\frac{\pi}{3})\right) = 0$ for all $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$. We note that $\sin x$ is positive for $x \in]0,\frac{\pi}{3}[$ and it is negative for $x \in]-\frac{\pi}{3},0[$, hence by also the previous conclusions, we find that $\Big(f^2(x)\Big)'$ is positive for any $x \in]-\frac{\pi}{3},\frac{\pi}{3}[$, thus f^2 is strictly increasing.

□

Lemma 3.16. *Let $\mathbf{E} = (E_1,E_2)$ be a standard planar double bubble of areas $\mathbf{a} = (a_1,a_2)$ with $a_1 \geq a_2$, then we have that*

$$p_{E_1} = \frac{2\sin\left(\frac{2\pi}{3}+\alpha\right)\sqrt{A(\frac{2\pi}{3}+\alpha)-A(\alpha)}}{\sqrt{a_1}},$$

$$p_{E_2} = \frac{2\sin\left(\frac{2\pi}{3} - \alpha\right)\sqrt{A(\frac{2\pi}{3} - \alpha) + A(\alpha)}}{\sqrt{a_2}},$$

where α denotes the angle between the joining line and the joining edge of the vertices A and B (see Figure 3.5), $A(\alpha) := \frac{\alpha - \sin\alpha\cos\alpha}{4\sin\alpha^2}$, that represents the area of a circular segment of amplitude 2α and unit distance between its vertices and finally p_{E_1} and p_{E_2} are the pressure of the big (E_1) and small bubble (E_2) respectively or equivalently the curvature of the external edge of the big and small bubble.

In particular the following estimate hold

$$\frac{\sqrt{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}}{\sqrt{a_2}} \geq p_{E_2} \geq p_{E_1} \geq \frac{\sqrt{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}}{\sqrt{a_1}}. \tag{3.35}$$

Furthermore if $p_{E_2} = p_{E_1}$, then $a_1 = a_2$.

Proof. As shown in Figure 3.5, $\alpha \in [0, \frac{\pi}{3}[$ (recall that the inner angles of bubbles are $\frac{2\pi}{3}$).

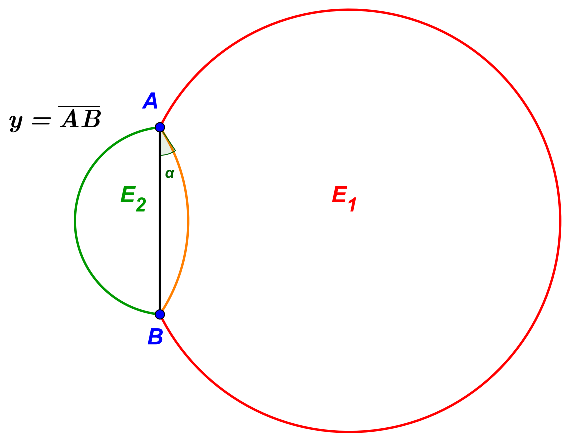

Figure 3.5: *A standard double bubble of areas a_1 and a_2.*

Using the formulas of Proposition 5.4 we have that

$$a_1 = y^2\left[A\left(\frac{2\pi}{3} + \alpha\right) - A(\alpha)\right],$$
$$a_2 = y^2\left[A\left(\frac{2\pi}{3} - \alpha\right) + A(\alpha)\right], \tag{3.36}$$

therefore we derive that

$$y = \sqrt{\frac{a_1}{A(\frac{2\pi}{3} + \alpha) - A(\alpha)}},$$
$$y = \sqrt{\frac{a_2}{A(\frac{2\pi}{3} - \alpha) + A(\alpha)}} \tag{3.37}$$

Again from formulas of Proposition 5.4, we know that for any circular sector of amplitude 2α, the curvature radius, R, is given by $R = \frac{y}{2\sin\alpha}$, where y is the length of the segment connecting its vertices. Thus it is clear that

$$p_{E_2} = \frac{2\sin(\frac{2\pi}{3} - \alpha)}{y},$$
$$p_{E_1} = \frac{2\sin(\frac{2\pi}{3} + \alpha)}{y}. \tag{3.38}$$

Now, since $\alpha \in [0, \frac{\pi}{3}[$, then $\frac{\pi}{3} < \frac{2\pi}{3} - \alpha \leq \frac{2\pi}{3} \leq \frac{2\pi}{3} + \alpha < \pi$. Therefore we obtain that $1 \geq \sin(\frac{2\pi}{3} - \alpha) \geq \frac{\sqrt{3}}{2} \geq \sin(\frac{2\pi}{3} + \alpha) > 0$, and so we get that

$$p_{E_2} \geq p_{E_1}. \tag{3.39}$$

From (3.37) and (3.38), it follows that

$$p_{E_2} = \frac{2\sin(\frac{2\pi}{3} - \alpha)\sqrt{A(\frac{2\pi}{3} - \alpha) + A(\alpha)}}{\sqrt{a_2}},$$
$$p_{E_1} = \frac{2\sin(\frac{2\pi}{3} + \alpha)\sqrt{A(\frac{2\pi}{3} + \alpha) - A(\alpha)}}{\sqrt{a_1}}.$$

From Lemma 3.15, we can see that

$$p_{E_2} = \frac{f(-\alpha)}{\sqrt{a_2}},$$
$$p_{E_1} = \frac{f(\alpha)}{\sqrt{a_1}}.$$

Furthermore by Lemma 3.15, we know that f is strictly increasing, therefore we get that

$$p_{E_2} \leq \frac{f(0)}{\sqrt{a_2}} = \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{a_2}},$$

$$p_{E_1} \geq \frac{f(0)}{\sqrt{a_1}} = \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{a_1}}. \tag{3.40}$$

Combining (3.39) and (3.40) we have (3.35).

Finally, since $\alpha \in [0, \frac{\pi}{3}[$ and by (3.38), it follows that if $p_{E_2} = p_{E_1}$ then $\alpha = 0$. Therefore by (3.36) we have that $a_1 = a_2$. $\square$

Chapter 4

Planar double and triple bubble with equal areas

In this chapter we prove the following theorems with the tools presented in Chapter 3:

Theorem 4.1. *[9] Every* $\mathbf{E} \in \mathcal{M}_{2,2}(a, a)$ *is standard;*

Theorem 4.2. *[16][17] Every* $\mathbf{E} \in \mathcal{M}_{2,3}(a, a, a)$ *is standard.*

Remark 4.3. Up to rescale the area in Theorems 4.1 and 4.2, we can consider that $|\,E_i\,| = 1$ for all i.

Remark 4.4. From Remark 1.51, in order to prove Theorems 4.1 and 4.2, it suffices to consider nonstandard clusters with exterior region connected and with non negative pressures and to prove they are not weakly minimizing.

4.1 Planar double bubble with equal areas

In this first section we prove Theorem 4.1; we will see that it is a direct consequence of Remark 4.7, that describes the composition of a disconnected region, and Corollary 4.8, that gives the minimum quantity of area that a small component must have.

First of all in the next remark we calculate the perimeter of a standard double bubble with unit areas.

Remark 4.5. By Proposition 5.4, we have that (see also Figure 4.1)

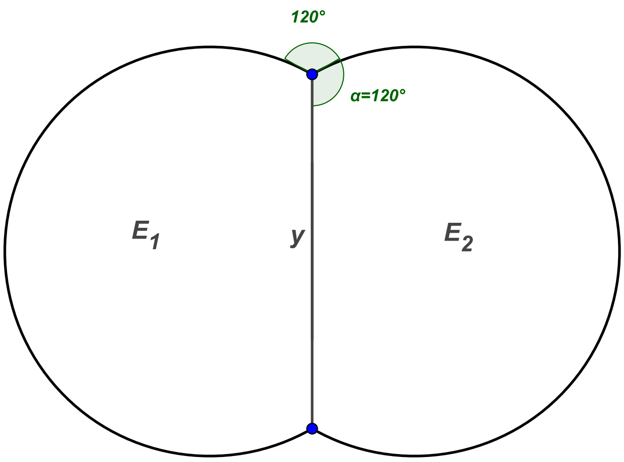

Figure 4.1: *A standard double bubble with areas* $|\,E_1\,| = |\,E_2\,| = 1$.

$1 = |\,E_1\,| = |\,E_2\,| = A(\frac{2\pi}{3}, y) = y^2 \cdot \frac{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}{3}$, therefore, indicating with PDB the perimeter of a standard double bubble,

$$PDB = y + 2L\left(\frac{2\pi}{3}, y\right) = \sqrt{\frac{3}{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}} \cdot \left(1 + \frac{8\pi}{3\sqrt{3}}\right) \approx 6.35913. \qquad (4.1)$$

Furthermore it holds that (note the definition of $\Phi_{i,\mathbf{a}}(p)$ in (3.8) where the vector $\mathbf{a} = (1,1)$)

$$1 < \Phi_{i,\mathbf{a}}(PDB) = \frac{PDB - \sqrt{\pi}(1+\sqrt{2})}{\sqrt{\pi}} < \sqrt{1 + \frac{2\sqrt{2}}{3}} < \sqrt{2} \qquad (4.2)$$

Thus, by (3.8), we can define

$$A_{1,2} = x_{i,\mathbf{a}}(PDB) = \frac{1 - \sqrt{\Phi^2_{i,\mathbf{a}}(PDB) \cdot (2 - \Phi^2_{i,\mathbf{a}}(PDB))}}{2} \approx 0.0369337. \quad (4.3)$$

Theorem 4.6. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,2}(1,1)$*. If* $\mathbf{E}$ *is not standard, then each disjoint union* U *of components of a disconnected region* E_i *satisfies that* $0 < |\,U\,| \leq A_{1,2}$ *or* $|\,U\,| \geq 1 - A_{1,2}$.

Proof. The proof is based on Theorem 3.5. We explicitly note that $a_i = 1$ for all $i = 1, 2$, so $a_0 = a_1 + a_2 = 2$. Finally we see that PDB, by (4.2) of Remark 4.5, satisfies the condition on the value of the perimeter in Theorem 3.5. This completes the proof.

□

Remark 4.7. Let $\mathbf{E} \in \mathcal{M}^*_{2,2}(1,1)$. From (4.2) of Remark 4.5, we see that PDB satisfies the condition on the value of the perimeter in Remark 3.8. Therefore it follows that if a region E_i of $\mathbf{E}$ is not connected then

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \dots E_i^{M(i)}, \tag{4.4}$$

with

a) $|\, E_i^0 \,| \geq |\, E_i \,| - A_{1,2} \geq 1 - A_{1,2} > \frac{2}{3}$;

b) $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,2} < \dfrac{1}{3}$,

where E_i^j is a component of E_i for any $j = 0, \dots, M(i)$ (note that $M(i)$ is finite by Theorem 1.10 and $M(i) > 1$, because E_i is disconnected). Furthermore any E_i^j is a small component and E_i^0 is the big component by Definition 1.18.

Now we present a corollary where we find the minimum area that a small component of a disconnected weak minimizer must have.

Corollary 4.8. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,2}(1,1)$. *If* $\mathbf{E}$ *is not standard, then the following inequality holds:*

$$|\, S_i \,| \geq 4\pi \cdot \left(\frac{1 - A_{1,2}}{PDB} \right)^2 := A_{2,2} \approx 0.288222, \tag{4.5}$$

where S_i *is a small component of some disconnected region.*

Proof. The proof is based on Lemma 3.11 and Remark 4.7. We note that $|\, E_i \,| \geq 1$ for all $i = 1, 2$ and $N = 2$, thus the number of regions, N_r, adjacent to any component is less or equal to 2. By Remark 4.7 the area of a small component S_i is such that $|\, S_i \,| \leq A_{1,2} \approx 0.0369337$ (therefore $1 - A_{1,2} > 0$) and finally from the minimality of $\mathbf{E}$ we have $P(\mathbf{E}) \leq PDB$. Linking these informations with the estimate 2) of Lemma 3.11 we have the claim. □

Theorem 4.9. *Let* $\mathbf{E}\in\mathcal{M}^*_{2,2}(1,1)$*. Then* $\mathbf{E}$ *is standard. Moreover if* $\mathbf{E}\in\mathcal{M}_{2,2}(1,1)$*, then* $\mathbf{E}$ *is standard.*

Proof. The proof is based on Remark 4.7 and on Corollary 4.8. We suppose by contradiction that $\mathbf{E}$ is not standard, therefore there exists at least one disconnected region E_i. From Remark 4.7 and Corollary 4.8, we have that $0.288222 \approx A_{2,2} \leq |\, S_i \,| \leq A_{1,2} \approx 0.0369337$ for any small component S_i of a disconnected region E_i. It is a contradiction and thus $\mathbf{E}$ is standard.

By Remark 4.4, we have that if $\mathbf{E} \in \mathcal{M}_{2,2}(1,1)$, then $\mathbf{E}$ is standard. $\square$

4.2 Planar three bubble with equal areas

In this section we prove Theorem 4.2; Theorem 4.11 and Remark 4.12 will be very important. The first gives some informations on the quantity of area of a disconnected region and the second describes the composition of a disconnected weakly minimizer. Another significant result is Corollary 4.15, that gives an estimate for the minimum quantity of area that a small component of a disconnected weakly minimizing must have. Finally we also underline Lemma 4.21, that describes a component of a disconnected region and a component of a connected region.

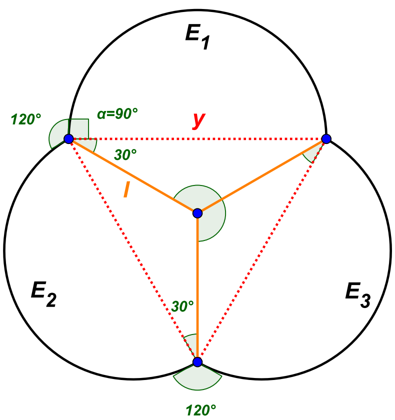

Figure 4.2: *A standard triple bubble with areas* $|\, E_1 \,| = |\, E_2 \,| = |\, E_3 \,| = 1$.

First of all in the next remark we calculate the perimeter of a standard triple bubble with unit areas.

Remark 4.10. By Proposition 5.4, it follows that (see also Figure 4.2) $1 = |E_1| = |E_2| = |E_3| = A(\frac{\pi}{2}, y) + \frac{y^2}{4\sqrt{3}} = \frac{y^2}{4} \cdot \left(\frac{\pi}{2} + \frac{1}{\sqrt{3}}\right)$, therefore, denoting by PTB the perimeter of a standard triple bubble,

$$PTB = 3l + 3L\left(\frac{\pi}{2}, y\right) = 6 \cdot \sqrt{\frac{\pi}{2} + \frac{1}{\sqrt{3}}} \approx 8.79393. \tag{4.6}$$

Furthermore it holds that (note the definition of $\Phi_{i,\mathbf{a}}(p)$ in (3.8) where the vector $\mathbf{a} = (1, 1, 1)$)

$$1 < \Phi_{i,\mathbf{a}}(PTB) = \frac{PTB - \sqrt{\pi}(2 + \sqrt{3})}{\sqrt{\pi}} < \sqrt{1 + \frac{2\sqrt{2}}{3}} < \sqrt{2} \tag{4.7}$$

Thus, by (3.8), we can define

$$A_{1,3} = x_{i,\mathbf{a}}(PTB) = \frac{1 - \sqrt{\Phi^2_{i,\mathbf{a}}(PTB) \cdot (2 - \Phi^2_{i,\mathbf{a}}(PTB))}}{2} \approx 0.0703324. \tag{4.8}$$

Theorem 4.11. *Let $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$. If $\mathbf{E}$ is not standard, then each disjoint union U of components of a disconnected region E_i satisfies that $0 < |U| \leq A_{1,3}$ or $|U| \geq 1 - A_{1,3}$.*

Proof. The proof is based on Theorem 3.5. We explicitly note that $a_i = 1$ for all $i = 1, 2, 3$, so $a_0 = a_1 + a_2 + a_3 = 3$. Finally we see that PTB, by (4.7) of Remark 4.10, satisfies the condition on the value of the perimeter in Theorem 3.5. This completes the proof. □

Remark 4.12. Let $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$. From (4.7) of Remark 4.10, we see that PTB satisfies the condition on the value of the perimeter in Remark 3.8. Therefore it follows that if a region E_i of $\mathbf{E}$ is not connected then

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \ldots E_i^{M(i)}, \tag{4.9}$$

with

a) $|E_i^0| \geq |E_i| - A_{1,3} \geq 1 - A_{1,3} > \frac{2}{3}$;

b) $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,3} < \frac{1}{3}$,

where E_i^j is a component of E_i for any $j = 0, \ldots, M(i)$ (note that $M(i)$ is finite by Theorem 1.10 and $M(i) > 1$, because E_i is disconnected). Furthermore any E_i^j is a small component and E_i^0 is the big component by Definition 1.18.

Remark 4.13. Let $\mathbf{E} \in \mathcal{M}_{2,3}^*(1,1,1)$, then any big component has area at least as large as $1 - A_{1,3} \approx 0.929668$.

First of all connected regions E_i are big components with $|\, E_i \,| \geq 1$.

While for all disconnected regions E_i, by Remark 4.12, E_i^0 is the big component and $|\, E_i^0 \,| \geq 1 - A_{1,3} \approx 0.929668$.

Corollary 4.14. *Let $\mathbf{E} \in \mathcal{M}_{2,3}^*(1,1,1)$, then any big component is external (i.e the component has an edge in common with E_0).*

Proof. We argue by contradiction and we suppose that there is at least one big inner component. By Remark 4.13 any big component has area at least $1 - A_{1,3} \approx 0.929668$. Let E_i be the region with its big component B_i inner (i.e B_i is disjoint to E_0), then by the isoperimetric inequality we have that

$$P(\mathbf{E}) \geq P(B_i) + P(E_0) \geq 2\sqrt{\pi}\Big(\sqrt{1 - A_{1,3}} + \sqrt{3}\Big) \approx 9.55793.$$

From the minimality of $\mathbf{E}$, we know that $P(\mathbf{E}) \leq PTB \approx 8.79393$. Since it is a contradiction, the proof is completed. □

Now we present a corollary where we find the minimum area that a small component of a disconnected weak minimizer must have.

Corollary 4.15. *Let $\mathbf{E} \in \mathcal{M}_{2,3}^*(1,1,1)$. If $\mathbf{E}$ is not standard, then the following inequality holds:*

$$|\, S_i \,| \geq 1 - \frac{9PTB^2}{32\pi}\left(\sqrt{1 + \frac{64\pi}{9PTB^2}} - 1\right) := A_{2,3} \approx 0.0633589. \tag{4.10}$$

where S_i is a small component of some disconnected region.

Proof. The proof is based on estimate (3.23) of Lemma 3.11. Let S_i be a small component of some disconnected region E_i (note that S_i exists by Remark 4.12), we choose PTB as perimeter of a weak competitor, then, since $N = 3$ and $a_i = 1$ for any $i = 1, 2, 3$, by estimate (3.23) of Lemma 3.11, we have the claim. $\square$

Proposition 4.16. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$*. Each disconnected region can have at most one small component.*

Proof. Let E_i be a disconnected region. By Remark 4.12

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \ldots \sqcup E_i^{M(i)}, \tag{4.11}$$

where $|\, E_i^0 \,| \geq 1 - A_{1,3}$, $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,3}$ and $M(i)$ denotes the number of small components E_i^j of E_i.

We have that, by Corollary 4.15, $|\, E_i^j \,| \geq A_{2,3}$ for all $j = 1 \ldots, M(i)$ and $i = 1, 2, 3$. Therefore we get that

$$M(i) \cdot A_{2,3} \leq \sum_{j=1}^{M(i)} |\, E_i^j \,| = \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,3},$$

obtaining that $M(i) \leq \frac{A_{1,3}}{A_{2,3}} \approx 1.11006$. Considering the integer part of 1.11006 we have the claim, $M(i) \leq 1$. $\square$

Theorem 4.17. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$*, then* $\mathbf{E}$ *has at least two connected regions.*

Proof. We argue by contradiction and we suppose that there are at least two disconnected regions. Let E_i and E_j be the disconnected regions. By Remark 4.12 and Proposition 4.16, it follows that

$$E_i = S_i \cup B_i,$$
$$E_j = S_j \cup B_j,$$

with S_i, S_j, that are the small components and B_i, B_j, instead which are the big components. Hence, by Remark 4.12 and Corollary 4.15 we have that $\min(|\, B_i \,|, |\, B_j \,|) \geq 1 - A_{1,3}$ and $\min(|\, S_i \,|, |\, S_j \,|) \geq A_{2,3}$ respectively.

So, by the isoperimetric inequality, we can give the following estimate for the perimeter of $\mathbf{E}$:

$$\begin{aligned} 8.79393 \approx PTB \geq P(\mathbf{E}) & \\ & \geq \frac{1}{2} \cdot \Big(\sum_{n=i,j} P(B_n) + \sum_{n=i,j} P(S_n) + P(E_k) + P(E_0) \Big) \\ & \geq \sqrt{\pi} \Big(2 \cdot (\sqrt{1 - A_{1,3}} + \sqrt{A_{2,3}}) + 1 + \sqrt{3} \Big) \approx 9.1527, \end{aligned}$$

where $k \neq i, j$. It is a contradiction, thus the proof is completed. □

Remark 4.18. Let $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$. If $\mathbf{E}$ is not standard, then, by Remark 4.12, Corollary 4.15, Proposition 4.16 and Theorem 4.17, $\mathbf{E}$ is composed by one and only disconnected region $E_i = S_i \sqcup B_i$ with $A_{2,3} \leq |\, S_i \,| \leq A_{1,3}$ and $|\, B_i \,| \geq 1 - A_{1,3}$. Moreover, since each region has unit area (up to a permutation of regions), $\mathbf{E}$ can only have this case of connection type: $I_{\mathbf{E}} = (1,0,0)$.

Proposition 4.19. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$*, then the total number of bounded components is at most four.*

Proof. If $\mathbf{E}$ is standard then $\mathbf{E}$ exactly has three bounded components, instead if $\mathbf{E}$ is not standard, then by Remark 4.18 $\mathbf{E}$ has four bounded components. □

Corollary 4.20. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$*, then* $\mathbf{E} \in \mathcal{M}_{2,3}(1,1,1)$.

Proof. The proof immediately comes from previous proposition and Theorem 1.50. □

We preset a simple lemma, that describes a component of the disconnected region and a component of a connected region.

Lemma 4.21. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$*. If* $\mathbf{E}$ *is not standard, then any component* C *of a disconnected region is external (i.e* C *has an edge in common with* E_0*) and it has three edges, while a connected region is external with at most four edges.*

Proof. By Remark 4.18 the connection type of $\mathbf{E}$ is $I_{\mathbf{E}} = (1,0,0)$ up to permutations. First of all we prove that any component C of E_1 is external. If it is inner, then, by Proposition 4.19 and by the fact $\mathbf{E}$ can not have redundant edges, it must have only two edges. But this contradicts Corollary 1.35.

Now we prove that C has three edges. It is clear because Proposition 4.19 applies and $\mathbf{E}$ is a minimum, so $\mathbf{E}$ can not have redundant edges.

Finally, by Corollary 4.14, we know that each connected region is external. By Proposition 4.19, by the minimality of $\mathbf{E}$, and by $I_{\mathbf{E}} = (1, 0, 0)$, it follows that each connected region can have at most four edges.

□

Remark 4.22. Let $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$. If $\mathbf{E}$ is not standard, then from Lemma 4.21 we have that any component of a region of $\mathbf{E}$ is external. Therefore the only possible case of $\mathbf{E}$ is represented in Figure 4.3

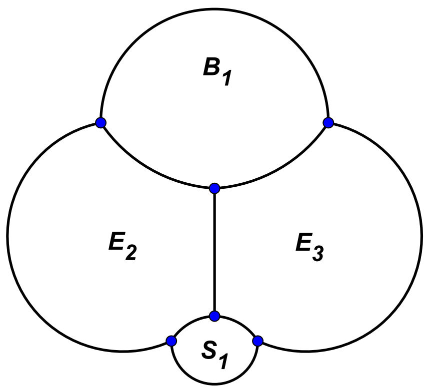

Figure 4.3: *The only possible case of disconnected* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1, 1, 1)$.

Lemma 4.23. *If* $\mathbf{E}$ *has the topology of Figure 4.3, then* $\mathbf{E} \notin \mathcal{M}^*_{2,3}(1, 1, 1)$.

Proof. We argue by contradiction and we suppose that $\mathbf{E}$ is a weak minimum. We recall that, by Corollary 4.20, $m(\mathbf{E}) = \mathbf{a}$ and $\mathbf{E} \in \mathcal{M}_{2,3}(1, 1, 1)$. We determine an estimate for the pressure of E_1 and for the lowest pressure of all regions. So we will be able to give an estimate for the perimeter of E, that will be bigger that PTB. We note that, by Remark 4.12, $|\, S_1 \,| \leq A_{1,3}$ and $|\, B_1 \,| \geq 1 - A_{1,3}$, where S_1 and B_1 are the small and the big component of E_1 respectively.

We start to obtain an estimate for the pressure of the disconnected region E_1. Let $l_{1,0}$, $l_{1,2}$ and $l_{1,3}$ be the edges of S_1. Since the turning angle of S_1 is π, we have that (recall that $p_{E_0} = 0$ and each pressure is non negative by Proposition 1.49)

$$p_{E_1} P(S_1) = \max_{k=0,2,3} (p_{E_1} - p_{E_k}) P(S_1) \geq \sum_{k=0,2,3} l_{1,k} \Big(p_{E_1} - p_{E_k} \Big) = \pi.$$

Therefore we obtain that

$$p_{E_1} \geq \frac{\pi}{P(S_1)}.$$

By the Definition 1.1 and by the isoperimetric inequality, it follows that

$$\begin{aligned} P(S_1) &= 2P(\mathbf{E}) - \Big(P(B_1) + P(E_2) + P(E_3) + P(E_0) \Big) \\ &\leq 2PTB - 2\sqrt{\pi}(\sqrt{1 - A_{1,3}} + 1 + 1 + \sqrt{3}). \end{aligned}$$

So we find that

$$p_{E_1} \geq \frac{\pi}{2PTB - 2\sqrt{\pi}(\sqrt{1 - A_{1,3}} + 2 + \sqrt{3})} := k_1 \approx 3.3417. \tag{4.12}$$

Certainly E_1 is the highest pressure region, indeed if there was another region with pressure at least k_1, then the perimeter of E would be at least (by Corollary 1.47 and by Proposition 1.49)

$$8.79393 \approx PTB \geq P(\mathbf{E}) = 2 \sum_{i=1}^{3} p_{E_i} \geq 4k_1 \approx 13.3668,$$

that is a contradiction. So the lowest pressure region is either E_2 or E_3. Without loss of generality we can consider that the lowest pressure region is E_3. It has four edges, therefore since its turning angle is $\frac{2\pi}{3}$, we have the following estimate for the lowest pressure p_{E_3}

$$p_{E_3} \cdot L_{3,0} = \max_{k \neq 3} (p_{E_3} - p_{E_k}) \geq \sum_{k \neq 3} L_{min,k} (p_{E_3} - p_{E_k}) = \frac{2\pi}{3},$$

where $L_{3,k}$ denotes an edge of E_3 in common with the region E_k. By Lemma 3.14, we have that $L_{3,0} \leq 2\sqrt{\pi}$ (recall that $\mathbf{E} \in \mathcal{M}_{2,3}(1,1,1)$, namely $|\, E_i \,| = 1$ for all i), hence

$$p_{E_3} \geq \frac{\sqrt{\pi}}{3} \approx 0.590818 \tag{4.13}$$

Then, by Corollary 1.47, (4.12) and (4.13),

$$8.79393 \approx PTB \geq P(\mathbf{E}) \geq 2\sum_{i=1}^{3} p_{E_i} \geq 2k_1 + 4\Big(\frac{\sqrt{\pi}}{3}\Big) \approx 9.04667.$$

It is a contradiction; so the proof is completed. □

Theorem 4.24. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$*. Then* $\mathbf{E}$ *is standard. In particular if* $\mathbf{E} \in \mathcal{M}_{2,3}(1,1,1)$*, then* $\mathbf{E}$ *is standard.*

Proof. The proof is immediate. Let $\mathbf{E} \in \mathcal{M}^*_{2,3}(1,1,1)$; we suppose by contradiction that $\mathbf{E}$ is not standard, then by Remark 4.22, $\mathbf{E}$ has the topology represented in Figure 4.3. By Lemma 4.23 $\mathbf{E} \notin \mathcal{M}^*_{2,4}(1,1,1,1)$; this is a contradiction, thus $\mathbf{E}$ is standard.

By Remark 4.4, we have that if $\mathbf{E} \in \mathcal{M}_{2,3}(1,1,1)$, then $\mathbf{E}$ is standard. □

Chapter 5

Planar four bubbles conjecture with equal area

In this chapter we present the problem, which is the core of the PhD thesis. It is a particular case of the problem (1.9), indeed it is the following:

$$\min \left\{ \mathrm{P}(\mathbf{E}) \,\middle|\, \mathbf{E} \in \mathcal{E}_{2,4},\, m(\mathbf{E}) = \mathbf{a} \right\}, \tag{5.1}$$

where $\mathbf{a} = (a, a, a, a)$ with the target to prove the corresponding planar soap bubble conjecture:

Theorem 5.1. *Every* $\mathbf{E} \in \mathcal{M}_{2,4}(\mathbf{a})$ *is standard.*

5.1 Necessary conditions on area of different components of the same region

Theorem 5.6 and Corollary 5.10 are the most important results in the first section of this chapter. The first gives some necessary conditions on the quantity of area that different components of the same region must have, while the second determines the minimum area that a small component of a disconnected weakly minimizer $\mathbf{E}$ must have.

Remark 5.2. From Remark 1.51, in order to prove Theorem 5.1 we can consider the corresponding weak problem of (5.1)

$$\min\left\{\mathrm{P}(\mathbf{E})\,\middle|\,\mathbf{E}\in\mathcal{E}_{2,4},\ m(\mathbf{E})\geq\mathbf{a}\right\},\tag{5.2}$$

where we must show that nonstandard 4-clusters with connected exterior region and with non negative pressures are not weakly minimizing.

Remark 5.3. Up to rescale for a we can consider $|\,E_i\,|=1$ for all i. Therefore from now for the problems (5.1) and (5.2) $\mathbf{a}=(1,1,1,1)$.

We begin with some basic formulas and omit the easy computations.

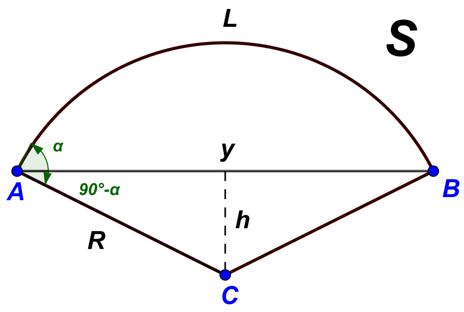

Figure 5.1: *The circular arc L has radius of curvature R, area A and length l.*

Proposition 5.4. *Let S be a circular sector. Define y the distance between the endpoints of the circular edge L of S , with α the angle between L and the line segment connecting its endpoints (see Figure 5.1). Then the radius of curvature R of L, the area A of the region between L and the line segment connecting its endpoints, and the length l of L are given by*

$$R(\alpha,y)=\frac{y}{2\sin\alpha},\quad A(\alpha,y)=y^2\cdot\frac{\alpha-\sin\alpha\cos\alpha}{4\sin^2\alpha},\quad l(\alpha,y)=y\cdot\frac{\alpha}{\sin\alpha}.\tag{5.3}$$

In this remark we show the construction of a possible connected competitor for the problem (5.1). We denote by $\tilde{p}$ the perimeter value of such competitor that is called *the competitor*.

Remark 5.5. *The competitor*, as in Figure 5.2,

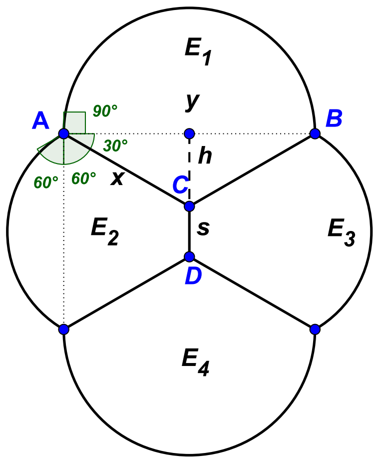

Figure 5.2: *The competitor* where $|\, E_i \,| = 1$ for all $i = 1, \ldots, 4$.

is composed by two adjacent regions of four sides and by two disjoint regions with three sides; each region is adjacent to the exterior region. We call a and b the following constants: $a := \frac{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}{3}$, $b := \frac{\sqrt{3}}{4}$ and x, h and s as in Figure 5.2. For the region with three sides the following identities hold: $y = x\sqrt{3}$ and $h = \frac{x}{2}$. Since the area of each region must be unit we have that the area of the regions with three sides is expressed in the following way $1 = (\frac{y}{2})^2 \cdot \frac{\pi}{2} + \frac{y \cdot h}{2} = x^2 \cdot (\frac{3\pi}{8} + \frac{\sqrt{3}}{4})$, getting that $x = \frac{1}{\sqrt{\frac{3\pi}{8} + \frac{\sqrt{3}}{4}}}$. Now, by the formulas (5.3), the area of each region with four sides is expressed in the following way

$$
\begin{aligned}
1 &= A((x+s), \pi/3) + \frac{(x+2s) \cdot \frac{\sqrt{3}x}{2}}{2} \\
&= (x+s)^2 \cdot \frac{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}{3} + \frac{(x+2s) \cdot \frac{\sqrt{3}x}{2}}{2} \\
&= (x+s)^2 \cdot a + (x+2s) \cdot x \cdot b,
\end{aligned}
$$

obtaining the next equation of second degree in the variable s

$$
a \cdot s^2 + 2s \cdot (ax + xb) + (x^2 \cdot (a+b) - 1) = 0.
$$

Since s must be positive, the solution of this equation is:

$$s = \frac{-(ax+xb) + \sqrt{(ax+xb)^2 - a \cdot (x^2 \cdot (a+b) - 1)}}{a} \approx 0.541492.$$

Considering the relations (5.3) we get that the perimeter of *the competitor* is

$$\begin{aligned} \tilde{p} &= 4x + s + 2 \cdot l(y, \pi/2) + 2 \cdot l(x+s, \pi/3) \\ &= 4x + s + 2 \cdot l(x\sqrt{3}, \pi/2) + 2 \cdot l(x+s, \pi/3) \approx 11.1946. \end{aligned} \tag{5.4}$$

Furthermore it holds that (note the definition of $\Phi_{i,\mathbf{a}}(p)$ in (3.8) where the vector $\mathbf{a} = (1,1,1,1)$)

$$1 < \Phi_{i,\mathbf{a}}(\tilde{p}) = \frac{\tilde{p} - 5\sqrt{\pi}}{\sqrt{\pi}} < \sqrt{1 + \frac{2\sqrt{2}}{3}} < \sqrt{2} \tag{5.5}$$

Thus, by (3.8), we can define

$$A_{1,4} = x_{i,\mathbf{a}}(\tilde{p}) = \frac{1 - \sqrt{\Phi^2_{i,\mathbf{a}}(\tilde{p}) \cdot (2 - \Phi^2_{i,\mathbf{a}}(\tilde{p}))}}{2} \approx 0.159132. \tag{5.6}$$

We present the most important Theorem for problem (5.2), that is a direct consequence of Theorem 3.5.

Theorem 5.6. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $\mathbf{E}$ is not standard, then each disjoint union U of components of a disconnected region E_i satisfies that $0 < |\,U\,| \le A_{1,4}$ or $|\,U\,| \ge 1 - A_{1,4}$.*

Proof. The proof is based on Theorem 3.5. We explicitly note that $a_i = 1$ for all $i = 1, \dots, 4$, so $a_0 = \sum_{i=1}^{4} a_i = 4$. Finally we see that $\tilde{p}$, by (5.5) of Remark 5.5, satisfies the condition on the value of the perimeter in Theorem 3.5. This completes the proof. $\square$

Remark 5.7. Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. From (5.5) of Remark 5.5, we see that $\tilde{p}$ satisfies the condition on the value of the perimeter in Remark 3.8. Therefore it follows that if a region E_i of $\mathbf{E}$ is not connected then

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \dots E_i^{M(i)}, \tag{5.7}$$

with

a) $|\,E_i^0\,| \geq |\,E_i\,| - A_{1,4} \geq 1 - A_{1,4} > \frac{2}{3}$;

b) $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,4} < \dfrac{1}{3}$,

where E_i^j is a component of E_i for any $j = 0, \ldots, M(i)$ (note that $M(i)$ is finite by Theorem 1.10 and $M(i) > 1$, because E_i is disconnected). Furthermore any E_i^j is a small component and E_i^0 is the big component by Definition 1.18.

Remark 5.8. Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, then any big component of $\mathbf{E}$ has area at least $1 - A_{1,4} \approx 0.840868$.

First of all connected regions E_i are big components with $|\,E_i\,| \geq 1$.

On the other hand for all disconnected regions E_i, by Remark 5.7, E_i^0 is the big component and $|\,E_i^0\,| \geq 1 - A_{1,4} \approx 0.840868$.

Corollary 5.9. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then there is at most one big inner component (i.e the component has not an edge in common with* E_0*).*

Proof. We argue by contradiction and we suppose that there are at least two big inner components (see Figure 5.3).

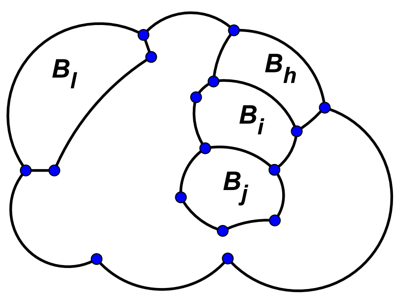

Figure 5.3: $\mathbf{E}$ 4-*Cluster which have two inner big components.*

By Remark 5.8 any big component has area at least $1 - A_{1,4} \approx 0.840868$. Let E_i and E_j be the regions with their big inner components B_i and B_j (i.e B_i and B_j are disjoint to E_0, therefore also $B_i \cup B_j$ is disjoint to E_0), then by the isoperimetric inequality we have the next estimate for the perimeter of $\mathbf{E}$:

$$P(\mathbf{E}) \geq P(B_i \cup B_j) + P(E_0) \geq 2\sqrt{\pi}\Big(\sqrt{2(1 - A_{1,4})} + 2\Big) \approx 11.6869.$$

From the minimality of $\mathbf{E}$, we know that $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$. Since this is a contradiction, the proof is completed. □

Now we present a corollary where we find the minimum area that a small component of a disconnected weak minimizer must have.

Corollary 5.10. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $\mathbf{E}$ is not standard, then the following inequalities apply:*

$$S \leq \frac{|\, S_i \,|}{2(1 - |\, S_i \,|)} \cdot \tilde{p}. \tag{5.8}$$

Furthermore the estimates hold:

$$|\, S_i \,| \geq 1 - \frac{(\tilde{p} \cdot N_r)^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi}{(\tilde{p} \cdot N_r)^2}} - 1\right), \tag{5.9}$$

and

$$|\, S_i \,| \geq 1 - \frac{\tilde{p}^2}{2\pi} \cdot \left(\sqrt{1 + \frac{4\pi}{\tilde{p}^2}} - 1\right) := A_{2,4} \approx 0.0238853, \tag{5.10}$$

where S_i is a small component of some disconnected region, S is the maximum sum of the lengths of the edges of S_i adjacent to the same region and N_r denotes the number of regions adjacent to S_i.

Proof. The proof is based on estimates $1)$, (3.22), (3.23) of Lemma 3.11. First of all, by Remark 5.7, we know that there are small components S_i of some disconnected region E_i. Let S_i be a small component of some disconnected region E_i, we choose $\tilde{p}$ as perimeter of a weak competitor.

Since $|\, E_i \,| \geq a_i = 1$ and by $1)$ of Lemma 3.11, we find (5.8).

Instead, since $a_i = 1$ for any $i = 1, \ldots, 4$ and by (3.22) of Lemma 3.11, we have (5.9) and finally, since also $N = 4$, by estimate (3.23) of Lemma 3.11, we get (5.10).

□

5.2 Possible cases of disconnected weakly minimizing 4-cluster

In this section, exploiting Remark 5.7 and Corollary 5.10, we consider all possible cases of disconnected weakly minimizing 4-cluster which we want to exclude in order to prove Theorem 5.1.

Proposition 5.11. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *Each disconnected region of* $\mathbf{E}$ *can have at most two small components.*

Proof. Let E_i be a disconnected region. By Remark 5.7

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \ldots E_i^{M(i)}, \tag{5.11}$$

where $|\, E_i^0 \,| \geq 1 - A_{1,4}$, $0 < \left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,4}$ and $M(i)$ denotes the number of small components of E_i.

We argue by contradiction and we suppose that $M(i) \geq 3$. By (5.10) of Corollary 5.10 we know that $|\, E_i^j \,| \geq A_{2,4}$ for all $j = 1 \ldots, M(i)$. Therefore we get, by the isoperimetric and by the definition of perimeter view in Definition 1.1, the following estimate for the perimeter of $\mathbf{E}$

$$\begin{aligned} P(\mathbf{E}) &= \frac{1}{2}\bigg(P(E_0) + P(E_i^0) + \sum_{j=1}^{M(i)} P(E_i^j) + \sum_{k \neq i} P(E_k)\bigg) \\ &\geq \sqrt{\pi}\bigg(2 + \sqrt{1 - A_{1,4}} + 3 \cdot \sqrt{A_{2,4}} + 3\bigg) \approx 11.3094. \end{aligned}$$

But by the minimality of $\mathbf{E}$, $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$ therefore we come to a contradiction that concludes the proof. $\square$

Remark 5.12. Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $\mathbf{E}$ is not standard, then $\mathbf{E}$ has at least one disconnected region E_i, thus, by Remark 5.7 and Proposition 5.11 $\mathbf{E}_i$ is so composed:

$$E_i = E_i^0 \sqcup E_i^1 \sqcup \ldots E_i^{M(i)},$$

where E_i^0 is the big component with $|\, E_i^0 \,| \geq 1 - A_{1,4}$ and $\bigsqcup_{j=1}^{M(i)} E_i^j$ is a disjoint union of $M(i)$ small components with $\left| \bigsqcup_{j=1}^{M(i)} E_i^j \right| \leq A_{1,4}$ and $1 \leq M(i) \leq 2$.

For any connected region E_i of $\mathbf{E}$ the number of small components $M(i)$ is zero. Therefore, since each region has unit area, up to a permutation of the regions, the only possible connection types $I_{\mathbf{E}} = (M(1), \ldots, M(4))$ for not standard $\mathbf{E}$ are the following: $(2,2,2,2)$, $(2,2,2,1)$, $(2,2,2,0)$, $(2,2,1,1)$, $(2,2,1,0)$, $(2,2,0,0)$, $(2,1,1,1)$, $(2,1,1,0)$, $(2,1,0,0)$, $(1,1,1,1)$, $(1,1,1,0)$, $(2,0,0,0)$, $(1,1,0,0)$, $(1,0,0,0)$ and $(0,0,0,0)$.

Lemma 5.13. *Let a, b, D be real positive constants with $a < b$ and let*

$$g : I = [a, b] \to \mathbb{R},$$

be a function such that

1) *g is convex,*

2) *$g'(a) > 0$,*

3) *$\sqrt{D} < g(a) < g(b) < \sqrt{2D}$.*

Then the function

$$f(x) := \frac{D - \sqrt{g(x)^2 \cdot (2D - g(x)^2)}}{2}, \qquad x \in I, \tag{5.12}$$

is strictly increasing and its first derivative is positive and strictly increasing.

In particular if $f(I) \subseteq I$ and $f'(b) < 1$, then one and only one fixed point l of f exists in I. Furthermore if $b < \frac{D}{2}$, l is a root of the function

$$F(x) := g(x)^2 \cdot \Big(2D - g(x)^2\Big) - (D - 2x)^2, \qquad x \in I, \tag{5.13}$$

where F is strictly increasing.

Proof. We define

$$T(x) := g(x)^2 \cdot \Big(2D - g(x)^2\Big), \qquad x \in I. \tag{5.14}$$

From the first property of g, we get that g' is increasing, therefore, by 2), we have that the function g is strictly increasing. By 3) we initially obtain that g is positive and finally that T is positive. With simple algebraic calculations it follows that

$$T'(x) = -4g(x) \cdot g'(x) \cdot \Big(g(x)^2 - D\Big).$$

Therefore by the positivity of g and g' and by $3)$ we have that T' is negative. Then T is strictly decreasing. Furthermore we obtain that the second derivative of T is

$$T''(x) = -4\left[g'(x)^2 \cdot \Big(g(x)^2 - D\Big) + g(x) \cdot g''(x) \cdot \Big(g(x)^2 - D\Big) + 2g(x)^2 \cdot g'(x)^2\right].$$

Again, by the positivity of g and g' and by the property $3)$ of g, we have that T'' is negative, therefore T is concave. Now we notice that the function f in (5.12) is equal to

$$f(x) = \frac{D - \sqrt{T(x)}}{2}. \tag{5.15}$$

Thus, since T is strictly decreasing, f is strictly increasing.

Furthermore we can see that

$$f'(x) = \frac{g'(x) \cdot \Big(g(x)^2 - D\Big)}{\sqrt{2D - g(x)^2}}. \tag{5.16}$$

We explicitly note that, by $3)$ and g is strictly increasing, $\sqrt{2D - g(x)^2}$ is positive. Hence by the positivity of g' and by $3)$ and g is strictly increasing, $g'(x) \cdot (g(x)^2 - D)$ is positive too. So we get that f' is positive.

We set

$$\begin{aligned} f_1(x) &:= g'(x) \cdot \Big(g(x)^2 - D\Big), \qquad x \in I; \\ f_2(x) &:= \sqrt{2D - g(x)^2}, \qquad x \in I. \end{aligned}$$

From what we have seen before we know that f_1 and f_2 are positive. By the assumption on the function g and by the monotonicity of g and g', we respectively deduce that f_1 is strictly increasing and f_2 is strictly decreasing. It is clear, by (5.16), that $f'(x) = \frac{f_1(x)}{f_2(x)}$ for any $x \in I$. Therefore f' is strictly increasing.

In particular if we have that $f(I) \subseteq I$ and $f'(b) < 1$ (recall that f' is strictly increasing), then, f is a contraction on I. By Banach fixed point Theorem, the function f has one and only fixed point l in I, namely $f(l) = l$.

If $b < \frac{D}{2}$, then from (5.15) and by the fact $f(l) = l$, it is clear that l is a root of the function

$$F(x) := T(x) - (D - 2x)^2, \qquad x \in I.$$

By the expression of T in (5.14), F is the same as in (5.13). We calculate the first and the second derivative of F and we obtain

$$\begin{aligned} F'(x) &= T'(x) + 4(D - 2x); \\ F''(x) &= T''(x) - 8x. \end{aligned}$$

Since T'' is negative and $0 < a \leq x \leq b$, F'' is negative, therefore F' is strictly decreasing. Thus $F'(x) \geq F'(b) = T'(b) + 4(D - 2b)$. Then, since $b < \frac{D}{2}$ and T' is positive, we have that F' is positive. Hence F is strictly increasing.

□

Remark 5.14. Let E_i be a region of $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, then, by Remark 5.12, E_i can be decomposed as:

$$E_i = \begin{cases} E_i, & \text{if } M(i) = 0 \\ E_i^0 \sqcup E_i^1, & \text{if } M(i) = 1 \\ E_i^0 \sqcup E_i^1 \sqcup E_i^2, & \text{if } M(i) = 2. \end{cases}$$

By Remark 5.7 we have that $|\, E_i^0 \,| \geq 1 - A_{1,4}$, while by (5.10) of Corollary 5.10 we know that it holds that $|\, E_i^j \,| \geq A_{2,4}$ for any $j \neq 0$. Therefore, by isoperimetric inequality, we get

$$P(E_i) \geq \begin{cases} 2\sqrt{\pi}, & \text{if } M(i) = 0 \\ 2\sqrt{\pi}\left(\sqrt{1 - A_{1,4}} + \sqrt{A_{2,4}}\right), & \text{if } M(i) = 1 \\ 2\sqrt{\pi}\left(\sqrt{1 - A_{1,4}} + 2\sqrt{A_{2,4}}\right), & \text{if } M(i) = 2, \end{cases}$$

where $2\sqrt{\pi}(\sqrt{1 - A_{1,4}} + 2\sqrt{A_{2,4}}) > 2\sqrt{\pi}(\sqrt{1 - A_{1,4}} + \sqrt{A_{2,4}}) > 2\sqrt{\pi}$, because $1.07154 \approx \sqrt{1 - A_{1,4}} + \sqrt{A_{2,4}} > 1$.

Now we present a theorem where we exclude all possible connection types $I_{\mathbf{E}}$ seen in Remark 5.12 except the cases $(2,0,0,0)$, $(1,1,0,0)$, $(1,0,0,0)$ and of course $(0,0,0,0)$. We denote by $\mathrm{n}(A)$ the cardinality of a set A.

Theorem 5.15. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then*

$$I_{\mathbf{E}} \in \{(0,0,0,0),(1,0,0,0),(1,1,0,0),(2,0,0,0)\}.$$

Proof. We call $P = \{(0,0,0,0),(1,0,0,0),(1,1,0,0),(2,0,0,0)\}$, and we suppose by contradiction that $I_{\mathbf{E}} \notin P$. Therefore the possible connection types $I_{\mathbf{E}} = (M(1),\dots,M(4))$ can be only one of the cases described in Remark 5.12 with the following properties:

1) $\displaystyle\sum_{i=1}^{4} M(i) \geq 3$;

2) $\mathrm{n}(\{j \in \{1,\dots,4\} \,|\, M(j) \geq 1\}) \geq 2$,

namely the possibilities for $I_{\mathbf{E}}$ are $(2,2,2,2)$, $(2,2,2,1)$, $(2,2,2,0)$, $(2,2,1,1)$, $(2,2,1,0)$, $(2,2,0,0)$, $(2,1,1,1)$, $(2,1,1,0)$, $(2,1,0,0)$, $(1,1,1,1)$, $(1,1,1,0)$. We denote by

$$I_d := \{i \in \{1,\dots,4\} \,|\, M(i) \geq 1\};$$
$$I_c := \{i \in \{1,\dots,4\} \,|\, M(i) = 0\};$$

the sets of indices, that represent the disconnected and connected regions respectively.

Now we divide the proof in two parts. In the first we determine the following inequality

$$\sqrt{x} + \sqrt{1-x} \leq \frac{\tilde{p} - \sqrt{\pi}\Big(3 + 2\sqrt{1 - A_{1,4}} + 2\sqrt{A_{2,4}}\Big)}{\sqrt{\pi}}, \tag{5.17}$$

where x represents the area of a disjoint union of small components, $\displaystyle\bigsqcup_{i=1}^{M(j)} E^i_j$. By Remark 5.7 and (5.10) of Corollary 5.10 we know that $A_{2,4} \leq x \leq A_{1,4}$.

Solving (5.17), we find a new estimate for x:

$$A_{2,4} \leq x \leq f_1(A_{1,4}) < A_{1,4}, \tag{5.18}$$

where

$$\begin{aligned}
f_1(x) &= \frac{1-\sqrt{g_1(x)^2\cdot\left(2-g_1(x)^2\right)}}{2}, \qquad x\in I:=[A_{2,4},A_{1,4}],\\
g_1(x) &:= C_1 - 2\sqrt{1-x}, \qquad x\in I, \qquad\qquad (5.19)\\
C_1 &:= \frac{\tilde{p}}{\sqrt{\pi}} - 3 - 2\sqrt{A_{2,4}}
\end{aligned}$$

In the second part we see that, for the cases $(2,2,2,2)$, $(2,2,2,1)$, $(2,2,2,0)$, $(2,2,1,1)$, $(2,2,1,0)$, $(2,2,0,0)$, $(2,1,1,1)$, $(2,1,1,0)$, the new estimate (5.18) on x is immediately a contradiction, while, for the following connection types $I_{\mathbf{E}}$, $(2,1,0,0)$, $(1,1,1,1)$, $(1,1,1,0)$, it will allow us to get an estimate for the perimeter of $\mathbf{E}$ greater than $\tilde{p}$. This is still a contradiction, thus the proof is completed.

Part I. By 2) there are at least two disconnected region. We take $j\in I_d$ such that

$$\sum_{k\in I_d^-} M(k)\geq 2,$$

where $I_d^- = I_d \setminus \{j\}$. We explicitly note that the choice of $j\in I_d$ is indifferent except in the case $I_{\mathbf{E}}=(2,1,0,0)$ where j must be 2 (i.e. the index j denotes the disconnected region with one and only one small component). Given j so done, $E_j = E_j^0 \bigsqcup \left(\bigsqcup_{i=1}^{M(j)} E_j^i\right)$. Therefore, by the minimality of $\mathbf{E}$, we get that

$$\tilde{p}\geq P(\mathbf{E})\geq \frac{1}{2}\left(P(E_0)+P(E_j^0)+P\left(\bigsqcup_{i=1}^{M(j)} E_j^i\right)+\sum_{k\in I_d^-}P(E_j)+\sum_{k\in I_c}P(E_j)\right), \tag{5.20}$$

where the following conditions apply (note that $0\leq M(i)\leq 2$ for any i by Proposition 5.11 and see also the properties 1 and 2))

a) $\mathrm{n}(I_d^-)\geq 1$;

b) $\mathrm{n}(I_d^-)+\mathrm{n}(I_c)=3$;

c) $\sum_{k \in I_d^-} M(k) \geq 2.$

By $a)$ $b)$ and $c)$ we get that $(\mathrm{n}(I_d^-), \mathrm{n}(I_c))$ can be $(1,2)$, $(2,1)$ and $(3,0)$. We notice that the case $(\mathrm{n}(I_d^-), \mathrm{n}(I_c)) = (1,2)$ is $I_{\mathbf{E}} \in \{(2,2,0,0),(2,1,0,0)\}$, while if $(\mathrm{n}(I_d^-), \mathrm{n}(I_c)) = (2,1)$ the possibilities for $I_{\mathbf{E}}$ are $(2,2,2,0)$, $(2,2,1,0)$, $(2,1,1,0)$, $(1,1,1,0)$, and finally if $(\mathrm{n}(I_d^-), \mathrm{n}(I_c)) = (3,0)$ the set of possible connection types $I_{\mathbf{E}}$ is $\{(2,2,2,2),(2,2,2,1),(2,2,1,1),(2,1,1,1),(1,1,1,1)\}$. We set

$$S(\mathbf{E}) \colon = \sum_{k \in I_d^-} P(E_k) + \sum_{k \in I_c} P(E_k).$$

We underline that, the condition $c)$ guarantees that in $\sum_{k \in I_d^-} P(E_k)$, there are disconnected regions such that the total number of their small components is at least two. Hence by Remark 5.14, we have that

$$S(\mathbf{E}) \geq \begin{cases} 2\sqrt{\pi}(\sqrt{1-A_{1,4}} + 2\sqrt{A_{2,4}}) + 4\sqrt{\pi}, & \text{if } (n(I_d^-), n(I_c)) = (1,2) \\ 4\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}) + 2\sqrt{\pi}, & \text{if } (n(I_d^-), n(I_c)) = (2,1) \\ 6\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}), & \text{if } (n(I_d^-), n(I_c)) = (3,0). \end{cases}$$

By Remark 5.7, $1 - A_{1,4} < 1$, thus we observe that

$$2\sqrt{\pi}(\sqrt{1-A_{1,4}} + 2\sqrt{A_{2,4}}) + 4\sqrt{\pi} > 4\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}) + 2\sqrt{\pi}.$$

Since $\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}} > 1$ (see Remark 5.14), we get that

$$6\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}) > 4\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}) + 2\sqrt{\pi}.$$

Thus we have that

$$S(\mathbf{E}) \geq 4\sqrt{\pi}(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}}) + 2\sqrt{\pi}, \tag{5.21}$$

By (5.20), it is clear that

$$\tilde{p} \geq P(\mathbf{E}) \geq \frac{1}{2}\left(P(E_0) + P(E_j^0) + P\left(\bigsqcup_{i=1}^{M(j)} E_j^i \right) + S(\mathbf{E}) \right).$$

We call $x = \left| \bigsqcup_{i=1}^{M(j)} E_j^i \right|$, thus $|\, E_j^0 \,| = |\, E_j \,| - x \geq 1 - x > 0$ (because, by Remark 5.7, $0 < x = \left| \bigsqcup_{i=1}^{M(j)} E_j^i \right| < A_{1,4} < \frac{1}{3}$). Therefore, from the isoperimetric inequality and (5.21), we get that

$$\tilde{p} \geq P(\mathbf{E}) \geq \sqrt{\pi}\left(2 + \sqrt{1-x} + \sqrt{x} + 2\sqrt{1 - A_{1,4}} + 2\sqrt{A_{2,4}} + 1\right), \quad (5.22)$$

which is equivalent to the inequality in (5.17), namely

$$\sqrt{x} + \sqrt{1-x} \leq \frac{\tilde{p} - \sqrt{\pi}\left(3 + 2\sqrt{1 - A_{1,4}} + 2\sqrt{A_{2,4}}\right)}{\sqrt{\pi}}, \quad (5.23)$$

where x represents the area of a disjoint union of small components, $\bigsqcup_{i=1}^{M(j)} E_j^i$, such that $A_{2,4} \leq x \leq A_{1,4}$, by Remark 5.7 and (5.10) of Corollary 5.10. Let I be the interval $I := [A_{2,4}, A_{1,4}]$ and let the functions g_1 and f_1 be as in (5.19). Therefore we have that

$$g_1'(x) = \frac{1}{\sqrt{1-x}},$$
$$g_1''(x) = \frac{1}{2(1-x)^{\frac{3}{2}}}.$$

Thus it follows that

$$\begin{aligned} &1)\ g_1 \text{ is convex on } I;\\ &2)\ g_1'(A_{2,4}) = \frac{1}{\sqrt{1 - A_{2,4}}} > 0;\\ &3)\ 1 < g_1(A_{2,4}) \approx 1.03083 < g_1(A_{1,4}) \approx 1.17283 < \sqrt{2}. \end{aligned} \quad (5.24)$$

We set $D = 1$, $a = A_{2,4}$ and $b = A_{1,4}$, so by Lemma 5.13, the function f_1 in (5.19) is strictly increasing on I. Furthermore we get that

$$0.0238853 \approx A_{2,4} < f_1(A_{1,4}) \approx 0.0365939 < A_{1,4} \approx 0.159132.$$

Now it is easy to see that the inequality in (5.23) is

$$\sqrt{x}+\sqrt{1-x}\le g_1(A_{1,4}). \tag{5.25}$$

We set $C=g_1(A_{1,4})$ and $D=1$, then by Lemma 3.1, the solution of previous inequality is

$$0<x\le f_1(A_{1,4})\text{ or }(1-f_1(A_{1,4}))\le x<1,$$

because, by (5.24), we have that $1<g_1(A_{1,4})<\sqrt{2}$ (see the condition on the constants in Lemma 3.1). We recall that $A_{2,4}\le x\le A_{1,4}$, furthermore $A_{2,4}<f_1(A_{1,4})<A_{1,4}$, thus $1-f_1(A_{1,4})>1-A_{1,4}>A_{1,4}$ because, by Remark 5.7, $A_{1,4}<\frac{1}{3}$. Then the inequality (5.25) reduce to:

$$A_{2,4}\le x\le f_1(A_{1,4})<A_{1,4}, \tag{5.26}$$

that is the estimate in (5.18).

Part II. From (5.26) we have that $A_{2,4}<x\le f_1(A_{1,4})<A_{1,4}$, where $x=\left|\bigsqcup_{i=1}^{M(j)}E_i^j\right|$ and $j\in I_d$ such that $\sum_{k\in I_d^-}M(k)\ge 2$. Therefore we can immediately exclude the following possibilities of connection type:$(2,2,2,2)$, $(2,2,2,1)$, $(2,2,2,0)$, $(2,2,1,1)$, $(2,2,1,0)$, $(2,2,0,0)$, $(2,1,1,1)$, $(2,1,1,0)$. In these cases we can choose j such that $M(j)=2$, then by (5.10) of Corollary 5.10, $|E_j^i|\ge A_{2,4}$ for any $i=1,2$, therefore by (5.26) it follows that

$$0.0477706\approx 2A_{2,4}\le\left|\bigsqcup_{i=1}^{2}E_i^j\right|\le f_1(A_{1,4})\approx 0.0365939.$$

This is a contradiction.

For the other cases of connection type, $(2,1,0,0)$, $(1,1,1,0)$ and $(1,1,1,1)$, we give an estimate for the perimeter $P(\mathbf{E})$, which will be greater that $\tilde{p}$.

We start with $I_{\mathbf{E}}\in\{(1,1,1,0),(1,1,1,1)\}$. From the first part we can say that $|E_i^1|\le f_1(A_{1,4})$ for any small component of a disconnected region, because it is true that $\sum_{k\in I_d^-}M(k)\ge 2$ for any $j\in I_d$. Therefore we have that

$$|E_i^0|=|E_i|-|E_i^1|\ge 1-f_1(A_{1,4})>1-A_{1,4}, \tag{5.27}$$

for any big component of a disconnected region. Moreover, by (5.10) of Corollary 5.10, it holds that

$$| E_i^1 | \geq A_{2,4}, \tag{5.28}$$

for any small component of a disconnected region. Hence, by (5.27), by (5.28) and by the isoperimetric inequality, we get the following estimate for the perimeter of $\mathbf{E}$:

$$11.1946 \approx \tilde{p} \geq P(\mathbf{E}) \geq \begin{cases} \frac{1}{2}\Big(P(E_0) + \sum_{i=1}^{3} P(E_i^0) + \sum_{i=1}^{3} P(E_i^1) + P(E_4)\Big) \\ \frac{1}{2}\Big(P(E_0) + \sum_{i=1}^{4} P(E_i^0) + \sum_{i=1}^{4} P(E_i^1)\Big) \end{cases}$$

$$\geq \begin{cases} \sqrt{\pi}\big(2 + 3\sqrt{1 - f_1(A_{1,4})} + 3\sqrt{A_{2,4}} + 1\big) \approx 11.3583, & \text{if } I_{\mathbf{E}} = (1,1,1,0), \\ \sqrt{\pi}\big(2 + 4\sqrt{1 - f_1(A_{1,4})} + 4\sqrt{A_{2,4}}\big) \approx 11.5995, & \text{if } I_{\mathbf{E}} = (1,1,1,1). \end{cases}$$

This is a contradiction.

Finally we consider the case when $I_{\mathbf{E}} = (2,1,0,0)$. From the first part we can say that $| E_2^1 | \leq f_1(A_{1,4})$, because the only choice of $j \in I_d$ such that $\sum_{k \in I_d} M(k) \geq 2$ is $j = 2$. Therefore we have that

$$| E_2^0 | = | E_2 | - | E_2^1 | \geq 1 - f_1(A_{1,4}) > 1 - A_{1,4}, \tag{5.29}$$

while, by (5.10) of Corollary 5.10, we get that

$$| E_2^1 | \geq A_{2,4}. \tag{5.30}$$

By Remark 5.7 and (5.10) of Corollary 5.10 we know that

$$\begin{aligned} &| E_1^0 | \geq 1 - A_{1,4}, \\ &| E_1^j | \geq A_{2,4}, \qquad \text{for any } j = 1,2. \end{aligned} \tag{5.31}$$

Hence, by (5.29), by (5.30), by (5.31) and by the isoperimetric inequality, we get the following estimate for the perimeter of $\mathbf{E}$:

$$\tilde{p} \geq P(\mathbf{E}) \geq \frac{1}{2}\Big(P(E_0) + \sum_{k=3}^{4} P(E_k) + P(E_1^0) + \sum_{j=1}^{2} P(E_1^j) + P(E_2^0) + P(E_2^1)\Big)$$

$$\geq \sqrt{\pi}\big(2+2+\sqrt{1-A_{1,4}}+2\sqrt{A_{2,4}}+\sqrt{1-f_1(A_{1,4})}+\sqrt{A_{2,4}}\big)\approx 11.2766$$

This is a contradiction because $\tilde{p} \approx 11.1946$.

□

Corollary 5.16. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $\mathbf{E}$ *has at least two connected regions. Moreover* $\mathbf{E}$ *has at most six bounded components, thus* $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.

Proof. If $\mathbf{E}$ is standard, then any region is connected therefore there are four bounded components.

If $\mathbf{E}$ is not standard, then by Remark 5.12 and Theorem 5.15, we know that its connection type $I_{\mathbf{E}}$ can be only $(2,0,0,0)$, $(1,1,0,0)$ or $(1,0,0,0)$. Therefore $\mathbf{E}$ has two connected regions and it can have at most six bounded components. Thus, by Theorem 1.50, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.

□

5.3 The cases $(2,0,0,0)$ and $(1,1,0,0)$

In this section we consider the cases $(2,0,0,0)$ and $(1,1,0,0)$. The most important results are Theorem 5.23 and Theorem 5.39, that exclude these possibilities.

Lemma 5.17. *Let* $f : I \to \mathrm{R}$ *be an increasing function (i.e.* $f(x) \leq f(y)$ *if* $x \leq y$*) where* I *is an interval of* R *and* $f(I) \subseteq I$*. Fixed arbitrarily* $x_0 \in I$*, we define the following recurrence sequence* $(u_n)_{n\in\mathbb{N}}$

$$\begin{cases} u_0 = x_0 \\ u_{n+1} = f(u_n), \qquad n \geq 0. \end{cases}$$

If $u_1 > x_0$ *then the sequence* $(u_n)_{n\in\mathbb{N}}$ *is increasing, while if* $u_1 < x_0$*, then* $(u_n)_{n\in\mathbb{N}}$ *is decreasing. In particular the limit of the sequence* $(u_n)_{n\in\mathbb{N}}$ *exists.*

Proof. If $x_0 = u_1$ then $f(u_1) = u_1$ thus, by induction, the sequence $(u_n)_{n\in\mathbb{N}}$ is constant and it holds that $u_n = x_0$ for all $n \in \mathbb{N}$, then the existence of the limit is obvious.

Thus we can consider $x_0 < u_1$ or $x_0 > u_1$. If $x_0 < u_1$, since f is increasing , it holds $u_1 = f(x_0) \leq f(u_1) = u_2$; now we prove by induction that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. For $n = 1$ we have that

$u_2 = f(u_1) \le f(u_2) = u_3$, since $u_1 \le u_2$ and f is increasing. We suppose true $u_n \le u_{n+1}$, then $u_{n+1} = f(u_n) \le f(u_{n+1}) = u_{n+2}$, thus we have $u_{n+1} \le u_{n+2}$.

If $x_0 > u_1$ we proceed in the same way, and now we obtain that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

Since in both cases the sequence $(u_n)_{n\in\mathbb{N}}$ is monotone, then the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ exists.

□

Lemma 5.18. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} \in \{(2,0,0,0),(1,1,0,0)\}$, *then* $\mathbf{E}$ *has ten vertices and fifteen edges.*

Proof. Let v, e and c be the numbers of the vertices, of the edges and of the connected components of $\mathbf{E}$ respectively, then, by the Euler's formula, it applies that $v - e + c = 2$. Since $\mathbf{E}$ is a minimum, each vertex of $\mathbf{E}$ is is a meeting point of exactly three edges (see Theorem 1.10), thus $3v = 2e$ (note that each edge has two vertices). Furthermore, $I_{\mathbf{E}} \in \{(2,0,0,0),(1,1,0,0)\}$, therefore $c = 7$. Solving the following linear system

$$\begin{cases} v - e = -5 \\ 3v = 2e, \end{cases}$$

we find the statement. □

5.3.1 The case $(2,0,0,0)$

We begin with the case $I_{\mathbf{E}} = (2,0,0,0)$. First of all we present a simple lemma where we describe a component of the disconnected region and a component of a connected region.

Lemma 5.19. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (2,0,0,0)$, *then a component* C *of* E_1 *has*

i) *three edges if it is inner;*

ii) *at most four edges if it is external.*

While a connected region has

iii) *at most five edges if it is inner;*

iv) *at most six edges if it is external.*

Proof. By Corollary 1.35, we know that every component has at least three edges. Furthermore from Proposition 1.33 we have that any two components of $\mathbf{E}$ may meet at most once along a single edge. We explicitly note that $\mathbf{E}$ has six bounded components with three connected regions.

We consider a component C of E_1; therefore if C is inner, by the minimality of $\mathbf{E}$, C must have three edges.

While if C is external, then it could have one and only one edges in common with $\mathbf{E}_0$, thus C can have at most four edges if it is also adjacent to the all others connected regions.

The argument is the same in the case that we take a connected region E_i, finding that E_i has at most five edges and most six edges if E_i is inner and external respectively.

$\square$

Lemma 5.20. *Let* $C_{_2} = \frac{\tilde{p}-\sqrt{\pi}(5+\sqrt{A_{2,4}})}{\sqrt{\pi}}$ *and*

$$f_2(x) := \frac{(1-x) - \sqrt{C_{_2}^2 \cdot \Big(2(1-x) - C_{_2}^2\Big)}}{2}, \qquad x \in I := [A_{2,4}, A_{1,4}],$$

then f_2 is strictly increasing on I with $f_2(I) \subset I$ and f_2 is a contraction on I.

Furthermore the unique fixed point l of f_2 on I is a root of the function

$$F_3(x) := (3x-1) + \sqrt{C_{_2}^2 \cdot \Big(2(1-x) - C_{_2}^2\Big)},$$

where F_3 is strictly increasing. In particular l is less than 0.042.

Proof. We explicitly note that f_2 is well defined because $2(1-x) - C_{_2}^2 > 0$ for all $x \in I$, indeed $2(1-x) \geq 2(1-A_{1,4}) \approx 1.68174 > C_{_2}^2 \approx 1.34874$. We initially prove that f_2 is strictly increasing and that it is a contraction on I. With simple algebraic calculations we can see that

$$f_2'(x) = \frac{1}{2} \cdot \left[\frac{C_{_2}^2}{\sqrt{C_{_2}^2 \cdot \Big(2(1-x) - C_{_2}^2\Big)}} - 1 \right].$$

Therefore f_2' is positive if and only if

$$C_2^2 > (1 - x).$$

But $x \in I$, so $(1 - x) \leq (1 - A_{2,4}) \approx 0.976115 < C_2^2 \approx 1.34874$. Thus we get that f_2 is strictly increasing on I. Hence we have that $f_2(I) \subset I$, indeed

$$\begin{gathered} 0.0369618 \approx f_2(A_{2,4}) \leq f_2(x) \leq f_2(A_{1,4}) \approx 0.0853508, \\ A_{2,4} \approx 0.0238853, \quad A_{1,4} \approx 0.159132. \end{gathered} \tag{5.32}$$

If we set $f_4(x) := 2(1-x) - C_2^2$, with $x \in I$, then f_4 is strictly decreasing on I, thus $f_4(x) > f_4(A_{1,4}) \approx 0.332994 > 0$. It is easy to see that

$$f_2'(x) = \frac{1}{2} \cdot \left[\frac{C_2^2 - \sqrt{C_2^2 \cdot f_4(x)}}{\sqrt{C_2^2 \cdot f_4(x)}} \right].$$

Since f_4 is strictly decreasing, we have that $C_2^2 - \sqrt{C_2^2 \cdot f_4(x)}$ is strictly increasing on I. Furthermore, since f_2' and f_4 are positive on I, we also get that $C_2^2 - \sqrt{C_2^2 \cdot f_4(x)}$ is positive on I. Therefore f_2' is strictly increasing on I. Thus we obtain that

$$0 < f_2'(x) < f_2'(A_{1,4}) \approx 0.506274. \tag{5.33}$$

Hence, by (5.32) and (5.33), we deduce that f_2 is a contraction on I.

Then, by Banach fixed point Theorem, we have that there exists one and only one fixed point l of f_2 on I such that $f_2(l) = l$. By $f_2(l) = l$, we deduce that l is a root of the function

$$F_3(x) := (3x - 1) + \sqrt{C_2^2 \cdot \Big(2(1 - x) - C_2^2\Big)}.$$

Its first derivative is

$$F_3'(x) = 3 - \frac{C_2^2}{\sqrt{C_2^2 \cdot \Big(2(1 - x) - C_2^2\Big)}}.$$

Now we can see that $F_3' > 0$ it is equivalent to

$$(1-x) > \frac{5C_{2}^2}{9}.$$

Since $x \in I$, then $(1-x) \geq (1-A_{1,4}) \approx 0.840868 > \frac{5C_{2}^2}{9} \approx 0.749301$, (note that the denominator of F_3' is just positive because it is the same denominator of f_2'). Therefore F_3' is positive on I and so F_3 is strictly increasing. Since $F_3(0.042) \approx 0.000691277 > 0$, we have the estimate $l < 0.042$.

$\square$

Lemma 5.21. *Let* $\mathbf{E} \in \mathcal{M}_{2,4}^*(1,1,1,1)$. *If* $I_{\mathbf{E}} = (2,0,0,0)$, *then the area of a small component can be at most the limit of the following sequence:*

$$\begin{cases} a_0 = A_{1,4} \\ a_{n+1} = f(a_n), \qquad n \geq 0, \end{cases} \tag{5.34}$$

where

$$f_2(x) := \frac{(1-x) - \sqrt{C_{2}^2 \cdot \left(2(1-x) - C_{2}^2\right)}}{2}, \qquad x \in I := [A_{2,4}, A_{1,4}] \tag{5.35}$$

$$\text{with } C_{2} := \frac{\tilde{p} - \sqrt{\pi}(5 + \sqrt{A_{2,4}})}{\sqrt{\pi}}.$$

In such case the limit l *is less than* 0.042.

Proof. First of all we note that, by Lemma 5.20, f_2 is strictly increasing and $f_2(I) \subset I$, in particular $f_2(A_{1,4}) < A_{1,4}$. So by Lemma 5.17, the sequence $(a_n)_{n\in\mathbb{N}}$ has a finite limit l and it is strictly decreasing. Furthermore, since by Lemma 5.20, f_2 is a contraction on I, then l is the unique fixed point of f_2 in I with $l < 0.042$. Since $I_{\mathbf{E}} = (2,0,0,0)$, then $E_1 = E_1^0 \sqcup E_1^1 \sqcup E_1^2$.

In order to show the statement of the Lemma we will prove by induction the following property:

$$|\, E_1^i \,| \leq a_n, \quad \forall n \in \mathbb{N}, \forall i = 1,2. \tag{5.36}$$

The case $n = 0$ is true since $a_0 = A_{1,4}$ and by Remark 5.7, by (5.10) of Corollary 5.10, we have that

$$A_{2,4} \leq |\, E_1^i \,| \leq A_{1,4} < \frac{1}{3}, \qquad \forall i = 1,2. \tag{5.37}$$

We suppose that (5.36) is true for n and now we prove it for $n+1$. Therefore the following estimates hold (note that $a_1 = f_2(A_{1,4}) < A_{1,4} = a_0$ and the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing, thus $a_n \leq A_{1,4}$ for all n $\in \mathbb{N}$)

$$A_{2,4} \leq |\, E_1^i \,| \leq a_n, \qquad \forall i = 1,2. \tag{5.38}$$

Let $x = |\, E_1^i \,|$ for $i = 1,2$, thus $|\, E_1^0 \,| = (|\, E_1 \,| - |\, E_1^j \,|) - |\, E_1^i \,| \geq (1 - a_n) - x > 0$ where $j \in \{1,2\} \setminus \{i\}$. By the minimality of **E**, by the isoperimetric inequality and by (5.38), we get that

$$\begin{aligned}\tilde{p} \geq P(\mathbf{E}) &= \frac{1}{2} \cdot \Big(P(E_0) + P(E_1^i) + P(E_1^0) + P(E_1^j) + \sum_{k=2}^{4} P(E_k) \Big) \\ &\geq \sqrt{\pi}\Big(2 + \sqrt{x} + \sqrt{(1-a_n) - x} + \sqrt{A_{2,4}} + 3 \Big).\end{aligned}$$

We find the following inequality

$$\sqrt{x} + \sqrt{(1-a_n) - x} \leq \frac{\tilde{p} - \sqrt{\pi}(5 + \sqrt{A_{2,4}})}{\sqrt{\pi}} = C_{_2}. \tag{5.39}$$

Since $f_2(I) \subset I$ and the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing, therefore $0 < A_{2,4} < a_n \leq A_{1,4} < 1$ for all $n \in \mathbb{N}$. Since

$$1 < C_{_2} \approx 1.16135 < \sqrt{2(1 - A_{1,4})} \approx 1.29682,$$

we get that

$$\sqrt{1 - a_n} < C_{_2} < \sqrt{2(1 - a_n)}.$$

So if we set $D = 1 - a_n$, by Lemma 3.1, the solution of (5.39) is

$$0 < x \leq \frac{D - \sqrt{C_{_2}^2(2D - C_{_2}^2)}}{2} \text{ or } \frac{D + \sqrt{C_{_2}^2(2D - C_{_2}^2)}}{2} \leq x < D.$$

From the expression of f_2 in (5.35), it is clear that the solution can be written as

$$0 < x \leq f_2(a_n) \text{ or } \Big(1 - a_n - f_2(a_n)\Big) \leq x < (1 - a_n).$$

But by (5.38), we know that $A_{2,4} \leq x \leq a_n$. The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing and the function f_2 is strictly increasing with $f_2(I) \subset I$, therefore $A_{2,4} < f_2(a_n) < a_n$. Moreover it follows that $1 - a_n - f_2(a_n) > 1 - a_n - a_n > a_n$, because $f_2(I) \subset I$ and by Remark 5.7, $A_{1,4} < \frac{1}{3}$. Thus the solution of (5.39) is

$$A_{2,4} \leq x \leq f_2(a_n) = a_{n+1}.$$

So (5.36) is true with $n+1$ in place of n. This completes the proof.

$\square$

Lemma 5.22. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (2,0,0,0)$, *then the small components of the disconnected region are external with four edges.*

Proof. Since $I_{\mathbf{E}} = (2,0,0,0)$, it follows that $E_1 = E_1^0 \sqcup E_1^1 \sqcup E_1^2$, where E_1^0 is the big component and E_1^j $(j \neq 0)$ are the small components. We argue by contradiction, thus, by Lemma 5.19, there exists E_1^i $(i = 1,2)$ such that it has three edges. Without loss of generality we can suppose that E_1^1 is a small three-sided component. Therefore, by (5.9) of Corollary 5.10, we have that

$$|\,E_1^1\,| \geq 1 - \frac{9\tilde{p}^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi}{9\tilde{p}^2}} - 1\right) := A_{3,4} \approx 0.0409878.$$

We summarize the conditions of area of small components of E_1; by (5.10) of Corollary 5.10 and Lemma 5.21 we get that

$$\begin{aligned} A_{3,4} \leq |\,E_1^1\,| \leq l < 0.042 \\ A_{2,4} \leq |\,E_1^2\,| \leq l < 0.042. \end{aligned} \tag{5.40}$$

We will show that the area of E_1^2 is smaller than $A_{2,4}$; therefore we would contradict (5.40), so the proof will be completed.

Let $x = |\,E_1^2\,|$, thus $|\,E_1^0\,| = (|\,E_1\,| - |\,E_1^1\,|) - |\,E_1^2\,| \geq (1 - 0.042) - x > 0$ by (5.40). By the minimality of $\mathbf{E}$, by the isoperimetric inequality and (5.40), we get that

$$\begin{aligned}
\tilde{p} \geq P(\mathbf{E}) &= \frac{1}{2}\Big(P(E_0) + P(E_1^2) + P(E_1^0) + P(E_1^1) + \sum_{k=2}^{4} P(E_k)\Big) \\
&\geq \sqrt{\pi}\Big(2 + \sqrt{x} + \sqrt{(1-0.042)-x} + \sqrt{A_{3,4}} + 3\Big).
\end{aligned}$$

We find the following inequality

$$\sqrt{x} + \sqrt{(1-0.042)-x} \leq \frac{\tilde{p} - \sqrt{\pi}(5+\sqrt{A_{3,4}})}{\sqrt{\pi}}. \tag{5.41}$$

We set $C_3 = \frac{\tilde{p}-\sqrt{\pi}(5+\sqrt{A_{3,4}})}{\sqrt{\pi}}$ and $D = 1-0.042$, then we can see that

$$0.978775 \approx \sqrt{1-0.042} < C_3 \approx 1.11345 < \sqrt{2(1-0.042)} \approx 1.3842.$$

So by Lemma 3.1, the solution of (5.41) is

$$0 < x \leq \frac{D - \sqrt{C_3^2(2D - C_3^2)}}{2} \text{ or } \frac{D + \sqrt{C_3^2(2D - C_3^2)}}{2} \leq x < D, \tag{5.42}$$

where

$$\begin{aligned}
\frac{D - \sqrt{C_3^2(2D - C_3^2)}}{2} &\approx 0.0211867 \\
\frac{D + \sqrt{C_3^2(2D - C_3^2)}}{2} &\approx 0.936813.
\end{aligned}$$

But from (5.40), $0.0238853 \approx A_{2,4} \leq x \leq A_{1,4} \approx 0.159132$. This contradicts the solution (5.42) of (5.41).

□

Now we are ready to eliminate the case $I_{\mathbf{E}} = (2,0,0,0)$ of Remark 5.12.

Theorem 5.23. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $I_{\mathbf{E}} \neq (2,0,0,0)$.

Proof. We suppose by contradiction that $I_{\mathbf{E}} = (2,0,0,0)$, therefore it follows that $E_1 = E_1^0 \bigsqcup E_1^1 \bigsqcup E_2^0$ where E_1^0 is the big component and E_1^i $(i = 1,2)$ are the small components. By Lemma 5.22, E_1^i $(i = 1,2)$ are external with four edges, while, by Lemma 5.19, E_1^0 has at least three edges and at most four edges. We denote by $v(C)$ the number of vertices of a component C of $\mathbf{E}$ and $v(\mathbf{E})$ is the number of the vertices of $\mathbf{E}$. Thus, we have that $(v(E_1^0), v(E_1^1), v(E_1^2))$ can be $(3,4,4)$ or $(4,4,4)$. Since E_1^i and E_1^j are disjoint two by two for any $i \neq j$ ($i,j \in \{0,1,2\}$), the vertices of E_1^i and E_1^j are all distinct (recall that $\mathbf{E}$ is a minimizer). Thus, we get that

$$v(\mathbf{E}) \geq \sum_{i=0}^{2} v(E_1^i) \geq 11,$$

but $v(\mathbf{E}) = 10$ by Lemma 5.18. This is a contradiction, so the proof is completed. □

5.3.2 The case $(1,1,0,0)$

We analyze the case $I_{\mathbf{E}} = (1,1,0,0)$ of Remark 5.12; first of all as in the case $I_{\mathbf{E}} = (2,0,0,0)$ of Remark 5.12 we describe a component of a disconnected region and a component of a connected region.

Lemma 5.24. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,1,0,0)$, then a component C of a disconnected region has*

i) *at most four edges if it is inner;*

ii) *at most five edges if it is external.*

While a connected region has

iii) *at most five edges if it is inner;*

iv) *at most six edges if it is external.*

Proof. By Corollary 1.35 we know that any component C of $\mathbf{E}$ has at least three edges. Let C be a component of a disconnected region. Since $\mathbf{E}$ is a minimum, by Proposition 1.33 and $I_{\mathbf{E}} = (1,1,0,0)$, then, C has at most four edges and at most five edges if C is inner and external respectively.

If E_i is a connected region, arguing as in the case that C is a component of a disconnected region, then E_i can have at most five edges and six edges if it is inner and external respectively.

□

In the next lemma we determine the new maximum value of the area of a small component.

Lemma 5.25. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*. If* $I_{\mathbf{E}} = (1,1,0,0)$*, then the area of a small component can be at most the limit of the following sequence:*

$$\begin{cases} a_0 = A_{1,4} \\ a_{n+1} = f(a_n), \qquad n \geq 0, \end{cases} \tag{5.43}$$

where

$$\begin{aligned} f_5(x) &:= \frac{1 - \sqrt{g_2(x)^2 \cdot \left(2 - g_2(x)^2\right)}}{2}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad x \in I := [A_{2,4}, A_{1,4}] \\ g_2(x) &:= \left(\frac{\tilde{p}}{\sqrt{\pi}} - 4 - \sqrt{A_{2,4}}\right) - \sqrt{1-x}. \end{aligned} \tag{5.44}$$

In such case the limit l *is less than* 0.042*.*

Proof. First of all we consider the function g_2 of (5.44), then its first and second derivatives are respectively

$$\begin{aligned} g_2'(x) &= \frac{1}{2\sqrt{1-x}}, \\ g_2''(x) &= \frac{1}{4(1-x)^{\frac{3}{2}}}. \end{aligned}$$

Therefore, since g_2' and g_2'' are positive, we get that

1) g_2 is convex on I;

2) g_2' is strictly increasing and $g_2'(A_{2,4}) \approx 0.50608 > 0$;

3) g_2 is strictly increasing and

$$1 < 1.17337 \approx g_2(A_{2,4}) < g_2(A_{1,4}) \approx 1.24436 < \sqrt{2}.$$

Thus if we set $a = A_{2,4}$, $b = A_{1,4}$ and $D = 1$, by Lemma 5.13, the function f_5 of (5.44) is strictly increasing and its first derivative is positive and strictly increasing. Furthermore it holds that $f_5(I) \subset I$, indeed ($A_{2,4} \approx 0.0238853$ and $A_{1,4} \approx 0.159132$)

$$0.0368513 \approx f_5(A_{2,4}) < f_5(A_{1,4}) \approx 0.0819064.$$

So by Lemma 5.17, the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing with a finite limit l which is a fixed point of f_5 in I. Moreover $f_5'(A_{1,4}) \approx 0.445023 < 1$, thus f_5 is a contraction on I, then l is the unique fixed point of f_5 on I. By Remark 5.7 it holds that $A_{1,4} < \frac{1}{3} < \frac{1}{2}$, hence again by Lemma 5.13, we obtain that l is a root of the function $F_6(x) = g_2(x)^2 \cdot \Big(2 - g_2(x)^2\Big) - (1-2x)^2$, where F_6 is strictly increasing on I. Since $F_6(0.042) \approx 0.00214782 > 0$, then

$$l < 0.042. \tag{5.45}$$

Since $I_{\mathbf{E}} = (1,1,0,0)$, then $E_1 = E_1^0 \sqcup E_1^1$ and $E_2 = E_2^0 \sqcup E_2^1$.

In order to show the statement of the Lemma we will prove by induction the following property:

$$|\, E_i^1 \,| \le a_n, \quad \forall n \in \mathbb{N}, \forall i = 1,2. \tag{5.46}$$

The case $n = 0$ is true since $a_0 = A_{1,4}$ and by remark 5.7, by (5.10) of Corollary 5.10, we have that

$$\begin{aligned} &A_{2,4} \le |\, E_i^1 \,| \le A_{1,4}, \qquad \forall i = 1,2, \\ &|\, E_i^0 \,| \ge 1 - A_{1,4}, \qquad \forall i = 1,2. \end{aligned} \tag{5.47}$$

We suppose that (5.46) is true for n and now we prove it for $n+1$. Therefore the following estimates hold

$$\begin{aligned} &A_{2,4} \le |\, E_i^1 \,| \le a_n, \qquad \forall i = 1,2, \\ &|\, E_i^0 \,| \ge 1 - a_n, \qquad \forall i = 1,2. \end{aligned} \tag{5.48}$$

We explicitly note that, since $f_5(I) \subset I$ and by the definition of the sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \le A_{1,4} < \frac{1}{3}$ for all $n \in \mathbb{N}$. Let $x = |\, E_i^1 \,|$ with $i = 1,2$, thus

$|E_i^0| = |E_i| - |E_i^1| \geq 1 - x > 0$. By the minimality of $\mathbf{E}$, by the isoperimetric inequality and (5.48), we get that

$$\begin{aligned}\tilde{p} \geq P(\mathbf{E}) &= \frac{1}{2}\Big(P(E_0) + P(E_i^1) + P(E_i^0) + P(E_j^1) + P(E_j^0) + \sum_{k=3}^{4} P(E_k)\Big)\\ &\geq \sqrt{\pi}\Big(2 + \sqrt{x} + \sqrt{1-x} + \sqrt{A_{2,4}} + \sqrt{1-a_n} + 2\Big).\end{aligned}$$

We find the following inequality

$$\sqrt{x} + \sqrt{1-x} \leq \frac{\tilde{p} - \sqrt{\pi}(4 + \sqrt{A_{2,4}} + \sqrt{1-a_n})}{\sqrt{\pi}}. \tag{5.49}$$

By definition of the function g_2 view in (5.44), it easy to see that the previous inequality can be written as

$$\sqrt{x} + \sqrt{1-x} \leq g_2(a_n).$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing with $A_{2,4} < a_n \leq A_{1,4}$ for all $n \in N$, while by 3) the function g_2 is strictly increasing with $1 < g_2(x) < \sqrt{2}$ for any $x \in I = [A_{2,4}, A_{1,4}]$. Thus $1 < g_2(a_n) \leq g_2(A_{1,4}) < \sqrt{2}$; so if we set $D = 1$ and $C = g_2(a_n)$, by Lemma 3.1, the solution of (5.49) is

$$0 < x \leq \frac{D - \sqrt{C^2(2D - C^2)}}{2} \text{ or } \frac{D + \sqrt{C^2(2D - C^2)}}{2} \leq x < D.$$

From the expression of f_5 in (5.44), it is clear that this can be written as

$$0 < x \leq f_5(a_n) \text{ or } \Big(1 - f_5(a_n)\Big) \leq x < 1.$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing with $A_{2,4} < a_n \leq A_{1,4}$ for all $n \in N$, while the function f_5 is strictly increasing on I with $f_5(I) \subset I$. Thus $A_{2,4} < f_5(a_n) < a_n$ and $1 - f_5(a_n) > 1 - a_n > a_n$, because $A_{2,4} < a_n \leq A_{1,4}$ for all $n \in \mathbb{N}$, $f_5(I) \subset I$ and by Remark 5.7, $A_{1,4} < \frac{1}{3}$. Furthermore by (5.48), $A_{2,4} \leq x = |E_i^1| \leq a_n$, therefore the solution of (5.49) is

$$A_{2,4} \leq x \leq f_5(a_n) = a_{n+1}.$$

This solves the case of (5.46) for $n+1$.

$\square$

In the following lemma we show that each component with three edges is a big component.

Lemma 5.26. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,1,0,0)$, *then the small components of the disconnected regions are surrounded by four different regions. In particular any three-sided component is a big component and each small component of a disconnected region is external.*

Proof. Let S_i and B_i the small and the big component respectively of a disconnected region $E_i = B_i \bigsqcup S_i$ $(i = 1, 2)$. We divide the proof in three steps.

Step I. We prove that the small components of the disconnected regions are surrounded by four different regions.

We argue by contradiction and we suppose that there exists a small component S_i of a disconnected region E_i such that it is surrounded by three different regions (note that S_i has at least three edges by Corollary 1.35, and recall that $\mathbf{E}$ is a minimizer). Without loss of generality we can assume that it is S_1. Hence, by (5.9) of Corollary 5.10, we have that

$$|\, S_1 \,| \geq 1 - \frac{9\tilde{p}^2}{32\pi} \cdot \left(\sqrt{1 + \frac{64\pi}{9\tilde{p}^2}} - 1 \right) := A_{3,4} \approx 0.0409878. \tag{5.50}$$

We summarize the conditions on area of small components of $\mathbf{E}$; by (5.10) of Corollary 5.10, by Lemma 5.25 and by (5.50) we get that

$$\begin{aligned} A_{3,4} \leq |\, S_1 \,| \leq l < 0.042 \\ A_{2,4} \leq |\, S_2 \,| \leq l < 0.042. \end{aligned} \tag{5.51}$$

We show that the area of S_2 is smaller than $A_{2,4}$; therefore by (5.51) the proof of **Step I** is completed.

We call $x = |\, S_2 \,|$, thus $|\, B_2 \,| = |\, E_2 \,| - |\, S_2 \,| \geq 1 - x$. We note that, by (5.51), $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| \geq 1 - 0.042$. By the minimality of $\mathbf{E}$, by the isoperimetric inequality and by (5.51), we get that

$$\begin{aligned}\tilde{p} \geq P(\mathbf{E}) &= \frac{1}{2}\Big(P(E_0) + P(S_2) + P(B_2) + P(S_1) + P(B_1) + \sum_{k=3}^{4} P(E_k)\Big) \\ &\geq \sqrt{\pi}\Big(2 + \sqrt{x} + \sqrt{1-x} + \sqrt{A_{3,4}} + \sqrt{(1-0.042)} + 2\Big).\end{aligned}$$

We find the following inequality

$$\sqrt{x} + \sqrt{1-x} \leq \frac{\tilde{p} - \sqrt{\pi}\left(4 + \sqrt{A_{3,4}} + \sqrt{(1-0.042)}\right)}{\sqrt{\pi}}. \tag{5.52}$$

We set $C_4 = \frac{\tilde{p} - \sqrt{\pi}\left(4 + \sqrt{A_{3,4}} + \sqrt{(1-0.042)}\right)}{\sqrt{\pi}}$ and $D = 1$, then we can see that

$$1 < C_4 \approx 1.13467 < \sqrt{2}.$$

So by Lemma 3.1, the solution of (5.52) is

$$0 < x \leq \frac{D - \sqrt{C_4^2(2D - C_4^2)}}{2} \text{ or } \frac{D + \sqrt{C_4^2(2D - C_4^2)}}{2} \leq x < D, \tag{5.53}$$

where

$$\begin{aligned}\frac{D - \sqrt{C_4^2(2D - C_4^2)}}{2} &\approx 0.0211071 \\ \frac{D + \sqrt{C_4^2(2D - C_4^2)}}{2} &\approx 0.978893.\end{aligned}$$

But from (5.51), $0.0238853 \approx A_{2,4} \leq x \leq A_{1,4} \approx 0.159132$. This contradicts the solution (5.53) of (5.52).

Step II. We prove that any three-sided component is a big component.

Let C be a three-sided component of $\mathbf{E}$; if $C = E_3$ or $C = E_4$, then $|\,C\,| = 1$, therefore, by Definition 1.18, C is a big component. While if C is a three-sided component of the disconnected regions E_1 or E_2, then, since $\mathbf{E}$ is a minimizer, C is surrounded by three different regions. From **Step I**, we get that C is a big component.

Step III. We prove that the small component of the disconnected regions are external. We suppose by contradiction that there exists a inner small component S_i of a disconnected region E_i ($i = 1, 2$). By Lemma 5.24 and **Step II** S_i has four edges. Moreover, since $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ with $I_{\mathbf{E}} = (1,1,0,0)$ (i.e S_i is disjoint to the big component B_i of the disconnected region E_i and there are four bounded regions E_i, $i = 1,2,3,4$) and S_i is inner (i.e S_i is disjoint from E_0), then S_i is surrounded by only three different regions. This contradicts **Step I**.

This completes the proof.

□

Corollary 5.27. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,1,0,0)$, then there is at most one inner component and it is big (eventually also a connected region). In particular $5 \le v(E_0) \le 6$ and $5 \le e(E_0) \le 6$, where $v(E_0)$ and $e(E_0)$ denote the number of the vertices which belong to E_0 and the number of the edges of E_0 respectively.*

Proof. By Lemma 5.26, we know that the small components of the disconnected regions E_i ($i = 1, 2$) are external, thus, only the big components can be inner. By Corollary 5.9 there is at most one big inner component, so there are at least five external bounded components. Since $I_{\mathbf{E}} = (1,1,0,0)$, $\mathbf{E}$ has six bounded components, therefore, by Proposition 1.33 the proof is completed. □

Lemma 5.28. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,1,0,0)$, then the following estimates for pressure p_C of each small component C are valid:*

1) $p_C \ge \dfrac{2\pi}{3\left(2\tilde{p} - 2\sqrt{\pi}\left(4 + 2\sqrt{1-0.042} + \sqrt{A_{2,4}}\right)\right)} := k_2 \approx 2.89895$, *if C has four edges;*

2) $p_C \ge \dfrac{\pi}{3\left(2\tilde{p} - 2\sqrt{\pi}\left(4 + 2\sqrt{1-0.042} + \sqrt{A_{2,4}}\right)\right)} = \frac{k_2}{2}$, *if C has five edges.*

Proof. From Lemma 5.24 and Lemma 5.26 any small component C can have at least four edges and at most five edges. Furthermore by (5.10) of Corollary 5.10 and from Lemma 5.25 we know that

$$A_{2,4} \le |\,C\,| \le 0.042. \tag{5.54}$$

Since C is a small component and $I_{\mathbf{E}} = (1, 1, 0, 0)$, then C is either a component of E_1 or E_2. Without loss of generality we can assume that C is the small component of E_1, otherwise we repeat the same argument if C is the small component of E_2. Since $I_{\mathbf{E}} = (1, 1, 0, 0)$, then $E_1 = B_1 \sqcup S_1$ and $E_2 = B_2 \sqcup S_2$, where B_i and S_i are respectively the big and the small component of E_i, $i = 1, 2$. Therefore $C = S_1$. We note that, by (5.54),

$$| B_i | = | E_i | - | S_i | \geq 1 - 0.042. \tag{5.55}$$

We start to prove 1); in this case C has four edges therefore its turning angle is $\frac{2\pi}{3}$ (see Lemma 1.38). Thus the highest turning angle of edges of C is positive, namely the pressure p_C is bigger than the pressure of at least one of the components adjacent to C (note that the signed curvature of an edge between C and any other component R is $p_C - p_R$). Thus, denoted A, B, D and F the components adjacent to C (A, B, D and F could be of the same region, for example if C is inner, since C has four edges and $\mathbf{E}$ is a minimum, then there must be two components of E_2) and L_A, L_B, L_D and L_F the lengths corresponding sides in common with C we have (recall that each pressure is non negative by Proposition 1.49):

$$p_C \cdot \mathrm{P}(C) = \sum_{R \in \{A,B,D,F\}} L_R \cdot p_C \geq \sum_{R \in \{A,B,D,F\}} L_R \cdot (p_C - p_R) = \frac{2\pi}{3}.$$

then

$$p_C \geq \frac{2\pi}{3} \cdot \frac{1}{P(C)}. \tag{5.56}$$

Moreover, by minimality of $\mathbf{E}$, we obtain that

$$\tilde{p} \geq P(\mathbf{E}) \geq \frac{\Big(P(C) + P(B_1) + P(B_2) + P(S_2) + \sum_{k=3}^{4} P(E_k) + P(E_0)\Big)}{2}.$$

By (5.54), (5.55) and the isoperimetric inequality, we get the following estimate for $P(C)$

$$P(C) \leq 2\tilde{p} - \Big(P(B_1) + P(B_2) + P(S_2) + \sum_{k=3}^{4} P(E_k) + P(E_0)\Big)$$

$$\leq 2\tilde{p} - 2\sqrt{\pi}\Big(2\sqrt{1-0.042} + \sqrt{A_{2,4}} + 2 + 2\Big)$$

hence, considering (5.56), we find 1).

The proof is the same for 2), indeed in this case the only change is the turning angle of C, which is $\frac{\pi}{3}$, because C has five edges (see also Lemma 1.38).

□

Corollary 5.29. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,1,0,0)$, then both the small components of E_1 and E_2 have not four edges.*

Proof. Let S_1 and S_2 be the small component of E_1 and E_2 respectively. We suppose by contradiction that S_1 and S_2 have four edges, therefore, from Lemma 5.28, we have that p_{E_1} and p_{E_2} are least k_2. Thus, since the pressure of each other region of $\mathbf{E}$ is non negative (see Proposition 1.49), by Corollary 1.47, we have that $P(\mathbf{E}) \geq 2\sum_{i=1}^{4} p_{E_i} \geq 4k_2 \approx 11.5958$. It leads to a contradiction since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$. This concludes the proof. □

Remark 5.30. Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ with $I_{\mathbf{E}} = (1,1,0,0)$ and let S_i, B_i be the small and the big component of the disconnected region E_i ($i = 1,2$) respectively. By Corollary 5.29, hereafter, we can assume that S_2 has five edges.

Lemma 5.31. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,1,0,0)$, then the big component of the disconnected region E_2 is external and it has at most four edges.*

Proof. Since $I_{\mathbf{E}} = (1,1,0,0)$, then $E_2 = B_2 \bigsqcup S_2$, where B_2 and S_2 are the big and the small component of E_2 respectively. We denote by $v(C)$ the number of the vertices of a subset C of $\mathbf{E}$ and $v(\mathbf{E})$ represents the number of the vertices of $\mathbf{E}$. By Lemma 5.24, B_2 has at least three edges, so $v(B_2) \geq 3$. By Remark 5.30, S_2 has five edges and it is external, thus, $v(S_2) = 5$ and $v(S_2 \setminus E_0) = 3$. If B_2 is inner (i.e B_2 is disjoint from E_0, so its vertices are not on E_0), then, from the previous considerations, by Lemma 5.18 and by Corollary 5.27, we get that

$$10 = v(\mathbf{E}) \geq v(E_0) + v(B_2) + v(S_2 \setminus E_0) \geq 5 + 3 + 3 = 11.$$

This is a contradiction, so B_2 is external.

If B_2 has five edges, then $v(B_2) = 5$ and by Lemma 5.24, B_2 is external, thus $v(B_2 \smallsetminus E_0) = 3$. By Lemma 5.18 and by Corollary 5.27, we also get that

$$10 = v(\mathbf{E}) \geq v(E_0) + v(S_2 \smallsetminus E_0) + v(B_2 \smallsetminus E_0) \geq 5 + 3 + 3 = 11.$$

It is a contradiction, so the proof is concluded.

□

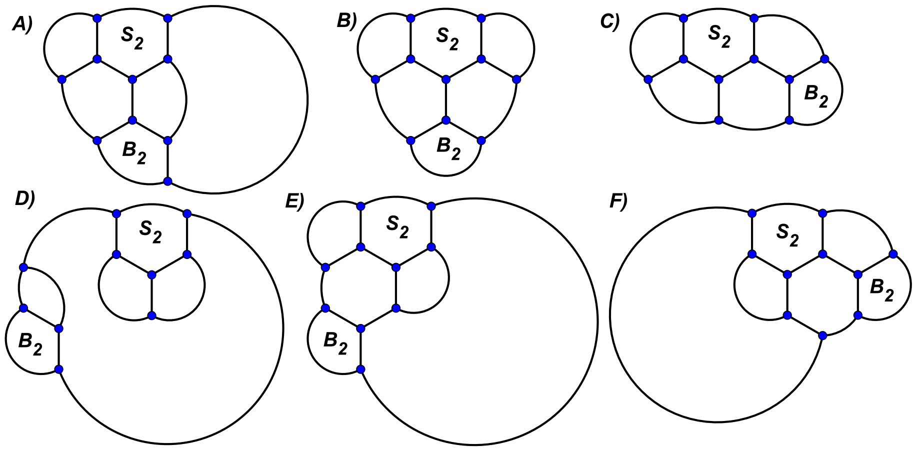

Figure 5.4: *The possible topologies of* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ *with* $I_{\mathbf{E}} = (1,1,0,0)$.

Lemma 5.32. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*. If* $I_{\mathbf{E}} = (1,1,0,0)$*, then the possible topologies of* $\mathbf{E}$ *are represented in Figure 5.4.*

Proof. By Remark 5.30 we have assumed that S_2 has five edges, while, from Lemma 5.24 and Lemma 5.31, we know that B_2 is external with three or four edges. First of all we explicitly note that the vertices of B_2 and S_2 are all distinct because $\mathbf{E}$ is a minimizer. Moreover vertices of a same component of $\mathbf{E}$ are not connected since there are no two-sided components by Corollary 1.35, thus all leaving edges from vertices of a same component of $\mathbf{E}$ are all different. Finally we recall that each vertex of $\mathbf{E}$ must be a meeting point of exactly three edges ($\mathbf{E}$ is a minimizer and see Theorem 1.10), thus we underline that, at the beginning of the creation of the topologies, each external vertex of S_2 and B_2 is already a meeting point of exactly three edges, while a inner vertex of S_2 and B_2 can be get another edge. We use the following notations:

1) $v(\mathbf{E})$ is the number of the vertices of $\mathbf{E}$;
2) $v(C)$ represents the number of the vertices of a component C of $\mathbf{E}$;
3) $v(\mathbf{E} \setminus C)$ denotes the number of the vertices which belong to $\mathbf{E}$ but not to the component C of $\mathbf{E}$;
4) $v(C_1 \setminus C_2)$ denotes the number of the vertices which belong to the component C_1 but not to the component C_2, where C_1 and C_2 are components of $\mathbf{E}$;
5) $v(C_1 \cap C_2)$ denotes the number of the vertices which belong to the components C_1 and C_2 of $\mathbf{E}$;
6) $e(\mathbf{E})$ is the number of the edges of $\mathbf{E}$;
7) $e(E_0)$ represents the number of the edges of E_0;
8) $e_i(C)$ denotes the number of the inner edges of a component C of $\mathbf{E}$;
9) $e_{l,i}(C)$ denotes the number of the leaving edges from a inner vertex of a component C of $\mathbf{E}$;
10) $e_{l,i}(v)$ is the number of the leaving inner edges from a vertex v of $\mathbf{E}$;
11) $e_{l,i}(v \cap C)$ is the number of the leaving inner edges from a vertex v which arrive in a vertex of a component C of $\mathbf{E}$.

From the previous notation, it immediately follows that $e_{l,i}(v) = 1$ if the vertex v is external and $e_{l,i}(v) = 3$ if the vertex v is inner. Furthermore we also have that $0 \leq e_{l,i}(v \cap C) \leq 3$. We recall that $v(\mathbf{E}) = 10$ and $e(\mathbf{E}) = 15$ by Lemma 5.18, while $v(E_0) \geq 5$ and $e(E_0) \geq 5$ by Corollary 5.27. Now we divided the proof in two parts depending on B_2 has three or four edges.

Part I. B_2 has four edges, thus $v(B_2) = 4$, $v(B_2 \setminus E_0) = 2$, $e_i(B_2) = 3$, while $v(S_2) = 5$ with $v(S_2 \setminus E_0) = 3$, $e_i(S_2) = 4$ and $e_{l,i}(S_2) = 3$. Since $v(\mathbf{E}) = 10$ and $v(S_2) + v(B_2) = 9$, we must add another vertex v_1. First of all we say that v_1 has to be external (i.e v_1 is on E_0). Indeed if v_1 is inner, we find that

$$\begin{aligned}10 = v(\mathbf{E}) &\geq v(E_0) + v(S_2 \setminus E_0) + v(B_2 \setminus E_0) + v(\mathbf{E} \setminus (S_2 \cup B_2)) \\ &\geq 5 + 3 + 2 + 1 = 11.\end{aligned}$$

This is a contradiction, thus v_1 is on E_0.

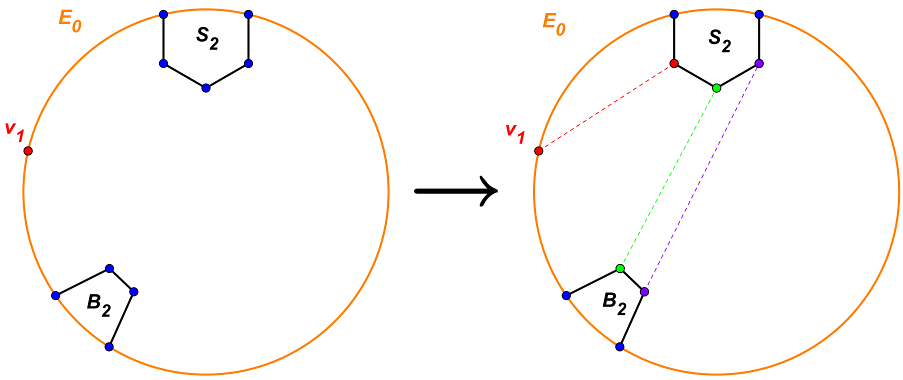

Figure 5.5: *S_2 and B_2 have five and four edges respectively and both are external. Since $v(\mathbf{E}) = 10$, there is another vertex v_1, which must be external and it has to be connected to only one inner vertex of S_2. Since the edges of $\mathbf{E}$ can not be intersect, there is only one way to link v_1 and the inner vertices of S_2 and B_2.*

Furthermore v_1 is linked to only one inner vertex of S_2, indeed if it was false then (v_1 is external, thus $e_{l,i}(v_1) = 1$ and note that $e(\mathbf{E}) = 15$ and $e(E_0) \geq 5$)

$$\begin{aligned}15 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_2) + e_i(B_2) + e_{l,i}(S_2) + e_{l,i}(v_1) \\ &\geq 5 + 4 + 3 + 3 + 1 = 16.\end{aligned}$$

This is a contradiction, so v_1 must be connected with only one inner vertex of S_2. Hence we are in the situation of Figure 5.5 where, since the edges of $\mathbf{E}$ can not intersect (if two arcs intersect, then a vertex would be created which is a meeting point of four arcs, which contradicts 2. of Theorem 1.10

since $\mathbf{E}$ is a minimizer), we have only one way to link v_1 and the inner vertices of S_2 and B_2. Thus, we obtain the case A) of Figure 5.4.

Part II. B_2 has three edges, so $v(B_2) = 3$, $v(B_2 \setminus E_0) = 1$, $v(B_2 \cap E_0) = 2$, $e_i(B_2) = 2$, while $v(S_2) = 5$ with $v(S_2 \setminus E_0) = 3$, $v(S_2 \cap E_0) = 2$, $e_i(S_2) = 4$ and $e_{l,i}(S_2) = 3$. Since $v(\mathbf{E}) = 10$ and $v(S_2) + v(B_2) = 8$, we must add another two vertices v_1 and v_2. Certainly one between v_1 and v_2 must be on E_0, otherwise it follows that

$$\begin{aligned} 10 = v(\mathbf{E}) &\geq v(E_0) + v(S_2 \setminus E_0) + v(B_2 \setminus E_0) + v(\mathbf{E} \setminus (S_2 \cup B_2)) \\ &\geq 5 + 3 + 1 + 2 = 11. \end{aligned}$$

This is impossible, so, without loss of generality, we can assume that v_1 is always external, therefore $e_{l,i}(v_1) = 1$ and we can have two cases; the first is v_1 and v_2 are external and the second is v_1 is external and v_2 is inner.

Part IIa. We take the case where v_1 and v_2 are external, therefore also $e_{l,i}(v_2) = 1$. Furthermore $e(E_0) = 6$ because

$$v(E_0) = v(S_2 \cap E_0) + v(B_2 \cap E_0) + v(\mathbf{E} \setminus (S_2 \cup B_2)) = 2 + 2 + 2.$$

There are two possibilities, the first is v_1 and v_2 are on opposite arcs respect to S_2 and the second is v_1 and v_2 are on the same arc respect to S_2 as represented in Figure 5.6 and in Figure 5.7 respectively. In both we say that v_1 and v_2 are connected each to only one inner vertex of S_2, indeed if it was false then one vertex between v_1 and v_2 would be not related to any inner vertex of S_2. Without loss of generality we can assume that it is v_1, so it would follow that

$$\begin{aligned} 15 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_2) + e_i(B_2) + e_{l,i}(S_2) + e_{l,i}(v_1) \\ &= 6 + 4 + 2 + 3 + 1 = 16. \end{aligned}$$

This is a contradiction.

Whether v_1 and v_2 are on opposite arcs respect to S_2 or they are on the same arc respect to S_2, there is only one way to link v_1, v_2 and the inner vertices of B_2 and S_2, since the edges of $\mathbf{E}$ can not intersect.

If v_1 and v_2 are external and they are on opposite arcs respect to S_2 we are in the situation of Figure 5.6 where we obtain the case B) of Figure 5.4.

If v_1 and v_2 are external and they are on the same arc respect to S_2 (with with the convention that v_1 is before of v_2 coming from S_2) we are in the situation of Figure 5.7 obtaining the case C) of Figure 5.4.

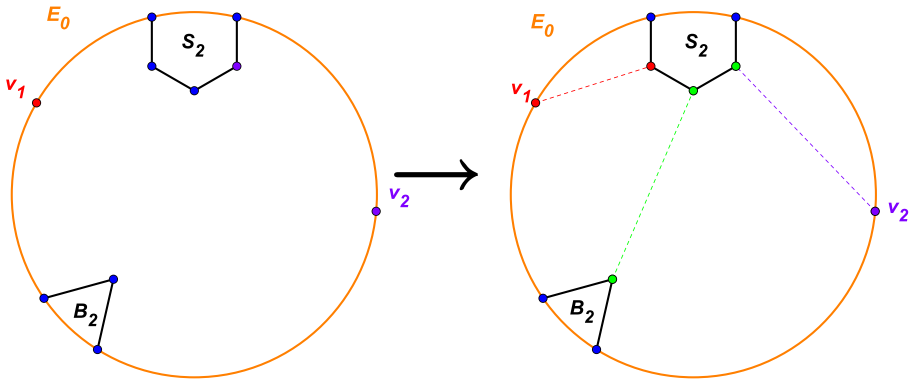

Figure 5.6: *S_2 and B_2 have five and three edges respectively and both are external. Since $v(\mathbf{E}) = 10$, there are another two vertex v_1 and v_2 one of which must be external. Here the vertices v_1 and v_2 are external and they are on opposite arcs respect to S_2. The vertices v_1 and v_2 must be connected each to only one inner vertex of S_2.*

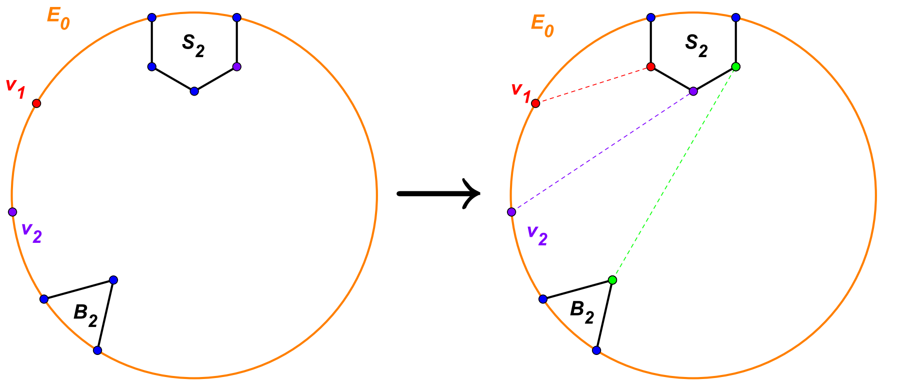

Figure 5.7: *S_2 and B_2 have five and three edges respectively and both are external. Since $v(\mathbf{E}) = 10$, there are another two vertex v_1 and v_2 one of which must be external. Here the vertices v_1 and v_2 are external and they are on the same arc respect to S_2. The vertices v_1 and v_2 must be connected each to only one inner vertex of S_2.*

Part IIb. Finally we consider the case where v_1 is external and v_2 is inner, thus $e_i(v_2) = 3$. We say that v_2 must be linked to at least two inner vertices of S_2 because it is false then we have that $e_{l,i}(v_2 \cap S_2) \leq 1$. Therefore it follows that

$$\begin{aligned}15 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_2) + e_i(B_2) + \big(e_{l,i}(S_2) + e_{l,i}(v_2) - e_{l,i}(v_2 \cap S_2)\big)\\ &\geq 5 + 4 + 2 + (3 + 3 - 1) = 5 + 4 + 2 + 5 = 16.\end{aligned}$$

This is a contradiction, therefore there are two possibilities depending on how many inner vertices of S_2 are connected with v_2, three or two.

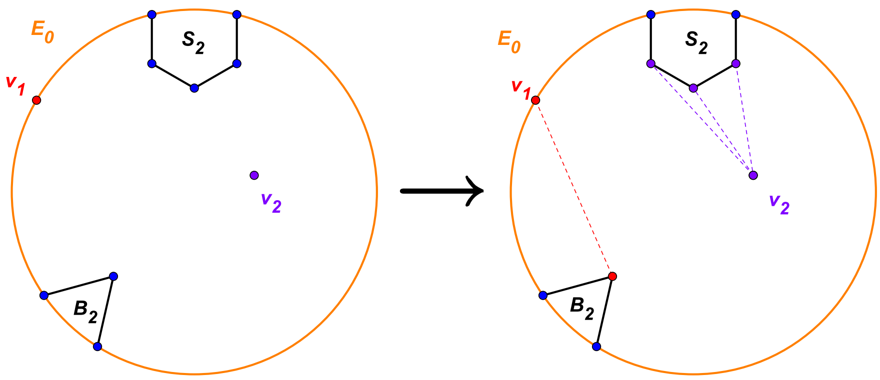

Figure 5.8: *S_2 and B_2 have five and three edges respectively and both are external. Since $v(\mathbf{E}) = 10$, there are another two vertex v_1 and v_2 one of which must be external. Here the vertex v_1 is external while the vertex v_2 is inner connected to all inner vertices of S_2. Since the edges of $\mathbf{E}$ can not be intersect, there is only one way to link the vertices v_1, v_2 and the inner vertices of S_2 and B_2.*

If v_2 is related to all inner vertices of S_2, then v_1 is connected to the inner vertex of B_2. It is shown in Figure 5.8 from which we obtain the case $D)$ of Figure 5.4.

While if v_2 is connect to only two inner vertices of S_2, since the edges of $\mathbf{E}$ can not be intersect, there are two ways to link v_1, v_2 and the inner vertices of B_2 and S_2. They are represented in Figure 5.9 from which we obtain the case $E)$ and $F)$ of Figure 5.4.

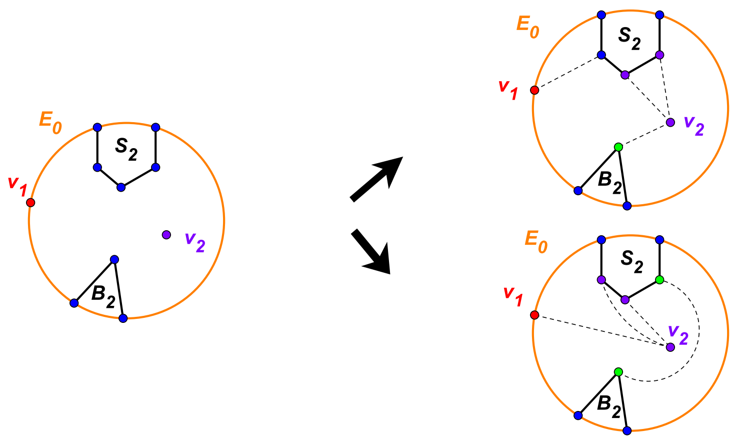

Figure 5.9: *S_2 and B_2 have five and three edges respectively and both are external. Since $v(\mathbf{E}) = 10$, there are another two vertex v_1 and v_2 one of which must be external. Here the vertex v_1 is external while the vertex v_2 is inner connected to two inner vertices of S_2. Since the edges of $\mathbf{E}$ can not be intersect, there are two ways to link the vertices v_1, v_2 and the inner vertices of S_2 and B_2.*

□

Lemma 5.33. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ with $I_{\mathbf{E}} = (1,1,0,0)$, then the big components of disconnected regions are adjacent (i.e the big components of the disconnected regions have a common edge).*

Proof. Let B_i and S_i the big and the small component of the disconnected region E_i $(i = 1, 2)$ respectively. By Lemma 5.25 we know that $|\,S_i\,| \leq 0.042$ for any $i = 1, 2$, therefore it holds that

$$|\,B_i\,| = |\,E_i\,| - |\,S_i\,| \geq 1 - 0.042, \qquad \forall i = 1, 2. \tag{5.57}$$

We suppose by contradiction that B_1 and B_2 are disjoint. By Corollary 5.9 we know that there is at most one big inner component, while, by Lemma 5.31, B_2 is external. So we have two possibilities depending on B_1 is external or inner. We prove that the two situations are impossible.

Case I. B_2 is external and B_1 is external disjoint from B_2. Let L_e^1 and L_e^2 be the lengths of the external edges of B_1 and B_2 respectively, then, by the minimality of $\mathbf{E}$, we have that

$$\tilde{p} \geq P(\mathbf{E}) \geq P(B_1) + P(B_2) + P(E_0) - (L_e^1 + L_e^1).$$

So by the isoperimetric inequality and by (5.57), we obtain the following estimate for the sum of the lengths of the external edges of B_1 and B_2

$$L_e^1 + L_e^2 \geq 2\sqrt{\pi}\left(2\sqrt{1-0.042}+2\right) - \tilde{p}: \;= \ell_1, \tag{5.58}$$

therefore there exists an index $i = 1, 2$ such that

$$L_e^i \geq \frac{\ell_1}{2}. \tag{5.59}$$

A small component S_i can have four edges or five edges by Lemma 5.24 and Lemma 5.26. Moreover, again by Lemma 5.26, S_i is always surrounded by four different regions. Therefore, applying Lemma 3.12 to the region E_i ($i = 1, 2$) removing S_i for B_i, we get this estimate for the its pressure

$$p_{E_i} \geq \frac{1}{2}\sqrt{\frac{\pi}{|\,S_i\,|}} - \frac{2}{L_e^i} \geq \frac{1}{2}\sqrt{\frac{1000\pi}{42}} - \frac{4}{\ell_1} := k_3 \approx 2.91316 \tag{5.60}$$

The region E_i is the highest pressure region, in fact if there was other region E_j ($j \neq i$) with $p_{E_j} \geq p_{E_i}$,then, by Corollary 1.47 and Proposition 1.49, the perimeter of E would be $P(\mathbf{E}) \geq 2\sum_{k=1}^{4} p_{E_k} \geq 4k_3 \approx 11.6526$. It is a contradiction because $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$. Now we take B_i; by Corollary 1.35 and Lemma 5.24, B_i has at least three edges and at most five edges, so its turning angle is at most π (see Lemma 1.38). Furthermore, since E_i is the higest pressure region, the turning angle of all edges of B_i is non negative, thus, by Lemma 1.38, we get that $p_{E_i} \cdot \frac{\ell_1}{2} \leq p_{E_i} \cdot L_e^i \leq \pi$, finding, by (5.60), that

$$2.91316 \approx k_3 \leq p_{E_i} \leq \frac{2\pi}{\ell_1} \approx 2.21668.$$

This is a contradiction.

Case II. B_2 is external and B_1 is inner disjoint from B_2. Also this case is impossible and the proof is the same of the **Case I**, where the estimate (5.58) will be the estimate for the length of the external edge L_e^2 of B_2 (note that in this case only B_2 is external). The considerations, done in the **Case I** for S_i are true for S_2 with the same argument, namely S_2 will be surrounded by four different regions (see Lemma 5.26). Therefore, applying Lemma 3.12 to the region E_2 removing S_2 for B_2, we get this estimate for the its pressure

$$p_{E_2} \geq \frac{1}{2}\sqrt{\frac{\pi}{|\,S_2\,|}} - \frac{2}{L_e^1} \geq \frac{1}{2}\sqrt{\frac{1000\pi}{42}} - \frac{2}{\ell_1} = k_3 + \frac{2}{\ell_1} \approx 3.61875. \tag{5.61}$$

Since $k_3 + \frac{2}{\ell_1}$ is greater than k_3 by (5.60) and by (5.61), it is clear that E_2 is the highest pressure region (you can reason as in the **Case I**). Again as in the **Case I** the property for B_i are true for B_2 with the same argument, namely the turning angle of B_1 is at most π. Therefore, the contradiction is the following

$$3.61875 \approx k_3 + \frac{2}{\ell_1} \leq p_{E_2} \leq \frac{\pi}{\ell_1} \approx 1.10834.$$

This completes the proof. $\square$

Lemma 5.34. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ *with* $I_{\mathbf{E}} = (1,1,0,0,)$. *If a connected region is inner, then it is adjacent to each big component of a disconnected region.*

Proof. Let E_3 be the inner connected region. We argue by contradiction and we suppose that there exists a big component B_i ($i = 1,2$) of a disconnected region disjoint to E_3. By Corollary 5.9, B_i is external. This situation is impossible and the proof is the same to the **Case II** of Lemma 5.33, replacing B_i with B_2 and E_3 with B_1 (note that $|\,E_3\,| \geq 1$). $\square$

Lemma 5.35. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,1,0,0)$ *then the possible topologies are only the cases* A) *and* C) *of Figure 5.4. Moreover* $\mathbf{E}$ *can be the three clusters of Figure 5.10 (up to the curvature of the edges of* $\mathbf{E}$*), where the unlabeled components are the connected regions.*

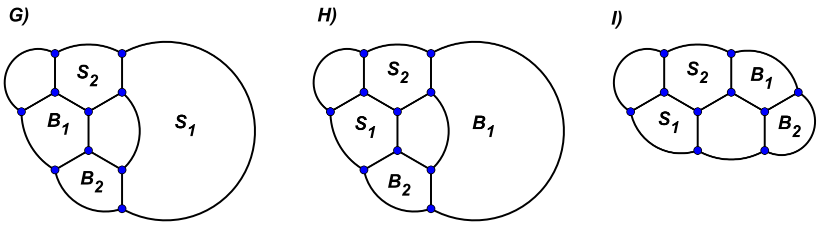

Figure 5.10: **E** *can be these three possible clusters.*

Proof. We divide the proof in two parts.

Part I. By Lemma 5.32 we know that **E** can have the topologies represented in Figure 5.4, thus we must exclude the cases $B)$, $D)$ $E)$ and $F)$ of Figure 5.4.

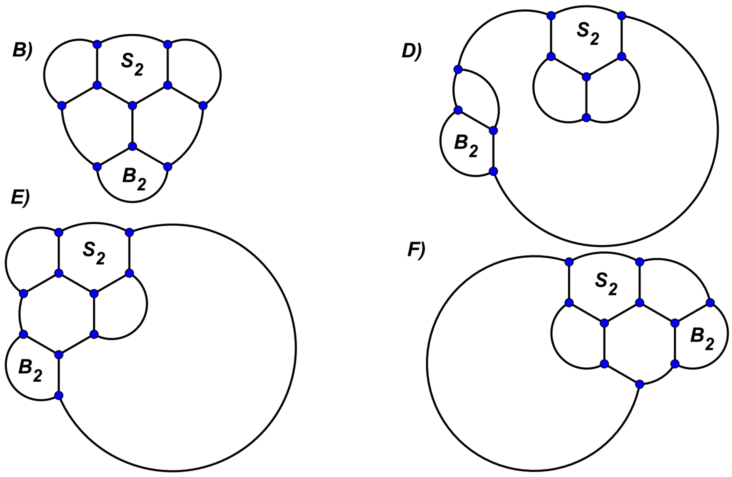

Figure 5.11: *The cases* $B)$, $D)$, $E)$ *and* $F)$ *of Figure 5.4.*

In the possibilities $B)$, $D)$ and $E)$ there are certainly two unlabeled three-sided component disjoint from B_2; by Lemma 5.26 and by Lemma 5.33 they are the connected regions E_3 and E_4. Thus, we have that, in these configurations, B_1 and S_1 are adjacent, but it is impossible since **E** is a minimum.

While in the case $F)$ there is an unlabeled three-sided component disjoint from B_2 which must be a connected region by Lemma 5.26 and by Lemma 5.33. But this is impossible by Lemma 5.34.

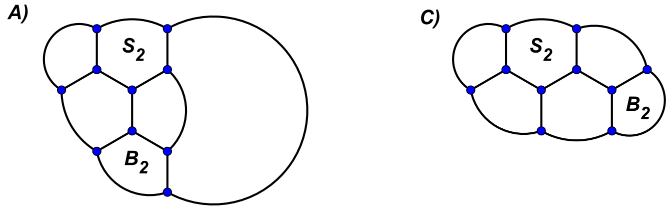

Figure 5.12: *The topologies* $A)$ *and* $C)$ *of Figure 5.4. .*

Part II. We prove that $\mathbf{E}$ can be the three clusters represented in Figure 5.10. The topologies A) and C) are recalled in Figure 5.12. At the beginning we consider the topology A) of Figure 5.4. The unlabeled three-sided component must be a connected region by Lemma 5.26 and by Lemma 5.33. So we can find the possibilities G) and H), which differ only for the change between S_1 and B_1. This change is significant because in G), S_1 has four edges, while in the case H), S_1 has five edges. Now we take the topology C) of Figure 5.4. Again from Lemma 5.26 and by Lemma 5.33 the unlabeled three-sided component is a connected region. So we determine the case I). □

Lemma 5.36. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,1,0,0)$, *then the following considerations apply:*

i) if S_i has four edges, then E_i is the highest pressure region;

ii) if S_i has five edges, then E_i is not the lowest pressure region,

where $i = 1, 2$ and S_i is the small component of E_i.

Proof. We show i); since S_i has four edges, then the pressure of E_i is at least k_2 by Lemma 5.28. So if E_i was not the highest pressure region, there would be at least one other region with pressure at least k_2. Thus the perimeter of $\mathbf{E}$ would be at least $4k_2 \approx 11.5958$ by Corollary 1.47 and by Proposition 1.49. This is a contradiction since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$.

We prove ii); here S_i has five edges, therefore the pressure of E_i is at least $\frac{k_2}{2}$ by Lemma 5.28. So if E_i was the lowest pressure region, then each pressure of any region would be at least $\frac{k_2}{2}$. By Corollary 1.47, we would have that the perimeter of $\mathbf{E}$ would be again at least $4k_2 \approx 11.5958$, but this is a contradiction since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$.

□

Lemma 5.37. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,1,0,0)$ *where the small components S_1 and S_2 have four edges and five edges respectively, then the pressure of the connected regions is less than $\frac{k_2}{2}$.*

Proof. From Remark 5.30 we have assumed that S_2 has five edges. We proceed by contradiction and we suppose that there is at least one connected region with its pressure bigger or equal to $\frac{k_2}{2}$, the lower limit for the pressure of the disconnected region E_2 (note that its small component S_2 has five edges). Without loss of generality let E_3 which has a pressure that is bigger or equal to $\frac{k_2}{2}$. Moreover, by Lemma 5.28, we know that $p_{E_1} \geq k_2$ and $p_{E_2} \geq \frac{k_2}{2}$. Thus, by Corollary 1.47, we have the following estimate for the perimeter of $\mathbf{E}$ (note the each pressure is non negative by Proposition 1.49),

$$P(\mathbf{E}) \geq 2\sum_{i=1}^{4} p_{E_i} \geq 2k_2 + 4\left(\frac{k_2}{2}\right) \approx 11.5958.$$

This contradicts the minimality of $\mathbf{E}$, which gives that $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$. $\square$

Proposition 5.38. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,1,0,0)$, *then* $\mathbf{E}$ *can be the clusters* H) *and* I) *of Figure 5.10.*

Proof. By Lemma 5.35, we know that $\mathbf{E}$ can be the three clusters of Figure 5.10, therefore, in order to prove the statement of the lemma, we just exclude the cluster G), which is recalled in Figure 5.13.

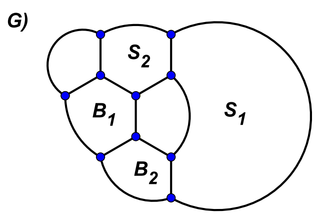

Figure 5.13: *The cluster* G) *of Figure 5.10.* .

In this case S_1 and S_2 have four and five edges respectively. Thus, by Lemma 5.37, E_1 is the highest pressure region and E_2 is the second region with higher pressure. Furthermore the inner four-sided connected region

E_i $(i = 3, 4)$ is surrounded only by components of E_1 and E_2, thus the signed curvature of all its edges are non positive. This contradicts that the turning angle of E_i is $\frac{2\pi}{3}$ (note that E_i has four edges and see Lemma 1.38). $\square$

Theorem 5.39. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1, 1, 1, 1)$*, then* $I_{\mathbf{E}} \neq (1, 1, 0, 0)$.

Proof. We suppose by contradiction that $I_{\mathbf{E}} = (1, 1, 0, 0)$, then, by Proposition 5.38, $\mathbf{E}$ can be the clusters $H)$ and $I)$ of Figure 5.10. We prove that these two possibilities are impossible.

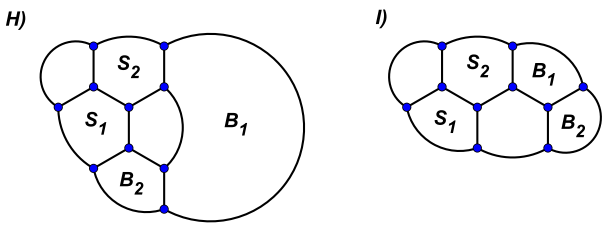

Figure 5.14: *The cases $H)$ and $I)$ of Figure 5.10.*

Part I. First of all, we exclude the case $H)$. Let S_i and B_i be the small and the big component respectively of the disconnected region E_i $(i = 1, 2)$. Here S_1 and S_2 have both five edges and there is a inner connected region surrounded only by components of the disconnected regions E_1 and E_2. Without loss of generality we can assume that the inner connected region is E_3 and the remaining unlabeled three-sided component is the connected region E_4. By Corollary 5.16, $\mathbf{E} \in \mathcal{M}_{2,4}(1, 1, 1, 1)$, thus $m(\mathbf{E}) = (1, 1, 1, 1)$. From Lemma 5.25 we know that $|\,S_i\,| \leq 0.042$, therefore, it follows that $|\,B_i\,| = |\,E_i\,| - |\,S_i\,| = 1 - 0.042$. Let L^1_e and L^2_e be the lengths of the external edges of B_1 and B_2 respectively, by the minimality of $\mathbf{E}$, we have that

$$\tilde{p} \geq P(\mathbf{E}) \geq P(B_1 \cup B_2 \cup E_3) + P(E_0) - (L^1_e + L^2_e).$$

So by the isoperimetric inequality, we obtain the following estimate for the sum of the lengths of external edges of B_1 and B_2

$$L_e^1 + L_e^2 \geq 2\sqrt{\pi}\left(\sqrt{2(1-0.042)+1}+2\right) - \tilde{p}\colon\ = \ell_2 \approx 1.94856.$$

Hence there exists an index $i = 1, 2$ such that

$$L_e^i \geq \frac{\ell_2}{2}.$$

Thus, applying Lemma 3.12 to the region E_i removing S_i for B_i (note that S_i is surrounded by four different regions by Lemma 5.26), we get this estimate for its pressure

$$p_{E_i} \geq \frac{1}{2}\sqrt{\frac{\pi}{|\,S_i\,|}} - \frac{2}{L_e^i} \geq \frac{1}{2}\sqrt{\frac{1000\pi}{42}} - \frac{4}{\ell_2} := k_4 \approx 2.27155.$$

Thus we have that $\max(p_{E_1}, p_{E_2}) \geq k_4$, while, by Lemma 5.28, we get that $\min(p_{E_1}, p_{E_2}) \geq \frac{k_2}{2}$ (notice that S_i has five edges independently from $i = 1$ or $i = 2$). From Lemma 1.38, E_3 has turning angle $\frac{2\pi}{3}$, so the highest turning angle of edges of E_3 is positive, namely the pressure p_{E_3} is bigger than the pressure of at least one of the components adjacent to E_3 (note that the signed curvature of an edge between E_3 and any other component C is $p_{E_3} - p_C$). Thus, by E_3 is inner and it is surrounded by only components of E_1 and E_2, it follows that $p_{E_3} \geq \min(p_{E_1}, p_{E_2}) \geq \frac{k_2}{2}$. Furthermore we claim that $p_{E_4} < \min(p_{E_1}, p_{E_2})$, because if $p_{E_4} \geq \min(p_{E_1}, p_{E_2})$, then, by Corollary 1.47, the perimeter of $\mathbf{E}$ would be at least

$$P(\mathbf{E}) = 2p_{E_1} + 2p_{E_2} + 2p_{E_3} + 2p_{E_4} \geq 2k_4 + 6\left(\frac{k_2}{2}\right) \approx 13.2400;$$

this contradicts the minimality of $\mathbf{E}$, in fact $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$. Thus, denote by L_4, $L_{4,1}$ and $L_{4,2}$ the edges of E_4 in common with E_0, S_1 and S_2 respectively, by Lemma 1.38, we have that (note that the turning angle of E_4 is π)

$$L_4\, p_{E_4} \geq L_4\, p_{E_4} + L_{4,1}\,(p_{E_4} - p_{E_1}) + L_{4,2}\,(p_{E_4} - p_{E_2}) = \pi.$$

Hence we obtain, by Lemma 3.14, the following estimate for the pressure p_{E_4} (note that $|\,E_4\,| = 1$):

$$p_{E_4} \geq \frac{\pi}{L_4} \geq \frac{\sqrt{\pi}}{2}.$$

Then, by Corollary 1.47, we can estimate the perimeter of $\mathbf{E}$, obtaining

$$P(\mathbf{E}) = 2p_{E_1} + 2p_{E_2} + 2p_{E_3} + 2p_{E_4} \geq 2k_4 + 4\left(\frac{k_2}{2}\right) + \sqrt{\pi} \approx 12.1135.$$

But for the minimality of $\mathbf{E}$, $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$, thus we get a contradiction.

Part II. Finally we eliminate the case $I)$. We note again that, by Corollary 5.16, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, thus $m(\mathbf{E}) = (1,1,1,1)$. In this configuration S_1 has four edges and S_2 has five edges, thus, by Lemma 5.36 and Lemma 5.37 we know that E_1 is the highest pressure region and E_2 is the second region of higher pressure with $p_{E_1} \geq k_2$ and $p_{E_2} \geq \frac{k_2}{2}$. We can assume that the unlabeled region with three edges is E_3 and the other is E_4; from Lemma 1.38, the turning angle of E_3 and E_4 are π and $\frac{\pi}{3}$ respectively. We set L_3, $l_{3,1}$ and $l_{3,2}$ the lengths of the edges of E_3 in common respectively with E_0, S_1 and S_2, thus we have that

$$L_3\, p_{E_3} \geq L_3\, p_{E_3} + l_{3,1}\,(p_{E_3} - p_{E_1}) + l_{3,2}\,(p_{E_3} - p_{E_2}) = \pi.$$

Then, by Lemma 3.14, we obtain the following estimate for p_{E_3} (recall that $|\, E_3 \,| = 1$)

$$p_{E_3} \geq \frac{\sqrt{\pi}}{2} := p_3'. \tag{5.62}$$

We repeat the same steps for E_4 (note that the turning angle of E_4 is $\frac{\pi}{3}$ and $|\, E_4 \,| = 1$) and so we have

$$p_{E_4} \geq \frac{\sqrt{\pi}}{6}.$$

Thus we get that

$$\min\,(p_{E_3}, p_{E_4}) \geq \frac{\sqrt{\pi}}{6} \approx 0.295409 := p'_{\text{min}}. \tag{5.63}$$

Now we find a new estimate for p_{E_1} and p_{E_2}, using (5.63) and Lemma 5.10. We show in detail only the case for p_{E_1}; the case for p_{E_2} is the same except that S_2 has five edges and so its turning angle is $\frac{\pi}{3}$. From (5.10) of Corollary 5.10 and Lemma 5.25 we know that $A_{2,4} \leq |\, S_i \,| \leq 0.042$, where

S_i is the small component of E_i with $i = 1, 2$. Therefore, for any big component B_i of a disconnecter region it follows that $|\,B_i\,| = |\,E_i\,| - |\,S_i\,| = 1 - 0.042$ ($i = 1, 2$). Furthermore, by (5.8) of Corollary 5.10 we know that the length of each edge of S_i is less than

$$\frac{42}{2000(1-0.042)}\,\tilde{p} := \ell_3. \tag{5.64}$$

We denote by l_1, $l_{1,4}$, $l_{1,2}$ and $l_{1,3}$ the lengths of the the edges of S_1 in common respectively with E_0, E_4, E_2 and E_3, thus we know that

$$l_1\, p_{E_1} + l_{1,4}\,(p_{E_1} - p_{E_4}) + l_{1,2}\,(p_{E_1} - p_{E_2}) + l_{1,3}\,(p_{E_1} - p_{E_3}) = \frac{2\pi}{3};$$

so we find that

$$\begin{aligned} p_{E_1}\, P(S_1) &= \frac{2\pi}{3} + l_{1,4}\, p_{E_4} + l_{1,2}\, p_{E_2} + l_{1,3}\, p_{E_3} \\ &\geq \frac{2\pi}{3} + (l_{1,4} + l_{1,2} + l_{1,3}) \min{(p_{E_4}, p_{E_2}, p_{E_3})} \\ &= \frac{2\pi}{3} + (P(S_1) - l_1)\frac{\sqrt{\pi}}{6}. \end{aligned}$$

Hence we obtain that

$$p_{E_1} \geq \frac{2\pi}{3}\,\frac{1}{P(S_1)} + \left(1 - \frac{l_1}{P(S_1)}\right)\frac{\sqrt{\pi}}{6}. \tag{5.65}$$

Now, since $|\,S_1\,| \geq A_{2,4}$ and by the isoperimetric inequality, it applies that $P(S_1) \geq 2\sqrt{\pi\, A_{2,4}}$. Furthermore, by the minimality of $\mathbf{E}$ and by the isoperimetric inequality we have that

$$\begin{aligned} P(S_1) &\leq 2P(\mathbf{E}) - \Big(P(B_1) + P(B_2) + P(S_2) + P(E_3) + P(E_4) + P(E_0)\Big) \\ &\leq 2\tilde{p} - 2\sqrt{\pi}\Big(2\sqrt{1-0.042} + \sqrt{A_{2,4}} + 1 + 1 + 2\Big) := \ell_4. \end{aligned}$$

So, by (5.64) and by (5.65), we obtain the following estimate for p_{E_1}:

$$p_{E_1} \geq \frac{2\pi}{3}\,\frac{1}{\ell_4} + \left(1 - \frac{\ell_3}{2\sqrt{\pi\, A_{2,4}}}\right)\frac{\sqrt{\pi}}{6} := k_5 \approx 3.06204. \tag{5.66}$$

While, repeating the same argument for S_2, we get that

$$p_{E_2} \geq \frac{\pi}{3}\,\frac{1}{\ell_4} + \left(1 - \frac{\ell_3}{2\sqrt{\pi\, A_{2,4}}}\right)\frac{\sqrt{\pi}}{6} := k_6 \approx 1.61257. \tag{5.67}$$

Then, by Corollary 1.47, by (5.62) , (5.63), (5.66) and (5.67) we have

$$P(\mathbf{E}) = 2\sum_{i=1}^{4} p_{E_i} \geq 2k_5 + 2k_6 + \sqrt{\pi} + \frac{\sqrt{\pi}}{3} \approx 11.7125.$$

This contradicts the minimality of $\mathbf{E}$ (indeed $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$), so the proof is completed. □

Furthermore we have the following corollary.

Corollary 5.40. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, then $\mathbf{E}$ has at least three connected regions. Moreover $\mathbf{E}$ has at most five bounded components, thus $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.*

Proof. If $\mathbf{E}$ is standard, then any region is connected therefore there are four bounded components.

If $\mathbf{E}$ is not standard, then by Remark 5.12, Theorem 5.15, Theorem 5.23 and Theorem 5.39, its connection type can be only $I_{\mathbf{E}} = (1,0,0,0)$. Therefore $\mathbf{E}$ can have at most five bounded components. Thus, by Theorem 1.50, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.

□

5.4 The case $(1,0,0,0)$

In this last section, we will exclude the case $(1,0,0,0)$; so we will complete the proof of Theorem 5.1. Initially we present a simple lemma, that describes a component of the disconnected region and a component of connected region.

Lemma 5.41. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,0,0,0)$, then a component C of a disconnected region has*

i) *three edges if it is inner;*

ii) *at most four edges if it is external.*

While a connected region has

iii) *at most four edges if it is inner;*

iv) *at most five edges if it is external.*

Proof. By Corollary 1.35 we know that any component C of $\mathbf{E}$ has at least three edges. Let C be a component of E_1. Since $\mathbf{E}$ is a minimum, by Proposition 1.33 and $I_{\mathbf{E}} = (1,0,0,0)$, then, if C is inner it has three edges, while if C is external it can have at most four edges.

If E_i is a connected region, arguing as in the case that C is a component of E_1, then E_i can have at most four edges and five edges if it is inner and external respectively.

□

Lemma 5.42. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,0,0,0)$, *then* $\mathbf{E}$ *has eight vertices and twelve edges.*

Proof. Let v, e and c be the numbers of the vertices, of the edges and of the connected components of $\mathbf{E}$ respectively, then, by the Euler's formula, it applies that $v - e + c = 2$. Since $\mathbf{E}$ is a minimum, each vertex of $\mathbf{E}$ is is a meeting point of exactly three edges (see Theorem 1.10), thus $3v = 2e$ (note that each edge has two vertices). Since $I_{\mathbf{E}} = (1,0,0,0)$, it follows $c = 6$. Solving the following linear system

$$\begin{cases} v - e = -4 \\ 3v = 2e, \end{cases}$$

we find the claim. □

Lemma 5.43. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,0,0,0)$, *then* $3 \le v(E_0) \le 5$ *and* $3 \le e(E_0) \le 5$, *where* $v(E_0)$ *and* $e(E_0)$ *denote the number of the vertices which belong to* E_0 *and the number of the edges of* E_0 *respectively.*

Proof. By Corollary 5.9 we know that there is at most one inner big component, thus, since $I_{\mathbf{E}} = (1,0,0,0)$ (i.e there is only one small component), there are at most two inner components, so there are at least three external bounded components. Since $I_{\mathbf{E}} = (1,0,0,0)$, $\mathbf{E}$ has five bounded components, therefore, by Proposition 1.33 the proof is completed. □

Lemma 5.44. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. *If* $I_{\mathbf{E}} = (1,0,0,0)$, *then* $\mathbf{E}$ *can be the clusters of the Figure 5.15, up to rigid motion and curvature of the edges.*

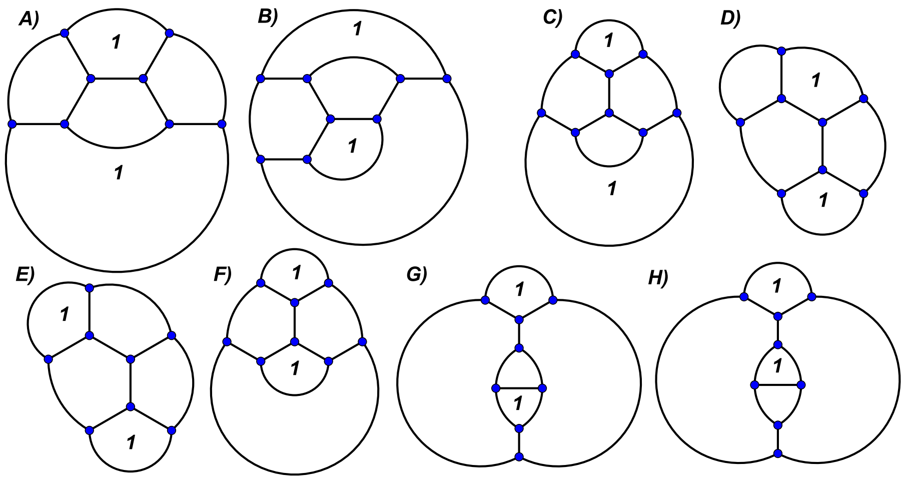

Figure 5.15: *The possible clusters for* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ *with* $I_{\mathbf{E}} = (1,1,0,0)$. *We label with* 1 *the components of the disconnected region* E_1*, while the unlabeled components represent the connected regions.*

Proof. By Lemma 5.41 we know that the components of the disconnected region E_1 can have three or four edges and if someone has four edges then it is external. We denote by S_1 and B_1 the components of E_1 where the names S_1, B_1 have not a link with the area quantity which own. First of all we explicitly note that the vertices of B_1 and S_1 are all distinct because $\mathbf{E}$ is a minimizer. Moreover vertices of a same component of $\mathbf{E}$ are not connected since there are no two-sided components by Corollary 1.35, thus all leaving edges from vertices of a same component of $\mathbf{E}$ are all different. Finally we recall that each vertex of $\mathbf{E}$ must be a meeting point of exactly three edges ($\mathbf{E}$ is a minimizer and see Theorem 1.10), thus we underline that, at the beginning of the creation of the clusters, each external vertex of S_1 and B_1 is already a meeting point of exactly three edges, while a inner vertex of S_1 and B_1 can be get another edge. We use the following notations:

1) $v(\mathbf{E})$ is the number of the vertices of $\mathbf{E}$;

2) $v(C)$ represents the number of the vertices of a component C of $\mathbf{E}$;

3) $v(\mathbf{E} \setminus C)$ denotes the number of the vertices which belong to $\mathbf{E}$ but

not to the component C of $\mathbf{E}$;

4) $v(C_1 \setminus C_2)$ denotes the number of the vertices which belong to the component C_1 but not to the component C_2, where C_1 and C_2 are components of $\mathbf{E}$;

5) $v(C_1 \cap C_2)$ denotes the number of the vertices which belong to the components C_1 and C_2 of $\mathbf{E}$;

6) $e(\mathbf{E})$ is the number of the edges of $\mathbf{E}$;

7) $e(E_0)$ represents the number of the edges of E_0;

8) $e(C)$ represents the number of the edges of a component C of $\mathbf{E}$;

9) $e_i(C)$ denotes the number of the inner edges of a component C of $\mathbf{E}$;

10) $e_{l,i}(C)$ denotes the number of the leaving edges from a inner vertex of a component C of $\mathbf{E}$;

11) $e_{l,i}(v)$ is the number of the leaving inner edges from a vertex v of $\mathbf{E}$;

12) $e_{l,i}(v \cap C)$ is the number of the leaving inner edges from a vertex v which arrive in a vertex of a component C of $\mathbf{E}$.

From the previous notation, it immediately follows that, less than to exchange S_1 and B_1, $(e(S_1), e(B_1))$ can be $(4,4)$, $(4,3)$ and $(3,3)$. Furthermore, $e_{l,i}(v) = 1$ if the vertex v is external and $e_{l,i}(v) = 3$ if the vertex v is inner. Finally we also have that $0 \leq e_{l,i}(v \cap C) \leq 3$. We recall that $v(\mathbf{E}) = 8$ and $e(\mathbf{E}) = 12$ by Lemma 5.42, while $3 \leq v(E_0) \leq 5$ and $3 \leq e(E_0) \leq 5$ by Lemma 5.43. Now we divided the proof in three parts depending on $(e(S_1), e(B_1))$ is $(4,4)$, $(4,3)$ and $(3,3)$.

Part I. Let $(e(S_1), e(B_1)) = (4,4)$, then B_1 and S_1 are external; moreover $v(B_1) = v(S_1) = 4$, thus, since $v(\mathbf{E}) = 8$, there are all the vertices of $\mathbf{E}$. Hence we are in the situation of Figure 5.16 where, since the edges of $\mathbf{E}$ can not intersect (if two arcs intersected, then a vertex would be created which is a meeting point of four arcs, which contradicts 2. of Theorem 1.10 since $\mathbf{E}$ is a minimizer), we have only one way to link the inner vertices of S_1 and B_1. So, we obtain the case A) of Figure 5.15.

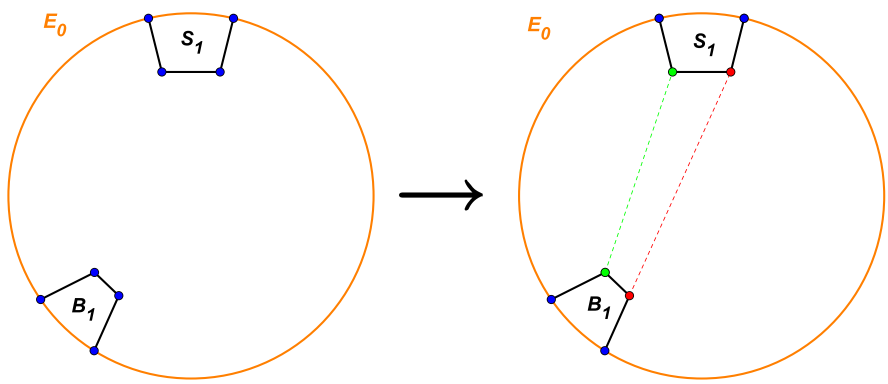

Figure 5.16: S_1 *and* B_1 *have four edges and they are external. Since* $v(\mathbf{E}) = 8$*, there are all vertices of* $\mathbf{E}$*, thus, since the edges of* $\mathbf{E}$ *can not be intersect, there is only one way to link the inner vertices of* S_1 *and* B_1*.*

Part II. Let $(e(S_1), e(B_1)) = (4, 3)$, then S_1 is external while B_1 can be inner or external. Furthermore we have that $v(S_1) = 4$ and $v(B_1) = 3$, thus, since $v(\mathbf{E}) = 8$, we must add another vertex v_1.

Part IIa. We consider the case where B_1 is inner. We know that $e_i(S_1) = 3$, $e_i(B_1) = 3$, $e_{l,i}(B_1) = 3$. First of all the vertex v_1 is external because $v(S_1 \setminus E_0) = 2$ and $v(E_0) \geq 3$, thus $e_{l,i}(v_1) = 1$. Furthermore v_1 is linked to only one inner vertex of B_1, indeed if it was false we would get that

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + e_{l,i}(B_1) + e_{l,i}(v_1) \\ &\geq 3 + 3 + 3 + 3 + 1 = 13. \end{aligned}$$

This is a contradiction, so, we are in the situation of Figure 5.17 where, since the edges of $\mathbf{E}$ can not intersect and up to rotation of the inner component B_1, we have only one way to link v_1 and the inner vertices of S_1 and B_1. So, we obtain the case $B)$ of Figure 5.15.

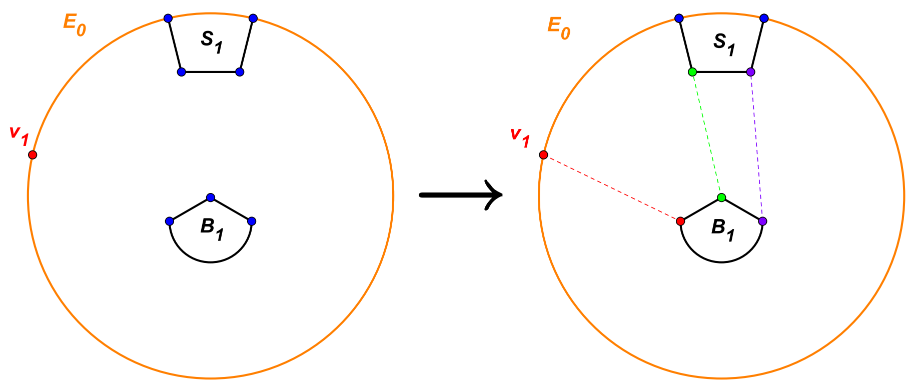

Figure 5.17: *S_1 and B_1 have four and three edges respectively and S_1 is certainly external. Since $v(\mathbf{E}) = 8$, there is another vertex v_1. Here B_1 is inner, thus v_1 must be external and it has to be connected to only one inner vertex of B_1. Since the edges of $\mathbf{E}$ can not be intersect and up to rotate B_1, there is only one way to link v_1 and the inner vertices of S_1 and B_1.*

Part IIb. We consider the case where B_1 is external, therefore $e_i(B_1) = 2$ and $v(B_1 \smallsetminus E_0) = 1$. In this situation it can happen that the vertex v_1 can be inner or external.

If v_1 is inner, then $e(E_0) = 4$ because $v(E_0) = 4$, since S_1 and B_1 are external. We say that v_1 must be related to the inner vertices of S_1, indeed if it was false then $e_{l,i}(v_1 \cap S_1) \leq 1$, thus

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + (e_{l,i}(S_1) + e_{l,i}(v_1) - \varepsilon_{l,i}(v_1) \cap S_1) \\ &\geq 4 + 3 + 2 + (2 + 3 - 1) = 13. \end{aligned}$$

It is an absurd. Furthermore v_1 is connected with the inner vertex of B_1, too, indeed it is false then

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + e_{l,i}(B_1) + e_{l,i}(v_1) \\ &\geq 4 + 3 + 2 + 1 + 3 = 13. \end{aligned}$$

This is a contradiction, so we are in the situation of Figure 5.18 where we have only one way to link v_1 and the inner vertices of S_1 and B_1. So, we obtain the case C) of Figure 5.15.

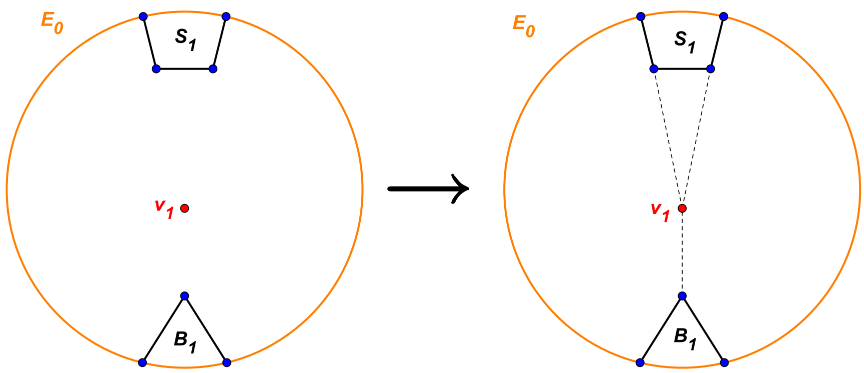

Figure 5.18: *S_1 and B_1 have four and three edges respectively and both are external. Since $v(\mathbf{E}) = 8$, there is another vertex v_1. If v_1 is inner, then it must be connected to the inner vertices of S_1 and to the inner vertex of B_1.*

If v_1 is external, then $e_{l,i}(v_1) = 1$ and $e(E_0) = 5$ because $v(E_0) = 5$, since S_1 and B_1 are external (i.e $v(E_0 \setminus S_1) = v(E_0 \setminus B_1) = 2$). We say that v_1 must be related to one inner vertex of S_1, indeed if it was false then

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + e_{l,i}(S_1) + e_{l,i}(v_1) \\ &\geq 5 + 3 + 2 + 2 + 1 = 13. \end{aligned}$$

It is a contradiction, so we are in the situation of Figure 5.19 where, since the edges of $\mathbf{E}$ can not intersect, we have only one way to link v_1 and the inner vertices of S_1 and B_1. So, we obtain the case D) of Figure 5.15.

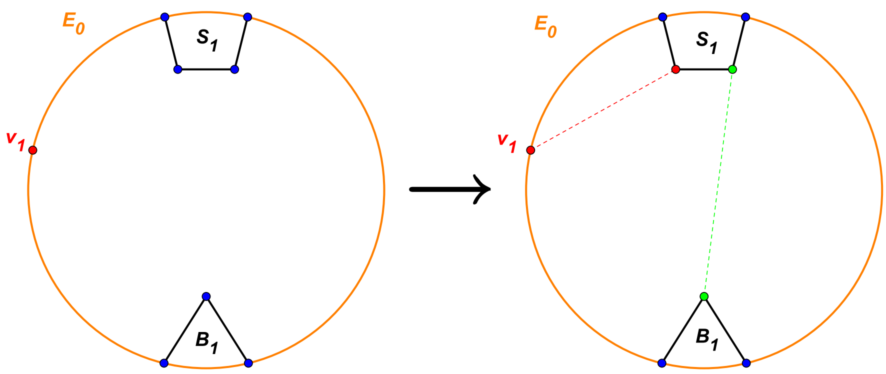

Figure 5.19: *S_1 and B_1 have four and three edges respectively and both are external. Since $v(\mathbf{E}) = 8$, there is another vertex v_1. If v_1 is external, then it must be connected to one inner vertex of S_1.*

Part III. Let $(e(S_1), e(B_1)) = (3, 3)$, thus $v(S_1) = v(B_1) = 3$, so, since $v(\mathbf{E}) = 8$, we must add another two vertices v_1 and v_2. Moreover one between S_1 and B_1 must be external, otherwise $v(E_0) = 2$, since $v(S_1 \setminus E_0) = v(S_1 \setminus E_0) = 3$ and $v(\mathbf{E}) = 8$, while we know that $v(E_0) \geq 3$ by Lemma 5.43. Without loss of generality we can assume that S_1 is always external, while B_1 can be external or inner.

Part IIIa. S_1 and B_1 are external, thus $v(S_1 \cap E_0) = v(B_1 \cap E_0) = 2$ and $e(E_0) = 4$. So, since $v(E_0) \leq 5$, only one vertex between v_1 and v_2 can be external.

Actually one vertex between v_1 and v_2 must be external, indeed if it was false then v_1 and v_2 would be inner. In this situation we say that v_1 and v_2 must be connected with two edges; if we prove it, we come a contradiction because there is two-sided component, which is impossible by Corollary 1.35.

Again we argue by contradiction, thus if v_1 and v_2 were inner, they would be linked with at most one edge, namely $e_{l,i}(v_1 \cap v_2) \leq 1$, therefore we would obtain that

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + (e_{l,i}(v_1) + e_{l,i}(v_2) - e_{l,i}(v_1 \cap v_2)) \\ &\geq 4 + 2 + 2 + (3 + 3 - 1) = 13. \end{aligned}$$

It is a contradiction, thus one vertex between v_1 and v_2 must be external and the other is inner.

Without loss of generality we can assume that v_1 is external while v_2 is inner, thus $e_{l,i}(v_1) = 1$, $e_{l,i}(v_2) = 3$ and $e(E_0) = 5$. We say that v_1 is connected to v_2, indeed if it was false then

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + e_{l,i}(v_1) + e_{l,i}(v_2) \\ &\geq 5 + 2 + 2 + 1 + 3 = 13. \end{aligned}$$

This is a contradiction.

We recall that each vertex of $\mathbf{E}$ must be a meeting point of exactly three edges, therefore v_2 must be linked to the inner vertices of S_1 and B_1. So, we are in the situation of Figure 5.20, obtaining the case $E)$ of Figure 5.15.

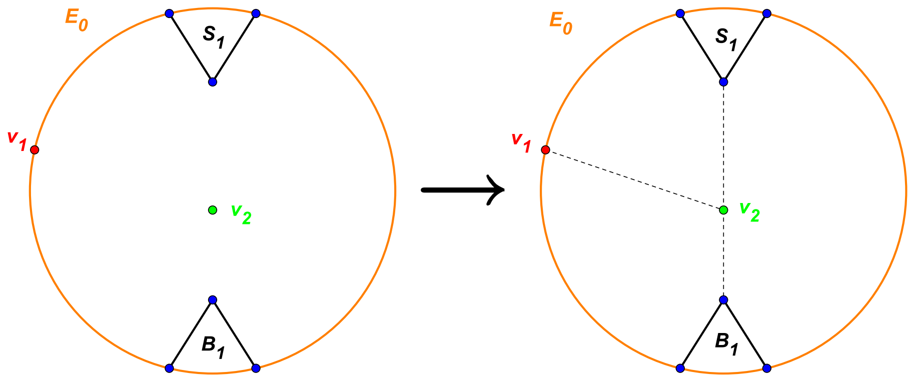

Figure 5.20: *S_1 and B_1 have three edges and they are external. Since $v(\mathbf{E}) = 8$, there are another two vertices v_1 and v_2. Only one vertex between v_1 and v_2 is external. Here v_1 is external and v_2 is inner and they must be linked. Thus, there is only one way to link v_1, v_2 and the inner vertices of S_1 and B_1.*

Part IIIb. S_1 is external and B_1 is inner, thus $v(S_1 \cap E_0) = 2$, $v(B_1) = 3$. So, since $v(E_0) \geq 3$, one vertex between v_1 and v_2 must be external. Without loss of generality we can assume that v_1 is external while v_2 can be external or inner.

If v_2 is external, therefore $e(E_0) = 4$. Furthermore we say that v_1 and v_2 are linked each to one inner vertex of B_1.

In fact if was false, then at least one vertex between v_1 and v_2 would be not connected to any inner vertex of S_1 obtaining that (let v_1 be not related to any inner vertex of S_1)

$$
\begin{aligned}
12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + e_{l,i}(S_1) + e_{l,i}(v_1) \\
&\geq 4 + 2 + 3 + 1 + 3 = 13.
\end{aligned}
$$

This is a contradiction, so, up to rotate the inner component B_1, we are in the situation of Figure 5.21, obtaining the case F) of Figure 5.15.

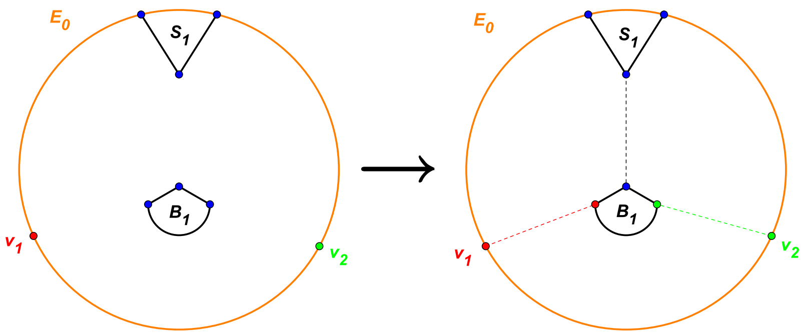

Figure 5.21: *S_1 and B_1 have three edges where S_1 is external and B_1 is inner. Since $v(\mathbf{E}) = 8$, there are another two vertices v_1 and v_2 of which one must be external. Here v_1 and v_2 are external and they have to linked each to one inner vertex of B_1.*

If v_2 is inner, therefore $e(E_0) = 3$ and $e_{l,i}(v_2) = 3$. Furthermore we say that v_2 is linked at least two inner vertices of B_1, otherwise $e_{i,l}(v_2 \cap B_1) \leq 1$, thus we would have that

$$\begin{aligned} 12 = e(\mathbf{E}) &\geq e(E_0) + e_i(S_1) + e_i(B_1) + (e_{l,i}(v_2) + e_{l,i}(B_1) - e_{l,i}(v_2 \cap B_1) -) \\ &\geq 3 + 2 + 3 + (3 + 3 - 1) = 13. \end{aligned}$$

This is a contradiction, so, v_2 can related with three or two inner vertices of B_1. If v_2 is linked to all vertices of B_1 we have the situation represented in Figure 5.22.

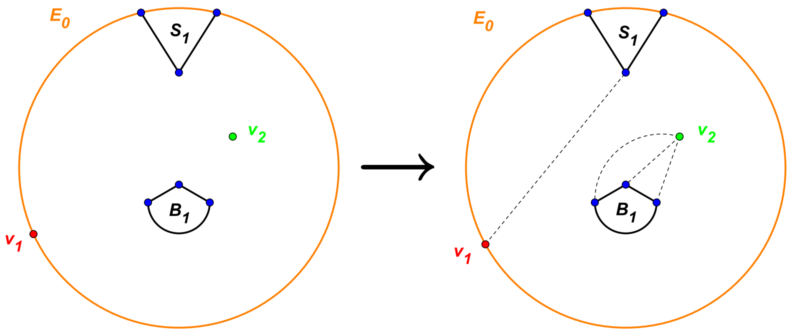

Figure 5.22: *S_1 and B_1 have three edges where S_1 is external and B_1 is inner. Since $v(\mathbf{E}) = 8$, there are another two vertices v_1 and v_2 of which one must be external. Here v_1 is external and v_2 is inner where v_2 is connected to all inner vertices of B_1.*

This possibility is not a cluster because there are two components such that the area of their intersection is not zero (see Definition 1.1). If v_2 is connected to two inner vertices of B_1 we have two possibility given if v_2 is related or not related with v_1. So, since the edges of $\mathbf{E}$ can not intersect, we are in the situation of Figure 5.23, obtaining the cases G) and H) of Figure 5.15.

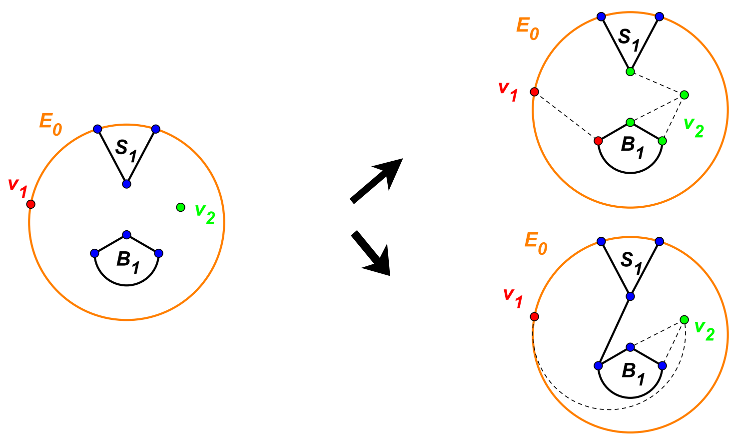

Figure 5.23: *S_1 and B_1 have three edges where S_1 is external and B_1 is inner. Since $v(\mathbf{E}) = 8$, there are another two vertices v_1 and v_2 of which one must be external. Here v_1 is external and v_2 is inner where v_2 is connected to two inner vertices of B_1. So, there are two possibility given if v_2 is linked or not linked to v_1.*

□

Remark 5.45. We explicitly note that in the cases B), G) and H) of Figure 5.15, there are two inner bounded components of which one is a connected region. Thus, by Corollary 5.9, the inner component of the disconnected region is a small component.

Proposition 5.46. *Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$. If $I_{\mathbf{E}} = (1,0,0,0)$, then $\mathbf{E}$ can not be the clusters G) and H) of Figure 5.15.*

Proof. The proof immediately comes from Lemma 5.41 because in these configurations there is a six-sided connected region. □

We show some important estimates for the pressure of the disconnected region before excluding the cases $A)$, $B)$, $C)$, $D)$, $E)$, $F)$ of Figure 5.15.

Lemma 5.47. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*. If* $I_{\mathbf{E}} = (1,0,0,0)$*, then the pressure* p_{E_1} *is*

1) $p_{E_1} \geq \frac{\pi}{2\tilde{p}-2\sqrt{\pi}(5+\sqrt{1-|\,S_1\,|})} + \min_{j\in J} p_{E_j}$*, if* S_1 *is inner;*

2) $p_{E_1} \geq \frac{\pi}{2\tilde{p}-2\sqrt{\pi}(5+\sqrt{1-|\,S_1\,|})} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{|\,S_1\,|}}{(1-|\,S_1\,|)}\right) \cdot \min_{j\in J} p_{E_j}$*, if* S_1 *is external with three edges;*

3) $p_{E_1} \geq \frac{2\pi}{3\left(2\tilde{p}-2\sqrt{\pi}(5+\sqrt{1-|\,S_1\,|})\right)} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{|\,S_1\,|}}{(1-|\,S_1\,|)}\right) \cdot \min_{j\in J} p_{E_j}$*, if* S_1 *is external with four edges;*

where S_1 *is the small component of* E_1 *and*

$$J := \left\{ j > 1 \middle| \mathcal{H}^1(\partial^* S_1 \cap \partial^* E_j) > 0 \right\}.$$

In particular, since $|\,S_1\,| \leq A_{1,4}$ *and denoting by* $k_7 := \frac{\pi}{2\tilde{p}-2\sqrt{\pi}(5+\sqrt{1-A_{1,4}})}$*,*

4) $p_{E_1} \geq k_7 + \min_{j\in J} p_{E_j}$*, if* S_1 *is inner;*

5) $p_{E_1} \geq k_7 + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{A_{1,4}}}{(1-A_{1,4})}\right) \cdot \min_{j\in J} p_{E_j}$*, if* S_1 *is external with three edges;*

6) $p_{E_1} \geq \frac{2k_7}{3} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{A_{1,4}}}{(1-A_{1,4})}\right) \cdot \min_{j\in J} p_{E_j}$*, if* S_1 *is external with four edges;*

Proof. First of all, by assumption $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, therefore, by Corollary 5.40, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$ and in particular $m(\mathbf{E}) = (1,1,1,1)$. We remind that, by Remark 5.7 and (5.10) of Corollary 5.10,

$$A_{2,4} \leq |\,S_1\,| \leq A_{1,4} < \frac{1}{3}, \tag{5.68}$$

thus $|\,B_1\,| = |\,E_1\,| - |\,S_1\,| \geq 1 - A_{1,4} > 0$, where B_1 denotes the big component of E_1. The regions surrounding S_1 are E_j with $j \in J$. We denote by $l_{1,j}$ the lengths of the edges of S_1 with $j \in J$.

First we prove 1). In this case S_1 is inner, therefore, by Lemma 5.41, S_1 has three edges. From Lemma 1.38, the turning angle of S_1 is π. Therefore we obtain that

$$\pi = \sum_{j \in J}(p_{E_1} - p_{E_j})l_{1,j} = p_{E_1} \cdot \sum_{j \in J} l_{1,j} - \sum_{j \in J} p_{E_j} l_{1,j}.$$

It follows that

$$p_{E_1} \cdot P(S_1) \geq \pi + P(S_1) \cdot \Big(\min_{j \in J} p_{E_j} \Big),$$

namely

$$p_{E_1} \geq \frac{\pi}{P(S_1)} + \min_{j \in J} p_{E_j}. \tag{5.69}$$

Now, by the minimality of $\mathbf{E}$, (5.68) and the isoperimetric inequality it follows that

$$\begin{aligned} P(S_1) &= 2P(\mathbf{E}) - \Big(P(B_1) + P(E_0) + \sum_{k=2}^{4} P(E_k) \Big) \\ &\leq 2\tilde{p} - 2\sqrt{\pi}(\sqrt{1 - |\, S_1 \,|} + 2 + 3). \end{aligned}$$

So, by (5.69), we get

$$p_{E_1} \geq \frac{\pi}{2\tilde{p} - 2\sqrt{\pi}(\sqrt{1 - |\, S_1 \,|} + 5)} + \min_{j \in J} p_{E_j},$$

that is 1).

Now we prove 2) and 3). In these cases S_1 is external, therefore, by Lemma 5.41, S_1 can have three or four edges. We show only 2), because in the case 3) the only difference is the turning angle of S_1, which is π in 2), while in 3) it is $\frac{2\pi}{3}$. Since S_1 is external, we obtain that (note that $p_{E_0} = 0$)

$$\pi = \sum_{j \in J \cup \{0\}} (p_{E_1} - p_{E_j})l_{1,j} = p_{E_1} \cdot P(S_1) - \sum_{j \in J} p_{E_j} l_{1,j}.$$

We find that

$$p_{E_1} \cdot P(S_1) = \pi + \sum_{j \in J} p_{E_j} l_{1,j} \geq \pi + \min_{j \in J} p_{E_j} \cdot \Big(P(S_1) - l_{1,0}\Big);$$

therefore it follows that

$$p_{E_1} \geq \frac{\pi}{P(S_1)} + \left(1 - \frac{l_{1,0}}{P(S_1)}\right) \cdot \min_{j \in J} p_{E_j}. \tag{5.70}$$

From (5.8) of Corollary 5.10, we know that

$$l_{1,0} \leq \frac{|\, S_1 \,|}{2(1 - |\, S_1 \,|)} \cdot \tilde{p}. \tag{5.71}$$

Furthermore by the minimality of $\mathbf{E}$ and the isoperimetric inequality we get the following estimates for $P(S_1)$ (note that $P(S_1) = 2P(\mathbf{E}) - P(B_1) - P(E_0) - \sum_{k=2}^{4} P(E_k)$ and $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| = 1 - |\, S_1 \,|$):

$$2\sqrt{\pi \,|\, S_1 \,|} \leq P(S_1) \leq 2\tilde{p} - 2\sqrt{\pi}(\sqrt{1 - |\, S_1 \,|} + 2 + 3). \tag{5.72}$$

We recall that the each pressure is non negative by Proposition 1.49, thus, by (5.70), (5.71) and (5.72) we obtain that

$$p_{E_1} \geq \frac{\pi}{2\tilde{p} - 2\sqrt{\pi}(5 + \sqrt{1 - |\, S_1 \,|})} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{|\, S_1 \,|}}{(1 - |\, S_1 \,|)}\right) \min_{j \in J} p_{E_j}. \tag{5.73}$$

This is 2).

From (5.68) we know that $A_{2,4} \leq |\, S_1 \,| \leq A_{1,4}$, therefore, denoting by $k_7 := \frac{\pi}{2\tilde{p} - 2\sqrt{\pi}(5 + \sqrt{1 - A_{1,4}})}$, by 1), 2) and 3) we find 4), 5) and 6) respectively.

$\square$

Remark 5.48. We explicitly note that the quantity $(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \frac{\sqrt{A_{1,4}}}{1 - A_{1,4}})$, view in Lemma 5.47, is positive, in fact $(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \frac{\sqrt{A_{1,4}}}{1 - A_{1,4}}) \approx 0.250923 > 0$.

We eliminate the case $A)$ of Figure 5.15. It depends by Corollary 2.16, Lemma 2.17, Lemma 2.18, Lemma 2.19.

Proposition 5.49. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, *then* $\mathbf{E}$ *is not as in the case* $A)$ *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case A). By Corollary 5.40 $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$ and in particular $m(\mathbf{E}) = (1,1,1,1)$. We respectively denote by S_1 and B_1 the small and the big component of E_1. Without loss of generality we can assume that we are in the situation described in Figure 5.24.

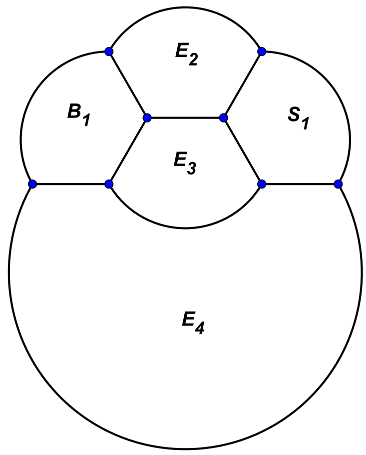

Figure 5.24: *The case* A).

By Remark 5.7, it follows that $|\, S_1 \,| \leq A_{1,4}$, thus $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| \geq 1 - A_{1,4}$. Furthermore, the connected region E_3 is not the lowest pressure region, because it is inner and it has four edges, then its turning angle is $\frac{2\pi}{3}$ (recall that the lowest pressure inner region has all concave edges, namely each edge has non positive signed curvature). Moreover, since S_1 is external with four edges, by Proposition 1.49, by 6) of Lemma 5.47 and Remark 5.48 it follows that $p_{E_1} \geq \frac{2k_7}{3}$. Therefore E_1 is not the lowest pressure region, because otherwise, by Corollary 1.47, the perimeter of $\mathbf{E}$ would be at least (each other region would have a pressure at least $\frac{2k_7}{3}$)

$$P(\mathbf{E}) = 2\sum_{i=1}^{4} p_{E_i} \geq \frac{16k_7}{3} \approx 11.8485,$$

and this is a contradiction, since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$.

Hence the lowest pressure region is either E_2 or E_4. Furthermore B_1 must have at least one strictly convex inner side (i.e. the edge has positive signed curvature). Then we can have the following cases, given by the relations between pressures of the regions adjacent to B_1:

$$
\begin{aligned}
&1) p_{E_1} > p_{E_2}, p_{E_1} > p_{E_3}, p_{E_1} > p_{E_4};\\
&2) p_{E_1} > p_{E_2}, p_{E_1} > p_{E_3}, p_{E_1} \le p_{E_4};\\
&3) p_{E_1} > p_{E_2}, p_{E_1} \le p_{E_3}, p_{E_1} > p_{E_4};\\
&4) p_{E_1} > p_{E_2}, p_{E_1} \le p_{E_3}, p_{E_1} \le p_{E_4};\\
&5) p_{E_1} \le p_{E_2}, p_{E_1} > p_{E_3}, p_{E_1} > p_{E_4};\\
&6) p_{E_1} \le p_{E_2}, p_{E_1} > p_{E_3}, p_{E_1} \le p_{E_4};\\
&7) p_{E_1} \le p_{E_2}, p_{E_1} \le p_{E_3}, p_{E_1} > p_{E_4}.
\end{aligned}
\tag{5.74}
$$

We immediately can eliminate the sixth case, because E_3 would be the lowest pressure region which is a contradiction. Furthermore we can see that the cases $2)$ and $5)$ are the same, it is sufficient to exchange the role of E_2 and E_4, and also the case $4)$ and $7)$ are the same for the same reason. Therefore we just need to exclude the cases $1)$, $3)$, $5)$ and $7)$ of (5.74). The idea is to prove that **E** is vertically symmetric, as illustrated in Figure 5.25.

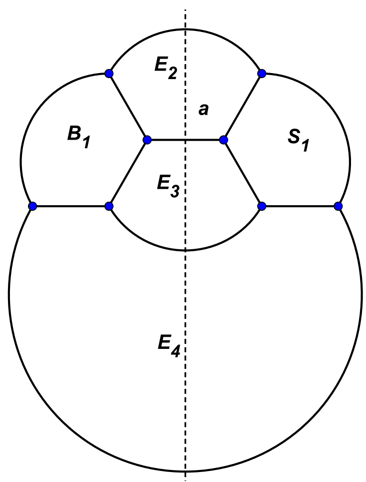

Figure 5.25: *The cluster* **E** *is vertically symmetric respect to the axes* a*.*

We start with the case $1)$ of (5.74). In this situation the disconnected region E_1 is the highest pressure region. The situation is illustrated in Figure

5.26 (the dashed sides are edges, of which we do not know exactly signed curvature).

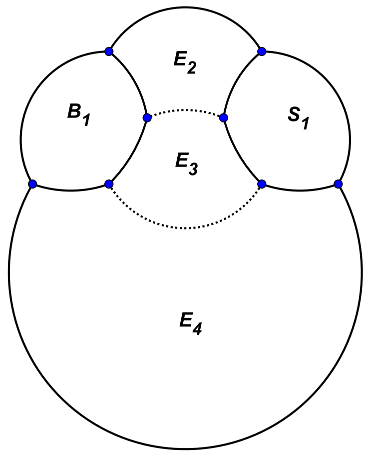

Figure 5.26: *The case* 1) *of* (5.74).

First we claim that $p_{E_3} > p_{E_2}$ and $p_{E_3} > p_{E_4}$, since otherwise $p_{E_2} \geq p_{E_3}$ or $p_{E_4} \geq p_{E_3}$, therefore either the region E_2 or E_4 would be as in Figure 5.27.

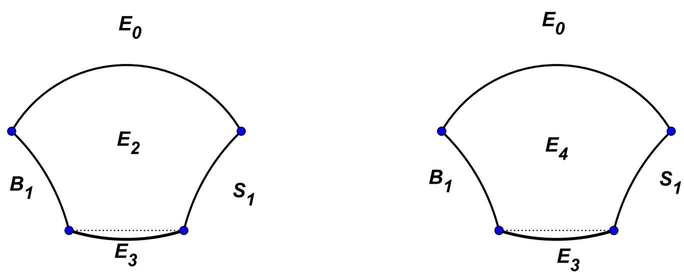

Figure 5.27: *If* $p_{E_2} \geq p_{E_3}$ *or* $p_{E_4} \geq p_{E_3}$*, then one between* E_2 *and* E_4 *would have the bottom edge straight of strictly convex.*

By Lemma 2.17, either E_2 or E_4 will not have cocircular lateral edges, therefore by Lemma 2.18, all other lateral edges of the connected region would be not cocircular. Thus, by Corollary 2.16 E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, **E** is vertically symmetric. So it follows that

$$0.159132 \approx A_{1,4} \geq |\,S_1\,| = |\,B_1\,| \geq 1 - A_{1,4} \approx 0.840868.$$

This is an absurd, hence $p_{E_3} > p_{E_2}$ and $p_{E_3} > p_{E_4}$ as claimed. Hence we are in this situation represented in Figure 5.28

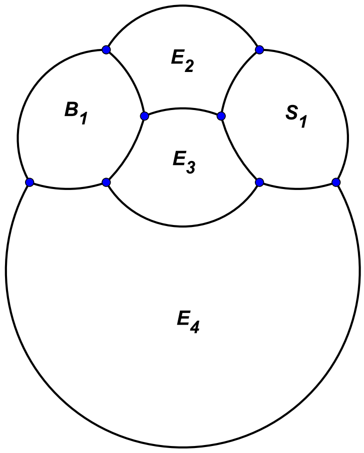

Figure 5.28: $p_{E_1} > p_{E_3} > p_{E_2}$ *and* $p_{E_1} > p_{E_3} > p_{E_4}$ *are the relations between the pressures in the case* 1).

We consider the inner region E_3: by Lemma 2.17 its lateral edges are not cocircular, hence by Lemma 2.18 each other lateral edge of E_2 and E_4 is not cocircular. Thus, by Corollary 2.16, E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, **E** is vertically symmetric. Then

$$0.159132 \approx A_{1,4} \geq |\,S_1\,| = |\,B_1\,| \geq 1 - A_{1,4} \approx 0.840868.$$

This is a contradiction, then the case 1) is excluded.

We consider the case 3) of (5.74). The situation is represented in Figure 5.29 (the dashed sides are edges, of which we do not know exactly signed curvature).

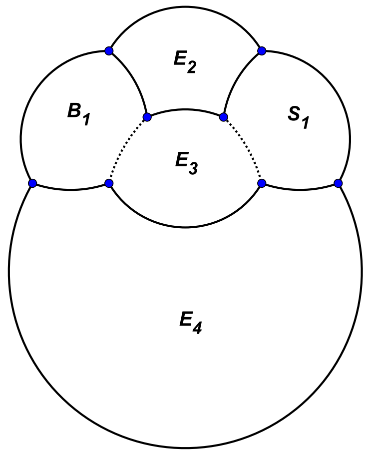

Figure 5.29: *The case* 3) *of* (5.74).

We note that E_3 is the highest pressure region, since $p_{E_3} \geq p_{E_1} > \max(p_{E_2}, p_{E_4})$. We consider the external region E_2, as shown in Figure 5.30 and we suppose that the lateral edges (the sides adjacent to E_1) of E_2 are cocircular. The radius of the circle containing the lateral edges of E_2 is $R = \frac{1}{p_{E_1} - p_{E_2}}$. Since the lateral edges of E_2 are cocircular and concave, then, by Lemma 2.17, the shape of E_2 is unique and it is represent in Figure 5.31.

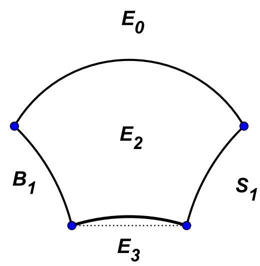

Figure 5.30: *The external region* E_2 *of case* 3).

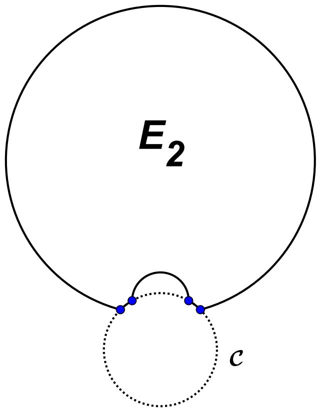

Figure 5.31: *The unique shape of E_2 if its lateral edges belong to the circle $\mathcal{C}$.*

We denote by L the bottom edge of E_2. We call P and Q the meeting points of the bottom edge of E_2 with the circle $\mathcal{C}$ and we respectively denote by α and θ, the angle between L and the chord line for its vertices P and Q and the angle determined by P on the circle (see Figure 5.32).

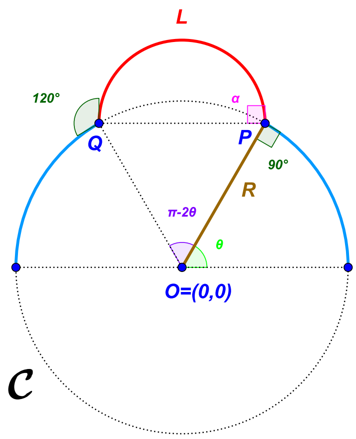

Figure 5.32: *The bottom edge L of E_2 when the opposite and concave adjacent edges to E_1 are cocircular.*

Since the bottom edge of E_2 is external to the circle, by Lemma 2.19, its curvature is given by the following function (see (2.16))

$$k^e_L(\theta) = (p_{E_1} - p_{E_2}) \cdot \frac{\sin\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}, \quad \theta \in] - \frac{\pi}{2}, \frac{\pi}{2}[.$$

Moreover since the inner angles between E_2 and the circumference are $\frac{2\pi}{3}$ (see Figure 5.32), then the external angles are $\frac{4\pi}{3}$, thus there is the following relation between α and θ:

$$\alpha = \frac{5\pi}{6} - \theta. \tag{5.75}$$

Hence, by (5.79), $\alpha \geq \frac{\pi}{3}$. The bottom edge of E_2 is the top edge of E_3, which is the highest pressure region and its turning angle is $\frac{2\pi}{3}$ (in the case A), each component has four edges and the turning angle of L is 2α), hence $\alpha \leq \frac{\pi}{3}$. Thus $\alpha = \frac{\pi}{3}$. Therefore the other sides of $\mathbf{E}_3$ should be straight, then $p_{E_3} = p_{E_4}$, but $p_{E_3} \geq p_{E_1} > p_{E_4}$. This is a contradiction. Hence the lateral edges of E_2 are not cocircular, so, by Lemma 2.18 the other lateral sides of E_3 and E_4 are not cocircular. Thus, by Corollary 2.16 E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, then $\mathbf{E}$ is vertically symmetric. So

$$0.159132 \approx A_{1,4} \geq |\, S_1 \,| = |\, B_1 \,| \geq 1 - A_{1,4} \approx 0.840868.$$

This is a contradiction, so the case 3) of (5.74) is excluded.

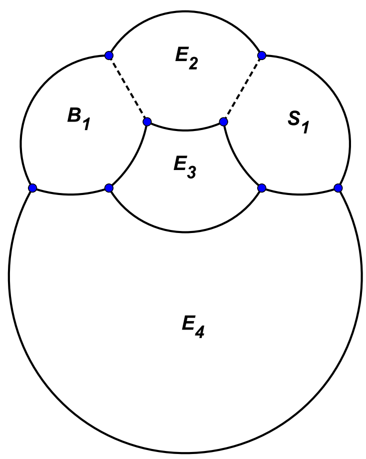

Figure 5.33: *The case* 5) *of* (5.74).

We consider the case $5)$ of (5.74). Since E_3 is not the lowest pressure region, we are in this situation $p_{E_2} \geq p_{E_1} > p_{E_3} > p_{E_4}$ and it is represented in Figure 5.33 (the dashed sides are edges, of which we do not know exactly the signed curvature). We consider the inner region E_3, as shown in Figure 5.34 and we suppose that the lateral edges (the sides adjacent to E_1) of E_3 are cocircular.

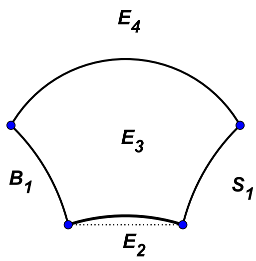

Figure 5.34: *The inner region E_3 of case $5)$.*

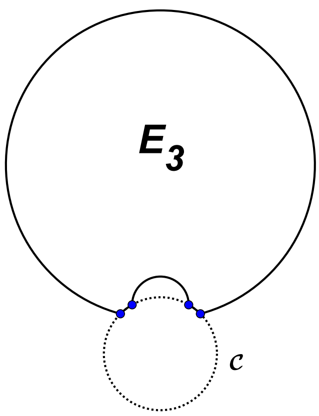

Figure 5.35: *The unique shape of E_3 if its lateral edges belong to the circle $\mathcal{C}$.*

The radius of the circle containing the lateral edges of E_3 is $R = \frac{1}{p_{E_1}-p_{E_3}}$. Since the lateral edges of E_3 are cocircular and concave, then, by Lemma 2.17, the shape of E_3 is unique and it is represent in Figure 5.35. We denote by L the bottom edge of E_3. We call P and Q the meeting points of the bottom edge of E_3 with the circle $\mathcal{C}$ and we respectively denote by α and θ, the angle between L and the chord line for its vertices P and Q and the angle determined by P on the circle (see Figure 5.36). Since the bottom edge of E_3 is external to the circle, by Lemma 2.19, its curvature is given by the following function (see (2.16))

$$k_L^e(\theta) = (p_{E_1} - p_{E_3}) \cdot \frac{\sin\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}, \quad \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[.$$

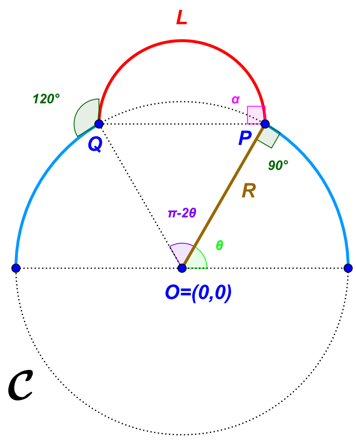

Figure 5.36: *The bottom edge L of E_3 when the opposite and concave adjacent edges to E_1 are cocircular.*

Moreover since the inner angles between E_3 and the circumference are $\frac{2\pi}{3}$ (see Figure 5.36), then the external angles are $\frac{4\pi}{3}$, thus there is the following relation between α and θ:

$$\alpha = \frac{5\pi}{6} - \theta. \tag{5.76}$$

Hence, by (5.79), $\alpha \geq \frac{\pi}{3}$. The bottom edge of E_3 is the top edge of E_2, which is the highest pressure region and its turning angle is $\frac{2\pi}{3}$ (in the case

A), each component has four edges and the turning angle of L is 2α), then $\alpha \le \frac{\pi}{3}$. Thus $\alpha = \frac{\pi}{3}$. Therefore the other sides of $\mathbf{E}_2$ should be straight, then $p_{E_2} = 0$, but $p_{E_2} \ge p_{E_1} \ge \frac{2k_7}{3} > 0$. This is a contradiction. Hence the lateral edges of E_3 are not cocircular, then, by Lemma 2.18 the other lateral sides of E_2 and E_4 are not cocircular. Thus, by Corollary 2.16 E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, then $\mathbf{E}$ is vertically symmetric. So

$$0.159132 \approx A_{1,4} \ge |\, S_1 \,| = |\, B_1 \,| \ge 1 - A_{1,4} \approx 0.840868.$$

This is a contradiction, so the case $5)$ of (5.74) is excluded.

Finally we consider the case $7)$ of (5.74). We are in the situation described in Figure 5.37 (the dashed sides are edges, of which we do not know exactly signed curvature).

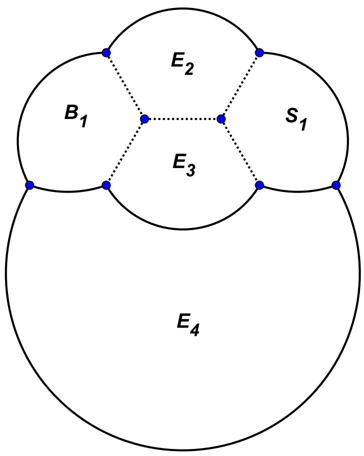

Figure 5.37: *The case* $7)$ *of* (5.74).

We first claim that $p_{E_2} > p_{E_3}$, otherwise $p_{E_3} \ge p_{E_2}$, therefore we have the following relations between the pressures $p_{E_3} \ge p_{E_2} \ge p_{E_1} > p_{E_4}$. We note that E_3 is the highest pressure region. We consider the external region E_4, as shown in Figure 5.38 and we suppose that the lateral edges (the sides adjacent to E_1) of E_4 are cocircular. The radius of the circle containing the lateral edges of E_4 is $R = \frac{1}{p_{E_1} - p_{E_4}}$. Since the lateral edges of E_4 are cocircular and concave, by Lemma 2.17, the shape of E_4 is unique and it is represented in Figure 5.39.

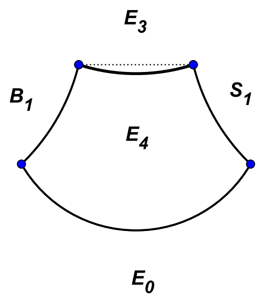

Figure 5.38: *The external region* E_4 *of case* 7)*.*

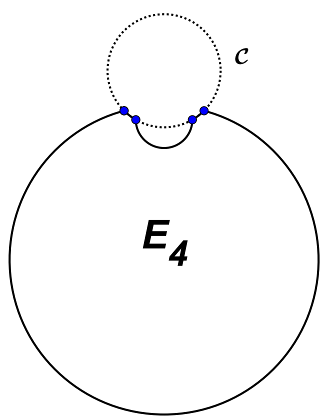

Figure 5.39: *The unique shape of* E_4 *if its lateral edges belong to the circle* $\mathcal{C}$*.*

We denote by L the bottom edge of E_4. We call P and Q the meeting points of the bottom edge of E_4 with the circle $\mathcal{C}$ and we respectively denote by α and θ, the angle between L and the chord line of its vertices P and Q and the angle determined by P on the circle (see Figure 5.40).

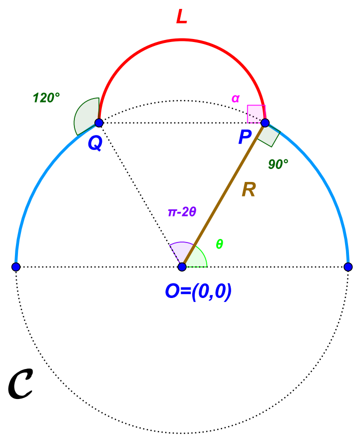

Figure 5.40: *The bottom edge L of E_4 when the opposite and concave adjacent edges to E_1 are cocircular.*

Since the bottom edge of E_4 is external to the circle, by Lemma 2.19, its curvature is given by the following function (see (2.16))

$$k_L^e(\theta) = (p_{E_1} - p_{E_4}) \cdot \frac{\sin\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}, \quad \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \tag{5.77}$$

where the function

$$\theta \mapsto g(\theta) := \frac{\sin\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}, \tag{5.78}$$

is strictly increasing with $g\left(\frac{\pi}{6}\right) = 1$. Moreover since the inner angles between E_4 and the circumference are $\frac{2\pi}{3}$ (see Figure 5.40), then the external angles are $\frac{4\pi}{3}$, thus there is the following relation between α and θ:

$$\alpha = \frac{5\pi}{6} - \theta. \tag{5.79}$$

Hence, by (5.79), $\alpha \geq \frac{\pi}{3}$. The bottom edge of E_4 is the top edge of E_3, which is the highest pressure region and its turning angle is $\frac{2\pi}{3}$ (in the case A), each component has four edges and the turning angle of L is 2α), then $\alpha \leq \frac{\pi}{3}$. Thus $\alpha = \frac{\pi}{3}$. Therefore the other sides of $\mathbf{E}_3$ should be straight, hence $p_{E_3} = p_{E_2} = p_{E_1}$. From (5.77) and (5.78), we find that $\theta = \frac{\pi}{6}$, thus, by (5.79), $\alpha = \frac{2\pi}{3}$. This is a contradiction, because E_3 has four edges (then, note that its turning angle is $\frac{2\pi}{3}$) and it is the highest pressure region (so each its edges is convex, thus $2\alpha \leq \frac{2\pi}{3}$).

Hence the lateral edges of E_4 are not cocircular, then, by Lemma 2.18 the other lateral sides of E_2 and E_3 are not cocircular. Thus, by Corollary 2.16 E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, $\mathbf{E}$ is vertically symmetric. Then

$$0.159132 \approx A_{1,4} \geq |\, S_1 \,| = |\, B_1 \,| \geq 1 - A_{1,4} \approx 0.840868.$$

This is a contradiction, so $p_{E_2} > p_{E_3}$ as claimed.

Thus the relations between the pressures are $p_{E_2} > p_{E_3} \geq p_{E_1} > p_{E_4}$. The situation is described in Figure 5.41.

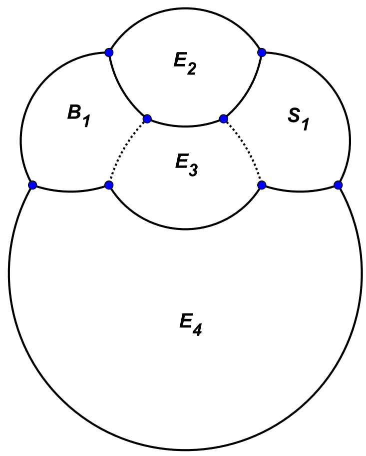

Figure 5.41: $p_{E_2} > p_{E_3} \geq p_{E_1} > p_{E_4}$ *are the relations between the pressures in the case* 7).

We consider the external region E_2 and we suppose that its lateral edges (the sides adjacent to E_1) are cocircular. We prove that we do not have enough circle length to make E_2. The radius of the circle containing the lateral edges of E_2 is $R = \frac{1}{p_{E_2} - p_{E_1}}$. The top and the bottom edge of E_2 meet the circle inside; their curvature are respectively given by the following functions (we respectively denote with T and B, the top and the bottom edge of E_2):

$$k_T^e(\theta_1) = (p_{E_2} - p_{E_1}) \cdot \frac{\sin\left(\frac{\pi}{6} - \theta_1\right)}{\cos\theta_1},$$
$$k_B^e(\theta_2) = (p_{E_2} - p_{E_1}) \cdot \frac{\sin\left(\frac{\pi}{6} - \theta_2\right)}{\cos\theta_2}, \tag{5.80}$$

where $\theta_1, \theta_2 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Let $g(\theta) := \frac{\sin\left(\frac{\pi}{6}-\theta\right)}{\cos\theta}$. Now the curvature of the top edge is

$$p_{E_2} > p_{E_2} - p_{E_1},$$

while the curvature of the bottom edge is

$$p_{E_2} - p_{E_3} \leq p_2 - p_{E_1}.$$

Then we respectively get that $g(\theta_1) > 1$ and $g(\theta_2) \leq 1$ for the top and the bottom edge. The function g, by Lemma 2.19 and Remark 2.20, is strictly decreasing and $g\Big(-\frac{\pi}{6}\Big) = 1$, thus we obtain that

$$i)\ \theta_1 \in]-\frac{\pi}{2}, -\frac{\pi}{6}[$$
$$ii)\ \theta_2 \in [-\frac{\pi}{6}, \frac{\pi}{2}[. \tag{5.81}$$

Therefore, by in order to draw the top edge, we must cut a center angle of at least

$$\pi - 2\theta_1 \overset{i)}{>} \frac{4\pi}{3},$$

while to draw the bottom edge, we must cut a center angle at most of

$$\pi - 2\theta_2 \overset{ii)}{\leq} \frac{4\pi}{3}.$$

So we do not have enough circle length to make E_2 with the lateral edges cocircular. Thus the lateral sides of E_2 are not cocircular, then, by Lemma 2.18 the other lateral sides of E_3 and E_4 are not cocircular. Thus, by Corollary 2.16 E_2, E_3 and E_4 are vertically symmetric. Since E_2, E_3 and E_4 are in sequence and the axes of a segment is unique, $\mathbf{E}$ is vertically symmetric too. So

$$0.159132 \approx A_{1,4} \geq |\, S_1 \,| = |\, B_1 \,| \geq 1 - A_{1,4} \approx 0.840868.$$

This is a contradiction, so also the case $7)$ is excluded.

Hence the case $A)$ of Figure 5.15 is excluded, so the proof is concluded.

□

Proposition 5.50. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, *then* $\mathbf{E}$ *is not as in the case* $B)$ *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case $B)$. By Corollary 5.40 $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, in particular $m(\mathbf{E}) = (1,1,1,1)$. We respectively call S_1 and B_1 the small and the big component of E_1. In this configuration the inner three-sided component of E_1 is its small component, because, by Corollary 5.9, there can be at most one big inner component. Without loss of generality we label the inner connected region with E_2. Furthermore immediately we have that E_2 and E_1 are not the lower pressure regions, because E_2 and S_1 are inner and their turning angle are respectively $\frac{2\pi}{3}$ and π (see Lemma 1.38). Therefore the lower pressure region is E_3 or E_4. We take the lower pressure region; it has five edges (so its turning angle is $\frac{\pi}{3}$) and each inner side is concave or straight (i.e the signed curvature of the edge is non positive), therefore by Lemma 1.38 one concludes that

$$L_{e,\min} \cdot p_{\min} \geq \frac{\pi}{3},$$

where we denote by $L_{e,\min}$ and $p_{\min}$ the length of the external edge and the pressure of the lower pressure region. So by Lemma 3.14 we establish the following estimate for the lower pressure

$$p_{\min} \geq \frac{\sqrt{\pi}}{6} \approx 0.295409. \tag{5.82}$$

Since S_1 is inner, by $4)$ of Lemma 5.47 and by (5.82), we get that

$$p_{E_1} \geq k_7 + \frac{\sqrt{\pi}}{6} \approx 2.51701. \tag{5.83}$$

So E_1 is the highest pressure region; indeed if it was false then we would have another region E_j ($j \neq 1$), such that $p_{E_j} \geq k_7$ and then, by Corollary 1.47 and by (5.82), the perimeter of $\mathbf{E}$ would be at least

$$P(\mathbf{E}) = 2\sum_{i=1}^{4} p_{E_i} \geq 4k_7 + 4\left(\frac{\sqrt{\pi}}{6}\right) \approx 11.2497.$$

This is a contradiction, since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$, by the minimality of $\mathbf{E}$.

Now we reduce $\mathbf{E}$ by applying Lemma 2.22 until we come to a standard double bubble, which, by Lemma 3.16, will allow us to determine an upper limit for the highest pressure, that will be smaller than k_7. We proceed with the reduction method (reduction of three-sided component), seen in Lemma 2.22 and described in Figure 5.42. The different steps of the reduction in Figure 5.42 are given by the arrows. At the beginning we reduce S_1, then we obtain that

$$E_4 \subseteq E_4'. \tag{5.84}$$

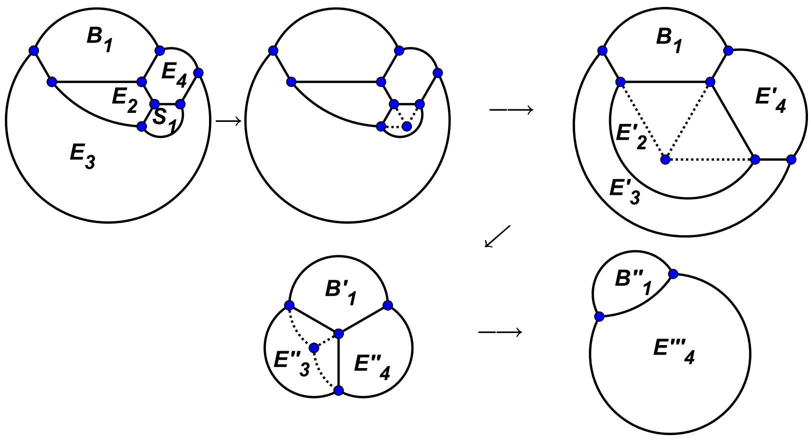

Figure 5.42: *Through the reduction of three-sided component of* $\mathbf{E}$, *we come to a standard double bubble, that allow us to determine an upper limit for the highest pressure.*

In the second step we reduce E_2', so we have that

$$E_4' \subseteq E_4'', \tag{5.85}$$

$$B_1 \subseteq B_1'.$$

In the last step we reduce E_3'', having that

$$E_4'' \subseteq E_4''', \qquad B_1' \subseteq B_1''. \tag{5.86}$$

At the end of these steps we have a standard double bubble of areas $|\, B_1'' \,|$ and $|\, E_4''' \,|$. We note explicitly that this method of reduction does not change the curvatures, because each side is only extended following its curvature and primarily, as shown in Lemma 2.22, the extended edges meet in an inner point satisfying the cocycle condition; therefore, at each step we create a planar regular cluster. Hence, since $p_{E_1} \geq p_{E_4}$, we have that $|\, E_4''' \,| \geq |\, B_1'' \,|$. From (5.84), (5.85) and (5.86), we determine that (recall that $|\, S_1 \,| \leq A_{1,4}$ by Remark 5.7, thus $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| \geq 1 - A_{1,4}$)

$$|\, B_1'' \,| \geq |\, B_1' \,| \geq |\, B_1 \,| \geq 1 - A_{1,4}.$$

By estimate (3.35) in Lemma 3.16, we have that

$$p_{E_1} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{|\, B_1'' \,|}} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - A_{1,4}}} \approx 1.7337.$$

This contradicts (5.83), so the proof is concluded.

□

Proposition 5.51. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, *then* $\mathbf{E}$ *is not as in the case* $C)$ *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case $C)$. By Corollary 5.40 $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, in particular $m(\mathbf{E}) = (1,1,1,1)$. Let S_1 and B_1 be the small and the big component of E_1 respectively. By Remark 5.7 and (5.10) of Corollary 5.10, we know that

$$0 < A_{2,4} \leq |\, S_1 \,| \leq A_{1,4} < \frac{1}{3}. \tag{5.87}$$

Thus $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| \geq 1 - A_{1,4}$. Certainly in this configuration, the connected inner region is not the lowest pressure region, because it has three edges (so its turning angle, by Lemma 1.38 is π) and it is inner.

First we suppose that S_1 has three edges and without loss of generality we can assume that $\mathbf{E}$ is as in Figure 5.43.

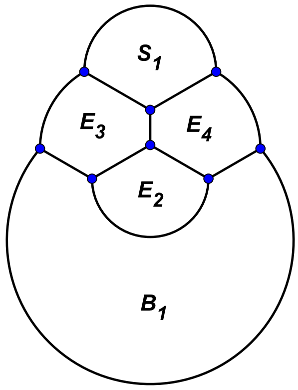

Figure 5.43: *The case* C) *when* S_1 *has three edges.*

Since S_1 is external, by Proposition 1.49, by 5) of Lemma 5.47 and Remark 5.48, it follows that

$$p_{E_1} \geq k_7 \approx 2.22160. \tag{5.88}$$

Now we will determine an upper limit for the $\max_{k=1,4} p_{E_k}$; we will find that it is less than k_7, thus we will get a contradiction. In order to do this, we reduce $\mathbf{E}$ through the reduction method of three-sided component, described in Lemma 2.22. The reduction is represented in Figure 5.44. The different steps of the reduction in Figure 5.44 are given by the arrows.

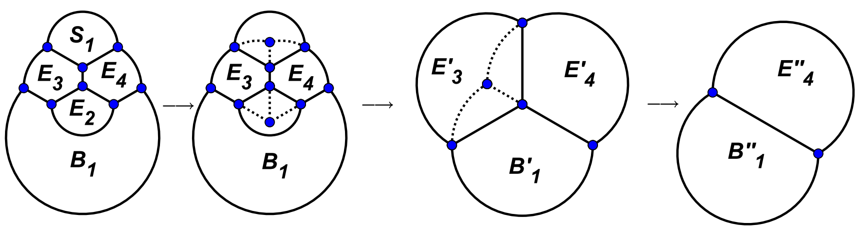

Figure 5.44: *This reduction determines an upper limit for the highest pressure between* p_{E_j} *with* $j = 1, 4$.

At the beginning we reduce S_1 and E_2 then we obtain that

$$\begin{aligned} E_4 &\subseteq E_4', \\ B_1 &\subseteq B_1'. \end{aligned} \tag{5.89}$$

In the second step we reduce E_3', so we have that

$$\begin{aligned} E_4' &\subseteq E_4'', \\ B_1' &\subseteq B_1''. \end{aligned} \tag{5.90}$$

At the end of these steps we have a standard double bubble of areas $\mid B_1'' \mid$ and $\mid E_4'' \mid$. By (5.89) and (5.90), it follows that

a) if $p_{E_4} \geq p_{E_1}$, then $\mid B_1'' \mid \geq \mid E_4'' \mid \geq \mid E_4' \mid \geq \mid E_4 \mid = 1$;

b) if $p_{E_1} \geq p_{E_4}$, then $\mid E_4'' \mid \geq \mid B_1'' \mid \geq \mid B_1' \mid \geq \mid B_1 \mid = 1 - A_{1,4}$.

Therefore, by Lemma 3.16, $\max(p_{E_4}, p_{E_1}) \leq \sqrt{\frac{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}{1-A_{1,4}}} \approx 1.7337$. This is in contradiction with (5.88), thus S_1 has four edges and B_1 three edges.

Now we do a reduction of $\mathbf{E}$ until we come to a standard double bubble, that will allow us to determine a lower limit for $\min_{k=3,4} p_{E_k}$. It is illustrated in Figure 5.45.

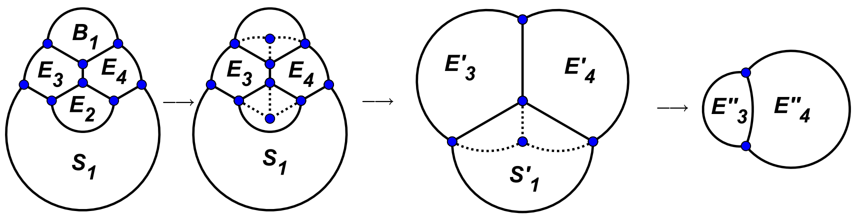

Figure 5.45: *This reduction determines a lower limit for the lowest pressure between p_{E_j} with $j = 3, 4$.*

The different steps of the reduction in Figure 5.45 are given by the arrows; at the beginning we reduce E_2 and B_1, then we obtain that

$$\begin{gathered} E_3' \cup S_1' \subseteq E_3 \cup S_1 \cup B_1 \cup E_2, \\ E_4' \cup S_1' \subseteq E_4 \cup S_1 \cup B_1 \cup E_2. \end{gathered} \tag{5.91}$$

In the second step we reduce S_1', so we have that

$$\begin{gathered} E_3'' \subseteq E_3' \cup S_1', \\ E_4'' \subseteq E_4' \cup S_1'. \end{gathered} \tag{5.92}$$

At the end of these steps we have a standard double bubble of areas $|\,E_3''\,|$ and $|\,E_4''\,|$. By (5.91) and (5.92), it follows that

c) if $p_{E_4} \geq p_{E_3}$, then

$$\begin{aligned} |\,E_4''\,| \leq |\,E_3''\,| \leq |\,E_3'\,| + |\,S_1'\,| &\leq |\,E_3\,| + |\,S_1\,| + |\,B_1\,| + |\,E_2\,| \\ &\leq |\,E_3\,| + |\,E_1\,| + |\,E_2\,| \leq 3; \end{aligned}$$

d) if $p_{E_3} \geq p_{E_4}$, then

$$\begin{aligned} |\,E_3''\,| \leq |\,E_4''\,| \leq |\,E_4'\,| + |\,S_1'\,| &\leq |\,E_4\,| + |\,S_1\,| + |\,B_1\,| + |\,E_2\,| \\ &\leq |\,E_4\,| + |\,E_1\,| + |\,E_2\,| \leq 3. \end{aligned}$$

Therefore, by Lemma 3.16, we can say that

$$\min_{k=3,4} p_{E_k} \geq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{3}} := k_8 \approx 0.917861 \tag{5.93}$$

Now S_1 is external, therefore, by $6)$ of Lemma 5.47, by Remark 5.48 and (5.93) (recall that the inner connected region can not be the lowest pressure region because it is inner and its turning angle is π),

$$p_{E_1} \geq \frac{2k_7}{3} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{A_{1,4}}}{(1 - A_{1,4})}\right) k_8 := k_9 \approx 1.71138. \tag{5.94}$$

We claim that

$$\max_{k=3,4} p_{E_k} \geq p_{E_1}. \tag{5.95}$$

Indeed if we suppose that the contrary holds; since B_1 is disjoint from E_2, by the minimality of $\mathbf{E}$, by (5.87) and the isoperimetric inequality, we would get that

$$\tilde{p} \geq P(\mathbf{E}) \geq P(B_1)+P(E_2)+P(E_0)-L_{1,e} \geq 2\sqrt{\pi}\Big(\sqrt{1-A_{1,4}}+1+2\Big) - L_{1,e},$$

where $L_{1,e}$ is the length of the external edge of B_1. Therefore we have the following estimate for $L_{1,e}$:

$$L_{1,e} \geq 2\sqrt{\pi}\Big(\sqrt{1-A_{1,4}}+3\Big) - \tilde{p} := \ell_5 \approx 2.69072. \tag{5.96}$$

Since $p_{E_1} > \max_{k=3,4} p_{E_k}$, then the edges of B_1 are convex (namely the signed curvature of its edges is non negative). By Lemma 1.38, the turning angle of B_1 is π, thus we have that

$$p_{E_1} \cdot \ell_5 \leq p_{E_1} \cdot L_{1,e} \leq \pi;$$

therefore, by (5.94), $1.71138 \approx k_9 \leq p_{E_1} \leq \frac{\pi}{\ell_5} \approx 1.16757$. It is a contradiction, hence (5.95) holds.

Finally we determine an estimate for the pressure of the inner connected region E_2. It has three edges and it is inner, therefore it is not the lowest pressure region, so at least one of its edges is convex (i.e the signed curvature of the edge is non negative). Hence, if $L_{2,3}$, $L_{2,4}$ and $L_{2,1}$ are the lengths of the sides of E_2 in common with E_3, E_4 and S_1 respectively, we have that

$$\begin{aligned}\Big(p_{E_2} - \min_{k=1,3,4} p_{E_k}\Big)P(E_2) &\geq \max_{k=1,3,4}\Big(p_{E_2} - p_{E_k}\Big)P(E_2)\\ &\geq \sum_{k=1,3,4} L_{2,k}\Big(p_{E_2} - p_{E_k}\Big) = \pi.\end{aligned}$$

Thus it follows that

$$p_{E_2} \geq \frac{\pi}{P(E_2)} + \min_{k=1,3,4} p_{E_k}. \tag{5.97}$$

By (5.87), by the isoperimetric inequality and the minimality of $\mathbf{E}$, we know that

$$\begin{aligned}P(E_2) &= 2P(\mathbf{E}) - P(B_1) - P(S_1) - P(E_3) - P(E_4) - P(E_0)\\ &\leq 2\tilde{p} - 2\sqrt{\pi}\Big(\sqrt{1-A_{1,4}} + \sqrt{A_{2,4}} + 1 + 1 + 2\Big) := \ell_6 \approx 4.41116.\end{aligned}$$

By (5.93) and (5.94), we have that $\min_{k=1,3,4} p_{E_k} \geq k_8$. So, by (5.97), we get that

$$p_{E_2} \geq \frac{\pi}{\ell_6} + k_8 := k_{10} \approx 1.63005. \tag{5.98}$$

Therefore the perimeter of $\mathbf{E}$ is at least, by (5.93), (5.94), (5.95) (5.98) and Corollary 1.47,

$$P(\mathbf{E}) = 2\sum_{i=1}^{4} p_{E_i} \geq 2(p_{E_1} + p_{E_2} + \max_{k=3,4} p_{E_k} + \min_{k=3,4} p_{E_k}) \geq 4k_9 + 2k_{10} + 2k_8 \approx 11.9413.$$

It is a contradiction, because $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$, so the proof is concluded. □

Proposition 5.52. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $\mathbf{E}$ *is not as in the case* $D)$ *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case $D)$. By Corollary 5.40 $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, thus $m(\mathbf{E}) = (1,1,1,1)$. We respectively denote with S_1 and B_1 the small and the big component of E_1. By Remark 5.7 and by (5.10) of Corollary 5.10, we know that

$$A_{2,4} \leq |\, S_1 \,| \leq A_{1,4} < \frac{1}{3}. \tag{5.99}$$

Thus $|\, B_1 \,| = |\, E_1 \,| - |\, S_1 \,| \geq 1 - A_{1,4}$.

First we suppose that S_1 has three edges and without loss of generality we can assume that $\mathbf{E}$ is as in Figure 5.46.

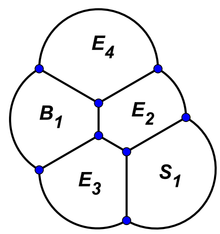

Figure 5.46: *The case* $D)$.

Since S_1 is external with three edges, then, by Proposition 1.49, by $5)$ of Lemma 5.47 and Remark 5.48, we have that

$$p_{E_1} \geq k_7 \approx 2.22160. \tag{5.100}$$

Now we reduce $\mathbf{E}$ by applying the reduction method of three-sided components described in Lemma 2.22. We reduce $\mathbf{E}$ until we come to a standard double bubble, which, by Lemma 3.16, will allow us to determine an upper limit for $\max\limits_{k=1,3} p_{E_k}$, that will be smaller than k_7. We proceed with the reduction, represented in Figure 5.47.

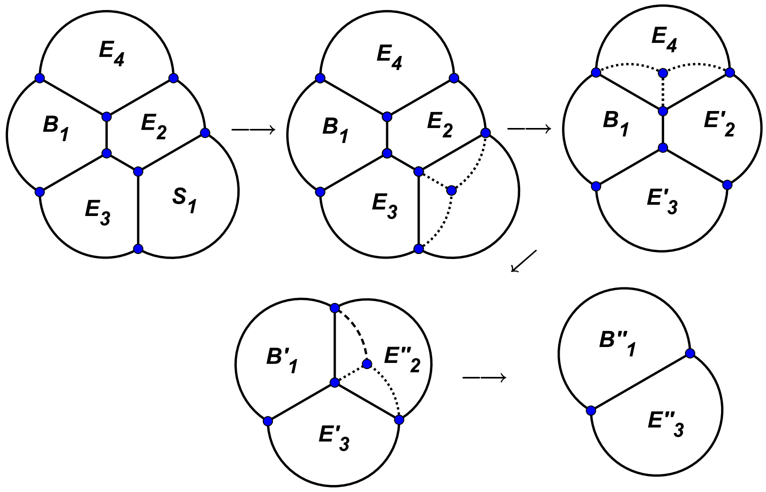

Figure 5.47: *This reduction determines an upper limit for* $\max\limits_{k=1,3} p_{E_k}$.

At the beginning we reduce only S_1, then we obtain that

$$E_3 \subseteq E_3'. \tag{5.101}$$

In the second step we reduce E_4, so we have that

$$B_1 \subseteq B_1'. \tag{5.102}$$

In the last step we reduce E_2'' having that

$$B_1' \subseteq B_1'',$$
$$E_3' \subseteq E_3''. \tag{5.103}$$

At the end of these steps we have a standard double bubble of areas $|B_1''|$ and $|E_3''|$. By (5.101), (5.102) and (5.103) it follows that

a) if $p_{E_3} \geq p_{E_1}$, then $|B_1''| \geq |E_3''| \geq |E_3'| \geq |E_3| = 1$;

b) if $p_{E_1} \geq p_{E_3}$, then $|E_3''| \geq |B_1''| \geq |B_1'| \geq |B_1| = 1 - A_{1,4}$.

Therefore, by Lemma 3.16, $\max(p_{E_3}, p_{E_1}) \leq \sqrt{\frac{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}{1-A_{1,4}}} \approx 1.7337$. This is in contradiction with (5.100), thus S_1 has four edges and B_1 three edges.

Initially we determine an estimate for $\min_{k=2,4} p_{E_k}$. In order to do this, we reduce $\mathbf{E}$ as showed in Figure 5.48, through the reduction method of three-sided component, seen in Lemma 2.22.

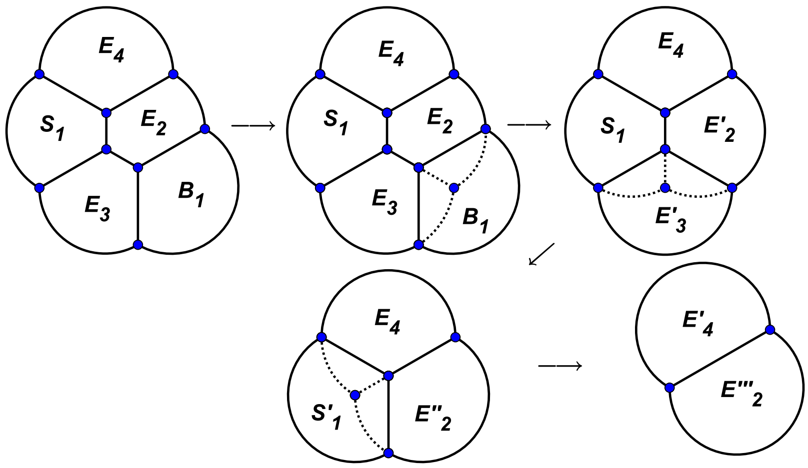

Figure 5.48: *This reduction determines an estimate for* $\min_{k=2,4} p_{E_k}$.

The different steps of the reduction in Figure 5.48 are given by the arrows; at the beginning we reduce only B_1, then we obtain that

$$\begin{aligned} &E_2' \subseteq E_2 \cup B_1, \\ &E_3' \subseteq E_3 \cup B_1 \\ &E_2' \cup E_3' \subseteq E_2 \cup E_3 \cup B_1. \end{aligned} \tag{5.104}$$

In the second step we reduce E_3', so we have that

$$E_2'' \subseteq E_2' \cup E_3', \tag{5.105}$$

$$S_1' \subseteq S_1 \cup E_3'.$$

In the last step we reduce S_1' having that

$$\begin{gathered} E_2''' \subseteq E_2'' \cup S_1', \\ E_4' \subseteq E_4 \cup S_1'. \end{gathered} \tag{5.106}$$

At the end of these steps we have a standard double bubble of areas $|\,E_2'''\,|$ and $|\,E_4'\,|$. By (5.104), (5.105) and (5.106) it follows that

c) if $p_{E_2} \geq p_{E_4}$, then

$$\begin{aligned} |\,E_2'''\,| \leq |\,E_4'\,| \leq |\,E_4\,| + |\,S_1'\,| &\leq |\,E_4\,| + |\,S_1\,| + |\,E_3'\,| \\ &\leq |\,E_4\,| + |\,S_1\,| + |\,B_1\,| + |\,E_3\,| \\ &\leq |\,E_4\,| + |\,E_1\,| + |\,E_3\,| \leq 3; \end{aligned}$$

d) if $p_{E_4} \geq p_{E_2}$, then

$$\begin{aligned} |\,E_4'\,| \leq |\,E_2'''\,| \leq |\,E_2''\,| + |\,S_1'\,| &\leq |\,E_2'\,| + |\,E_3'\,| + |\,S_1\,| \\ &\leq |\,E_2\,| + |\,E_3\,| + |\,B_1\,| + |\,S_1\,| \\ &\leq |\,E_2\,| + |\,E_3\,| + |\,E_1\,| \leq 3. \end{aligned}$$

Therefore, by Lemma 3.16, we can say that

$$\min_{k=2,4} p_{E_k} \geq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{3}} := k_8 \approx 0.917861. \tag{5.107}$$

Now we do two reductions of $\mathbf{E}$ through the reduction method of three-sided component, seen in Lemma 2.22; in the first we reduce $\mathbf{E}$ until we come to a standard double bubble, that, by Lemma 3.16, will allow us to determine an upper limit for the highest pressure between p_{E_j} with $j = 1, 2, 3$. Later we do the second reduction of $\mathbf{E}$ until we come to a standard double bubble, that will allow us to determine a lower limit for $\min_{k=2,3} p_{E_k}$. We begin with the first reduction; it is described in Figure 5.49. The different steps of the reduction in Figure 5.49 are given by the arrows; at the beginning we reduce only E_4, then we obtain that

$$E_2 \subseteq E_2',$$

$$
S_1 \subseteq S_1'. \tag{5.108}
$$

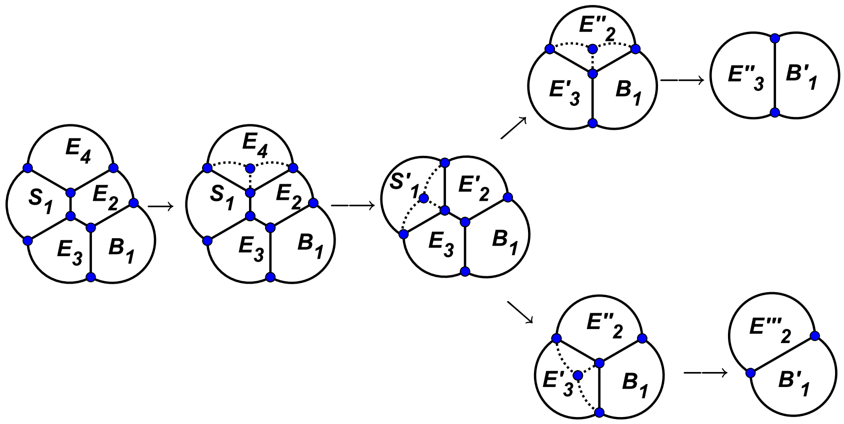

Figure 5.49: *This reduction determines an upper limit for the highest pressure between p_{E_j} with $j = 1, 2, 3$.*

In the second step we reduce S_1', so we have that

$$
\begin{gathered}
E_2' \subseteq E_2'', \\
E_3 \subseteq E_3'.
\end{gathered} \tag{5.109}
$$

In the last step we reduce E_2'' or E_3', so we reduce E_2'' we have that

$$
\begin{gathered}
B_1 \subseteq B_1', \\
E_3' \subseteq E_3''.
\end{gathered} \tag{5.110}
$$

while if we reduce E_3' we find that

$$
\begin{gathered}
E_2'' \subseteq E_2''', \\
B_1 \subseteq B_1'.
\end{gathered} \tag{5.111}
$$

If in the last step we reduce E_2'', at the end, we have a standard double bubble of areas $\mid B_1' \mid$ and $\mid E_3'' \mid$. By (5.108), (5.109) and (5.110) it follows that

c) if $p_{E_1} \geq p_{E_3}$, then

$$|\, E_3'' \,| \geq |\, B_1' \,| \geq |\, B_1 \,| \geq 1 - |\, S_1 \,|;$$

d) if $p_{E_3} \geq p_{E_1}$, then

$$|\, B_1' \,| \geq |\, E_3'' \,| \geq |\, E_3' \,| \geq |\, E_3 \,| \geq 1.$$

Therefore, by Lemma 3.16, we can say that

$$\max_{k=1,3} p_{E_k} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - |\, S_1 \,|}}. \tag{5.112}$$

If in the last step we instead reduce E_3', at the end of these steps we get a standard double bubble of areas $|\, B_1' \,|$ and $|\, E_2''' \,|$. By (5.108), (5.109) and (5.111) it follows that

c) if $p_{E_1} \geq p_{E_2}$, then

$$|\, E_2''' \,| \geq |\, B_1' \,| \geq |\, B_1 \,| \geq 1 - |\, S_1 \,|;$$

d) if $p_{E_2} \geq p_{E_1}$, then

$$|\, B_1' \,| \geq |\, E_2''' \,| \geq |\, E_2'' \,| \geq |\, E_2' \,| \geq |\, E_2 \,| \geq 1.$$

Therefore, by Lemma 3.16, we can say that

$$\max_{k=1,2} p_{E_k} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - |\, S_1 \,|}}. \tag{5.113}$$

So, by (5.112) and (5.113) we have that

$$\max_{k=1,2,3} p_{E_k} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - |\, S_1 \,|}}. \tag{5.114}$$

We call

$$f_7(x) := \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - x}}, \quad x \in [A_{2,4}, A_{1,4}], \tag{5.115}$$

then, by (5.114), $\max\limits_{1 \leq k \leq 3} p_{E_k} \leq f_7(|\, S_1 \,|)$. The function f_7 is strictly increasing, indeed its first derivative is $f_7'(x) = \frac{\sqrt{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}}{2} \cdot \frac{1}{(1-x)^{\frac{3}{2}}}$.

We present the second reduction; it is illustrated in Figure 5.50.

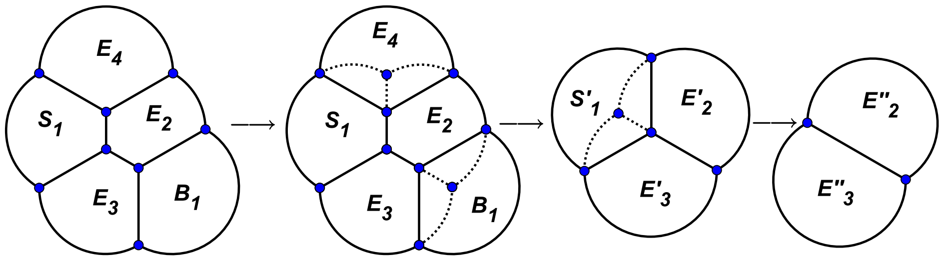

Figure 5.50: *This reduction determines a lower limit for the lowest pressure between* p_{E_j} *with* $j = 2, 3$.

The different steps of the reduction in Figure 5.50 are given by the arrows; at the beginning we reduce E_4 and B_1, then we obtain that

$$\begin{aligned} &E_2' \cup S_1' \subseteq E_2 \cup S_1 \cup B_1 \cup E_4, \\ &E_2' \subseteq E_2 \cup E_4 \cup B_1, \\ &E_3' \subseteq E_3 \cup B_1, \\ &S_1' \subseteq S_1 \cup E_4. \end{aligned} \tag{5.116}$$

In the second step we reduce S_1', so we have that

$$\begin{aligned} &E_2'' \subseteq E_2' \cup S_1', \\ &E_3'' \subseteq E_3' \cup S_1'. \end{aligned} \tag{5.117}$$

At the end of these steps we have a standard double bubble of areas $|\,E_3''\,|$ and $|\,E_2''\,|$. By (5.116) and (5.117), it follows that

c) if $p_{E_2} \geq p_{E_3}$, then

$$\begin{aligned} |\,E_2''\,| \leq |\,E_3''\,| \leq |\,E_3'\,| + |\,S_1'\,| &\leq |\,E_3\,| + |\,S_1\,| + |\,B_1\,| + |\,E_4\,| \\ &\leq |\,E_3\,| + |\,E_1\,| + |\,E_4\,| \leq 3; \end{aligned}$$

d) if $p_{E_3} \geq p_{E_2}$, then

$$\begin{aligned}|\,E_3''\,| \leq |\,E_2''\,| \leq |\,E_2'\,| + |\,S_1'\,| \leq |\,E_4\,| + |\,S_1\,| + |\,B_1\,| + |\,E_2\,| \\ \leq |\,E_4\,| + |\,E_1\,| + |\,E_2\,| \leq 3.\end{aligned}$$

Therefore, by Lemma 3.16, we can say that

$$\min_{k=2,3} p_{E_k} \geq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{3}} := k_8 \approx 0.917861. \tag{5.118}$$

Therefore, from (5.107), we have that

$$\min_{2\leq k\leq 4} p_{E_k} \geq k_8. \tag{5.119}$$

Now S_1 is external with four edges, then, by 3) of Lemma 5.47, the pressure of E_1 satisfies

$$p_{E_1} \geq \frac{2\pi}{3\cdot\left(2\tilde{p} - 2\sqrt{\pi}\cdot(5+\sqrt{1-|\,S_1\,|})\right)} + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}}\cdot\frac{\sqrt{|\,S_1\,|}}{(1-|\,S_1\,|)}\right)k_8, \tag{5.120}$$

because the function

$$f_8(x) := 1 - \frac{\tilde{p}}{4\sqrt{\pi}}\cdot\frac{\sqrt{x}}{1-x}, \quad x \in [A_{2,4}, A_{1,4}]$$

is positive. Indeed its first derivative is

$$f_8'(x) = -\frac{\tilde{p}}{8\sqrt{\pi}}\cdot\frac{1+x}{\sqrt{x}(1-x)^2},$$

therefore f_8 is strictly decreasing, thus $f_8(x) > f_8(A_{1,4}) \approx 0.250923 > 0$. We set

$$f_9(x) := \frac{2\pi}{3\cdot\left(2\tilde{p} - 2\sqrt{\pi}\cdot(5+\sqrt{1-x})\right)} + f_8(x)\cdot k_8, \quad x \in [A_{2,4}, A_{1,4}]. \tag{5.121}$$

It is clear, by (5.120), that $p_{E_1} \geq f_9(|\,S_1\,|)$. Furthermore f_9 is strictly decreasing, because its first derivative is

$$f_9'(x) = -\left(\frac{2(\pi)^{\frac{3}{2}}}{3\sqrt{1-x}\cdot\left(2\tilde{p} - 2\sqrt{\pi}(5+\sqrt{1-x})\right)^2} + \frac{\tilde{p}\cdot k_8}{8\sqrt{\pi}}\cdot\frac{1+x}{\sqrt{x}\cdot(1-x)^2}\right).$$

We note that, by (5.114), (5.115), (5.120) and (5.121),

$$f_9(|S_1|) \leq p_{E_1} \leq f_7(|S_1|), \quad |S_1| \in [A_{2,4}, A_{1,4}], \tag{5.122}$$

where f_7 and f_9 are strictly increasing and strictly decreasing respectively. Furthermore, by (5.122), we have that

$$|S_1| > 0.15, \tag{5.123}$$

otherwise

$$1.75724 \approx f_9(0.15) \leq p_{E_1} \leq f_7(0.15) \approx 1.72436$$

We derive an important upper limit for the sum of the lengths of inner edges of S_1. We have called with $l_{1,0}$, $l_{1,2}$, $l_{1,3}$, and $l_{1,4}$ the lengths of the edges of S_1 in common with E_0, E_2, E_3, and E_4 respectively, therefore, since the turning angle of S_1 is $\frac{2\pi}{3}$, we get that

$$p_{E_1}P(S_1) - l_{1,2}p_{E_2} - l_{1,3}p_{E_3} - l_{1,4}p_{E_4} = \sum_{k=0,2,3,4} l_{1,i}(p_{E_1} - p_{E_i}) = \frac{2\pi}{3}.$$

It follows that

$$l_{1,2}p_{E_2} + l_{1,3}p_{E_3} + l_{1,4}p_{E_4} = p_{E_1}P(S_1) - \frac{2\pi}{3}. \tag{5.124}$$

We respectively note that, by (5.122), by the fact that f_7 is strictly increasing and (5.119), $p_{E_1} \leq f_7(A_{1,4})$ and $\min_{2\leq k\leq 4} p_{E_k} \geq k_8$. Furthermore by the minimality of $\mathbf{E}$ and the isoperimetric inequality, we get the following estimate for $P(S_1)$ (note that $P(S_1) = 2P(\mathbf{E}) - P(B_1) - P(E_2) - P(E_3) - P(E_4) - P(E_0)$ and $|B_1| \geq 1 - A_{1,4}$, by (5.99)):

$$P(S_1) \leq 2\tilde{p} - 2\sqrt{\pi}(5 + \sqrt{1 - A_{1,4}}).$$

Thus, by (5.124), we have that

$$l_{1,2}+l_{1,3}+l_{1,4} \leq \frac{f_7(A_{1,4}) \cdot \left(2\tilde{p} - 2\sqrt{\pi}(5 + \sqrt{1 - A_{1,4}})\right) - \frac{2\pi}{3}}{k_8} := \ell_7 \approx 0.389221. \tag{5.125}$$

So, recalling that $|S_1| > 0.15$ by (5.123) and using the isoperimetric inequality we can obtain an estimate for the length of the external edge of S_1

$$l_{1,0} = P(S_1) - (l_{1,2} + l_{1,3} + l_{1,4}) \geq 2\sqrt{\pi \cdot 0.15} - \ell_7 := \ell_8 \approx 0.983716. \tag{5.126}$$

The configuration and the minimality of $\mathbf{E}$, we allow to say that (note that $|\,B_1\,| \geq 1 - A_{1,4}$ by (5.99), and $|\,S_1\,| > 0.15$ by (5.123))

$$\begin{aligned} \tilde{p} \geq P(\mathbf{E}) &\geq \sum_{k=2}^{4} P(E_k) + P(B_1) + P(S_1) - I(\mathbf{E}) \\ &\geq 2\sqrt{\pi}\Big(3 + \sqrt{1 - A_{1,4}} + \sqrt{0.15}\Big) - I(\mathbf{E}), \end{aligned}$$

where $I(\mathbf{E})$ denotes the sum of the lengths of the inner edges of $\mathbf{E}$. So we get the following estimate for the sum of the lengths of the inner edges of $\mathbf{E}$:

$$I(\mathbf{E}) \geq 2\sqrt{\pi}\Big(3 + \sqrt{1 - A_{1,4}} + \sqrt{0.15}\Big) - \tilde{p} := \ell_9 \approx 4.06365. \tag{5.127}$$

Therefore, the sum of the lengths of the inner sides of $\mathbf{E}$ minus the inner edges of S_1 (we denote them with $I(\mathbf{E} \setminus S_1)$) must be at least, by (5.125) and (5.127)

$$I(\mathbf{E} \setminus S_1) = I(\mathbf{E}) - (l_{1,2} + l_{1,3} + l_{1,4}) \geq \ell_9 - \ell_7 := \ell_{10} \approx 3.67443. \tag{5.128}$$

Finally we can conclude, because we able to give an estimate for $P(\mathbf{E})$; considering the configuration of $\mathbf{E}$, we get that

$$\begin{aligned} 11.1946 \approx \tilde{p} \geq P(\mathbf{E}) &\geq P(B_1 \cup E_3 \cup E_4 \cup E_2) + I(\mathbf{E} \setminus S_1) + l_{1,0} \\ &\overset{(5.126),(5.128)}{\geq} 2\sqrt{\pi}\Big(\sqrt{(1 - A_{1,4}) + 3}\Big) + \ell_{10} + \ell_8 \approx 11.6055. \end{aligned}$$

It is a contradiction, so the proof is completed.

$\square$

Proposition 5.53. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $\mathbf{E}$ *is not as in the case* $E)$ *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case $E)$. By Corollary 5.40, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$ thus $m(\mathbf{E}) = (1,1,1,1)$. Certainly in this configuration, the small component has three edges and without loss of generality we can assume that $\mathbf{E}$ is as in Figure 5.51.

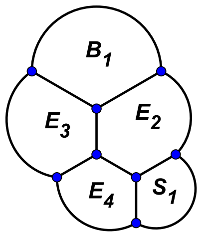

Figure 5.51: *The case* E*).*

Since S_1 is external, by Proposition 1.49, by $5)$ of Lemma 5.47, and Remark 5.48 we have that

$$p_{E_1} \geq k_7. \tag{5.129}$$

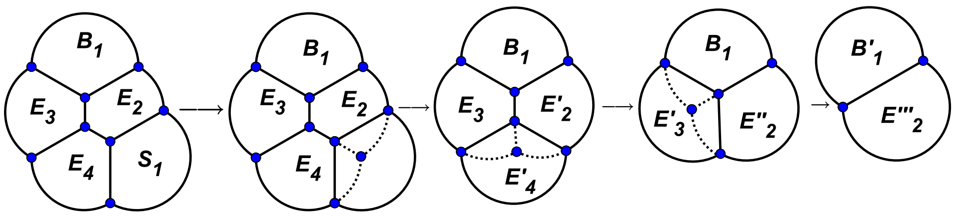

Figure 5.52: *This reduction determines an upper limit for* $\max_{k=1,2} p_{E_k}$.

Now we do a reduction of $\mathbf{E}$ until we come to a standard double bubble, that will allow us to determine an upper limit for $\max_{k=1,2} p_{E_k}$, that will be smaller than k_7. The different steps of the reduction in Figure 5.52 are given by the arrows. First we reduce only S_1, then we obtain that

$$E_2 \subseteq E_2'. \tag{5.130}$$

In the second step we reduce E_4', so we have that

$$E_2' \subseteq E_2''. \tag{5.131}$$

In the last step we reduce E_3', having that

$$\begin{aligned} B_1 &\subseteq B_1', \\ E_2'' &\subseteq E_2'''. \end{aligned} \tag{5.132}$$

At the end of these steps we have a standard double bubble of areas $|\, B_1' \,|$ and $|\, E_2''' \,|$. By (5.130), (5.131) and (5.132) it follows that

i) if $p_{E_1} \geq p_{E_2}$, then $|\, E_2''' \,| \geq |\, B_1' \,| \geq |\, B_1 \,| \geq 1 - A_{1,4}$;

ii) if $p_{E_2} \geq p_{E_1}$, then $|\, B_1' \,| \geq |\, E_2''' \,| \geq |\, E_2'' \,| \geq |\, E_2' \,| \geq |\, E_2 \,| = 1$.

Therefore, by Lemma 3.16,

$$\max\left(p_{E_1}, p_{E_2}\right) \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - A_{1,4}}} \approx 1.7337. \tag{5.133}$$

By (5.129) and (5.133) we obtain that

$$2.2216 \approx k_7 \leq p_{E_1} \leq \max_{k=1,2} p_{E_k} \leq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{1 - A_{1,4}}} \approx 1.7337.$$

It is a contradiction, so the proof is completed. □

Proposition 5.54. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $\mathbf{E}$ *is not as in the case* F) *of Figure 5.15.*

Proof. We suppose by contradiction that $\mathbf{E}$ is as in the case F). By Corollary 5.40, $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, thus $m(\mathbf{E}) = (1,1,1,1)$. We respectively call S_1 and B_1 the small and the big component of E_1. We remind that, by Remark 5.7 and (5.10) of Corollary 5.10,

$$A_{2,4} \leq |\, S_1 \,| \leq A_{1,4} < \frac{1}{3}. \tag{5.134}$$

First we determine the lower limit for the lowest pressure between p_{E_2}, p_{E_3} and p_{E_4}. In order to do this, we use the reduction method of three sided-component, described in Lemma 2.22. We proceed with the reduction, represented in Figure 5.53. The different steps of the reduction in Figure 5.53 are given by the arrows.

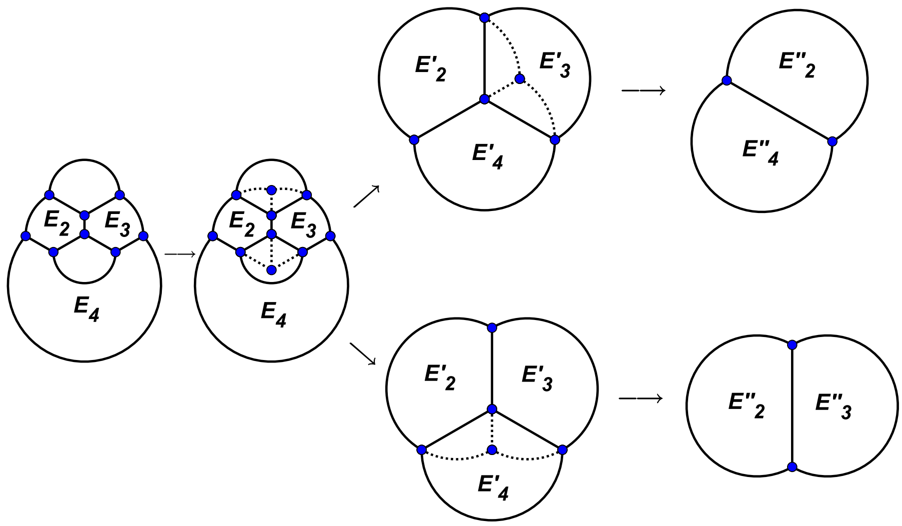

Figure 5.53: *This reduction determines a lower limit for the lowest pressure between p_{E_j} with $j = 2, 3, 4$.*

At the beginning we reduce the two components of E_1, S_1 and B_1, then we obtain that

$$\begin{aligned} E_2' \cup E_3' &\subseteq E_2 \cup E_3 \cup S_1 \cup B_1, \\ E_4' \cup E_3' &\subseteq E_4 \cup E_3 \cup S_1 \cup B_1, \\ E_2' \cup E_4' &\subseteq E_2 \cup E_4 \cup S_1 \cup B_1. \end{aligned} \tag{5.135}$$

In the second step we reduce E_3' or E_4', so if we reduce E_3' we have that

$$\begin{aligned} E_2'' &\subseteq E_2' \cup E_3', \\ E_4'' &\subseteq E_4' \cup E_3'. \end{aligned} \tag{5.136}$$

while if we reduce E_4' we find that

$$\begin{aligned} E_2'' &\subseteq E_2' \cup E_4', \\ E_3'' &\subseteq E_3' \cup E_4'. \end{aligned} \tag{5.137}$$

If in the second step we reduce E_3', at the end of these steps we have a standard double bubble of areas $|\,E_2''\,|$ and $|\,E_4''\,|$. By (5.135) and (5.136), it follows that

c) if $p_{E_4} \geq p_{E_2}$, then

$$\begin{aligned}|\,E_4''\,| \leq |\,E_2''\,| \leq |\,E_2'\,| + |\,E_3'\,| &\leq |\,E_2\,| + |\,E_3\,| + |\,S_1\,| + |\,B_1\,|\\ &\leq |\,E_2\,| + |\,E_3\,| + |\,E_1\,| \leq 3;\end{aligned}$$

d) if $p_{E_2} \geq p_{E_4}$, then

$$\begin{aligned}|\,E_2''\,| \leq |\,E_4''\,| \leq |\,E_4'\,| + |\,E_3'\,| &\leq |\,E_4\,| + |\,E_3\,| + |\,S_1\,| + |\,B_1\,|\\ &\leq |\,E_4\,| + |\,E_3\,| + |\,E_1\,| \leq 3.\end{aligned}$$

Therefore, by Lemma 3.16, we can say that

$$\min_{k=2,4} p_{E_k} \geq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{3}} := k_8 \approx 0.917861. \tag{5.138}$$

If in the second step we instead reduce E_4', at the end of these steps we get a standard double bubble of areas $|\,E_2''\,|$ and $|\,E_3''\,|$. By (5.135) and (5.137), it follows that

c) if $p_{E_3} \geq p_{E_2}$, then

$$\begin{aligned}|\,E_3''\,| \leq |\,E_2''\,| \leq |\,E_2'\,| + |\,E_4'\,| &\leq |\,E_2\,| + |\,E_4\,| + |\,S_1\,| + |\,B_1\,|\\ &\leq |\,E_2\,| + |\,E_4\,| + |\,E_1\,| \leq 3;\end{aligned}$$

d) if $p_{E_2} \geq p_{E_3}$, then

$$\begin{aligned}|\,E_2''\,| \leq |\,E_3''\,| \leq |\,E_3'\,| + |\,E_4'\,| &\leq |\,E_3\,| + |\,E_4\,| + |\,S_1\,| + |\,B_1\,|\\ &\leq |\,E_3\,| + |\,E_4\,| + |\,E_1\,| \leq 3.\end{aligned}$$

Therefore, by Lemma 3.16, we can say that

$$\min_{k=2,3} p_{E_k} \geq \sqrt{\frac{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}{3}} k_8 \approx 0.917861. \tag{5.139}$$

So, by (5.138) and (5.139) we have that

$$\min_{k=2,3,4} p_{E_k} \geq k_8. \tag{5.140}$$

Now we will find an estimate for the pressure of E_1. If S_1 is inner, then from $4)$ of Lemma 5.47 and by (5.140), we get that

$$p_{E_1} \geq k_7 + k_8 \approx 3.13946. \tag{5.141}$$

Therefore, by Corollary 1.47, we have that the perimeter of $\mathbf{E}$ is at least

$$P(\mathbf{E}) = 2\sum_{k=1}^{4} p_{E_k} \geq 2(k_7 + k_8) + 6k_8 \approx 11.7861.$$

This is a contradiction, since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$.

Therefore S_1 is external and B_1 is inner. Thus, by $5)$ of Lemma 5.47, by Remark 5.48 and (5.140), we have the following estimate for p_{E_1}

$$p_{E_1} \geq k_7 + \left(1 - \frac{\tilde{p}}{4\sqrt{\pi}} \cdot \frac{\sqrt{A_{1,4}}}{(1 - A_{1,4})}\right) k_8 := k_{11} \approx 2.45191. \tag{5.142}$$

We find an estimate for $\max\limits_{k>1} p_{E_k}$; in order to do this we use the fact that E_1 has two three-sided components (so note that their turning angle is π), one small and the other big. Therefore, denoted by $L_{1,2}$, $L_{1,3}$, $L_{1,4}$ the lengths of the edges of B_1 in common with E_2, E_3 and E_4 respectively and let $l_{1,2}$, $l_{1,3}$ and $l_{1,0}$ be the lengths of the edges of S_1 respectively in common with E_2, E_3 and E_0, we have that

$$\sum_{k=0,2,3} l_{1,k}(p_{E_1} - p_{E_k}) = \pi = \sum_{k=2,3,4} L_{1,k}(p_{E_1} - p_{E_k}) \geq (p_{E_1} - \max_{k>1} p_{E_k}) P(B_1).$$

Thus we obtain that (by Proposition 1.49 each pressure is non negative)

$$\begin{aligned} p_{E_1} P(S_1) &\geq (p_{E_1} - \max_{k>1} p_{E_k}) P(B_1) + l_{1,2} p_{E_2} + l_{1,3} p_{E_3} \\ &\geq (p_{E_1} - \max_{k>1} p_{E_k}) P(B_1). \end{aligned}$$

Dividing by p_{E_1} and $P(B_1)$ (recall that p_{E_1} is positive by (5.142)), we obtain that

$$\max_{k>1} p_{E_k} \geq p_{E_1}\left(1 - \frac{P(S_1)}{P(B_1)}\right). \tag{5.143}$$

By (5.134), by the isoperimetric inequality and by the minimality of $\mathbf{E}$ we get that (remind that $P(S_1) \leq 2P(\mathbf{E}) - (P(B_1) + P(E_2) + P(E_3) + P(E_4) + P(E_0))$ and $1 - A_{1,4} > 0$ by (5.134)) $P(S_1) \leq 2\tilde{p} - 2\sqrt{\pi} \cdot (5 + \sqrt{1 - A_{1,4}})$ and $P(B_1) \geq 2\sqrt{\pi(1 - A_{1,4})}$. Thus by (5.143), we find that

$$\max_{k>1} p_{E_k} \geq p_{E_1} \left(1 - \frac{2\tilde{p} - 2\sqrt{\pi} \cdot (5 + \sqrt{1 - A_{1,4}})}{2\sqrt{\pi(1 - A_{1,4})}}\right).$$

The quantity $1 - \frac{2\tilde{p} - 2\sqrt{\pi} \cdot (5 + \sqrt{1 - A_{1,4}})}{2\sqrt{\pi(1 - A_{1,4})}} \approx 0.564974$ is positive, therefore, since $p_{E_1} \geq k_1$

$$\max_{k>1} p_{E_k} \geq k_{11} \cdot \left(1 - \frac{2\tilde{p} - 2\sqrt{\pi} \cdot (5 + \sqrt{1 - A_{1,4}})}{2\sqrt{\pi(1 - A_{1,4})}}\right) := k_{12} \approx 1.38527. \tag{5.144}$$

So, by (5.140), (5.142) and (5.144) and Corollary 1.47, we have the following estimate for $P(\mathbf{E})$:

$$P(\mathbf{E}) = 2\sum_{i=1}^{4} p_{E_i} \geq 2\big(p_{E_1} + \max_{k>1} p_{E_k} + 2\min_{k>1} p_{E_k}\big) \geq 2k_{11} + 2k_{12} + 4k_8 \approx 11.3458.$$

This contradicts the minimality of $\mathbf{E}$, since $P(\mathbf{E}) \leq \tilde{p} \approx 11.1946$.

□

Theorem 5.55. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*, then* $I_{\mathbf{E}}$ *is not* $(1,0,0,0)$.

Proof. We proceed by contradiction and we suppose that $I_{\mathbf{E}} = (1,0,0,0)$, then by Propositions 5.46, 5.49, 5.50, 5.51, 5.52, 5.53 and 5.54 we come a contradiction, so the statement is true. □

Theorem 5.56. *Let* $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$*. Then* $\mathbf{E}$ *is standard. In particular if* $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$*, then* $\mathbf{E}$ *is standard.*

Proof. The proof is immediate. Let $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$; we suppose by contradiction that $\mathbf{E}$ is not standard. Its possible connection types $I_{\mathbf{E}}$ are illustrated in Remark 5.12. By Theorems 5.15, 5.23, 5.39 and 5.55 we come to contradict that $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, thus $\mathbf{E}$ is standard.

By Remark Remark 5.2, we have that if $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, then $\mathbf{E}$ is standard. □

Remark 5.57. $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ if and only if $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, indeed if $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$, then $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, by Corollary 5.16.

If $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, then $\mathbf{E} \in \mathcal{M}^*_{2,4}(1,1,1,1)$ by Remark 1.8 and by Corollary 5.16 (recall that $\mathcal{M}^*_{2,4}(1,1,1,1) \neq \emptyset$ by Corollary 1.13).

Remark 5.58. We explicitly note that the exterior region E_0 of $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$ is connected by Proposition 1.49 and by Remark 5.57.

Now let us investigate the possible topologies of $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.

Lemma 5.59. *Let $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, then a region E_i ($i = 1, \ldots, 4$) has:*

i) *three edges if it is inner;*

ii) *at most four edges if it is external.*

Proof. First of all, we know that any region E_i ($i = 1, \ldots, 4$) of $\mathbf{E}$ has at least three edges and it is connected by Corollary 1.35 and by Theorem 5.56 respectively. Since $\mathbf{E}$ is a minimum, by Proposition 1.33 and $\mathbf{E}$ is standard (see Theorem 5.56), then, C has three edges and at most four edges if E_i is inner and external respectively. □

Lemma 5.60. *Let $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, then $\mathbf{E}$ has six vertices and nine edges.*

Proof. Let v, e and c be the numbers of the vertices, of the edges and of the connected components of $\mathbf{E}$ respectively, then, by the Euler's formula, one has that $v-e+c = 2$. Since $\mathbf{E}$ is a minimum, each vertex of $\mathbf{E}$ is is a meeting point of exactly three edges (see Theorem 1.10), thus $3v = 2e$ (note that each edge has two vertices). Furthermore, by Theorem 5.56 $\mathbf{E}$ is standard (i.e each region is connected), therefore $c = 5$. Solving the following linear system

$$\begin{cases} v - e = -3 \\ 3v = 2e, \end{cases}$$

we find the statement. □

Lemma 5.61. *Let $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$, then $3 \leq v(E_0) \leq 4$ and $3 \leq e(E_0) \leq 4$, where $v(E_0)$ and $e(E_0)$ denote the number of the vertices which belong to E_0 and the number of the edges of E_0 respectively.*

Proof. By Corollary 5.9 and Remark 5.57 we know that there is at most one inner big component. Moreover, by Theorem 5.56, $I_{\mathbf{E}} = (0,0,0,0)$, so there are at least three external bounded components. Since $\mathbf{E}$ is a minimum, by Proposition 1.33, the result follows. $\square$

Theorem 5.62. *Let* $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$*, then the topology* $\mathbf{E}$ *is one of the two topologies represented in Figure 5.54.*

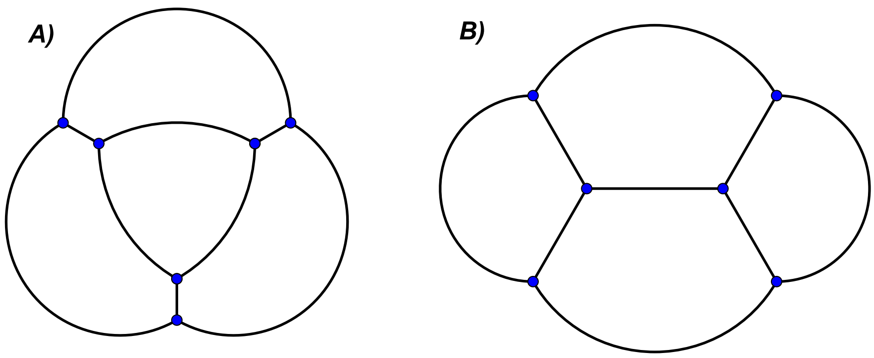

Figure 5.54: *The possible topologies for* $\mathbf{E} \in \mathcal{M}_{2,4}(1,1,1,1)$.

Proof. By Corollary 5.9, Theorem 5.56 and by Remark 5.57, there is at most one inner connected region E_i, thus, we divide the proof in two parts: in the first there is a inner connected region E_i and in the second all connected regions are external.

Part I. There is a inner connected region E_i, therefore, by Lemma 5.59, E_j has three internal vertices. Denoting by $v(\mathbf{E})$ and $v(E_0)$ the number of the vertices of $\mathbf{E}$ and of E_0 respectively, since $v(\mathbf{E}) = 6$ and $v(E_0) \geq 3$ (see Lemma 5.60 and Lemma 5.43 respectively), there are other three external vertices v_1, v_2 and v_3 and there are no other internal vertices. So, we are in the situation of Figure 5.55. Since an edge leaving the internal region can not go to another vertex of the internal region, each internal vertex must be linked to one external vertex v_1, v_2 and v_3. So, we obtain the topology $A)$ of Figure 5.54.

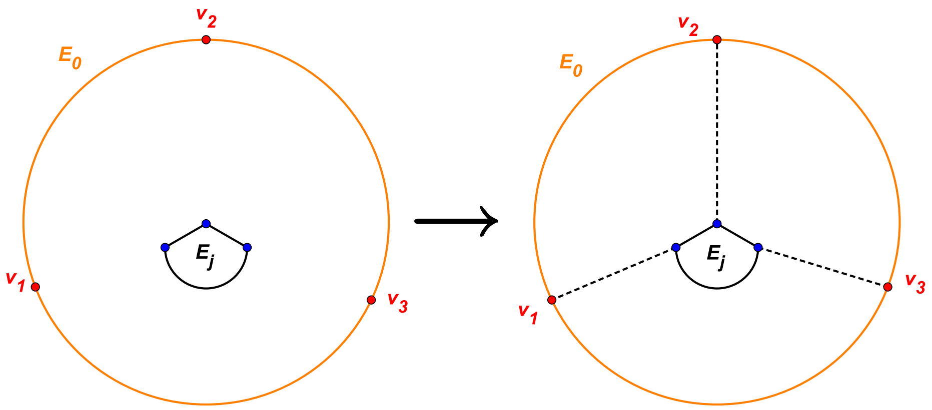

Figure 5.55: $\mathbf{E}$ *has one inner connected region* E_j*, thus* E_j *has three edges. So, since* $v(\mathbf{E}) = 6$ *and* $v(E_0) \geq 3$*, there are only other three external vertices* v_1*,* v_2 *and* v_3*. Since an edge leaving the internal region can not go to another vertex of the internal region, each internal vertex must be linked to one external vertex* v_1*,* v_2 *and* v_3*.*

Part II. Here all the connected regions E_i are external, therefore, $v(E_0) \geq 4$. So, since $v(\mathbf{E}) = 6$ and $v(E_0) \leq 4$, there are other two inner vertices v_1 and v_2. Furthermore $e(E_0) = 4$ because, by Lemma 5.61 $e(E_0) \leq 4$ and, since each connected region E_j is external, $e(E_0) \geq 4$.

First of all we say that v_1 and v_2 are linked, otherwise we would have at least $3 + 3 = 6$ internal edges (they are the leaving edges from the vertices v_1 and v_2 respectively) and 4 external edges (they are the edges of E_0). But we know that the number of the edges of $\mathbf{E}$ is 9 by Lemma 5.60, thus, we obtain a contraction.

So, we are in the situation of Figure 5.56 where, since the edges of $\mathbf{E}$ can not intersect and up to rotate the edge which links the vertices v_1 and v_2, we have only one way to link the inner vertices v_1 and v_2 with the external vertices. Therefore, we obtain the topology B) of Figure 5.54.

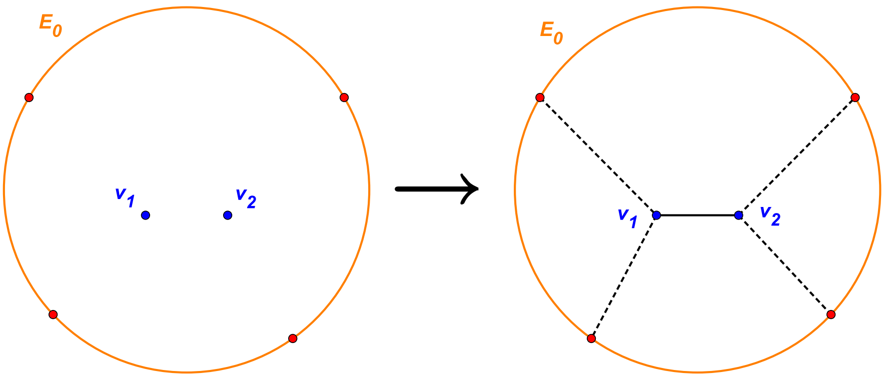

Figure 5.56: *Each connected region E_j is external, then $e(E_0) = v(E_0) = 4$. So, since $v(\mathbf{E}) = 6$, there are another two inner vertices v_1 and v_2, which must be connected. Since the edges of* $\mathbf{E}$ *can not be intersect and up to rotate the edge which links the inner vertices v_1 and v_2, there is only one way to link the vertices v_1 and v_2 with the external vertices.*

$\square$

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