ANALISI MATEMATICA B
LEZIONE 15

FORME INDETERMINATE

Sono un non-concetto.

Teorema se $f(x) \to \ell$, $g(x) \to m$

allora
$$f(x)+g(x) \to \ell+m$$
$$f(x)\cdot g(x) \to \ell\cdot m$$
$$f(x)-g(x) \to \ell-m$$
$$\frac{f(x)}{g(x)} \to \frac{\ell}{m} \quad \text{se } m\neq 0$$

Se h è continua:
$$\boxed{h(f(x)) \to h(\ell)}$$

Tutto questo se non siamo di fronte ad una forma indeterminata:

non so il segno

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ (+\infty)-(+\infty),\ (\pm\infty)\cdot 0,\ \boxed{\frac{\pm\infty}{0},\ \frac{1}{0}}$$

Nel teorema non c'è $\boxed{f(x)^{g(x)}}$

$$1^{\infty} \quad 0^{0}$$

[(c'è $a^{f(x)}$, $[f(x)]^{\alpha}$ perché sono funzioni composte).]

$$\boxed{e^{g(x)\cdot \ln f(x)}}$$

Esempio

$$\lim_{x\to+\infty} \frac{\frac{1}{x}+1}{\sqrt{2^x}} = \frac{\frac{1}{+\infty}+1}{\sqrt{2^{+\infty}}} = \frac{1}{+\infty} = 0$$

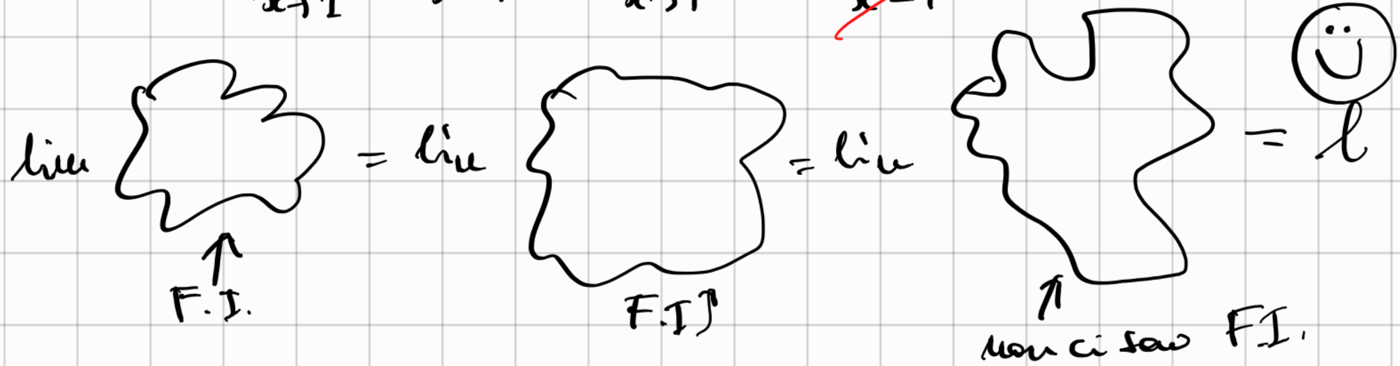
non ci sono forme indeterminate

Esempio

⚠ NON si può porre il limite solo a una parte dell'espressione.

ES
$$\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \neq \lim_{n\to+\infty}(1+0)^n = 1}$$
No!

ES
$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$



ES
$$\lim_{x\to+\infty}\frac{(x^2+1)\cdot(x-1)}{x^3+1} = \lim_{x\to+\infty}\frac{x^3-x^2+x-1}{x^3+1}$$

$$= \lim_{x\to+\infty}\frac{\cancel{x^3}\cdot\left[1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right]}{\cancel{x^3}\left[1+\frac{1}{x^3}\right]}$$

$$= \lim_{x\to+\infty}\frac{1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}}{1+\frac{1}{x^3}} = \frac{1}{1} = 1$$

Notazione preferita: per $\boxed{x\to+\infty}$

$$\frac{(x^2+1)(x-1)}{x^3+1} = \frac{x^3-x^2+x-1}{x^3+1} = \frac{\cancel{x^3}\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}{\cancel{x^3}\left(1+\frac{1}{x^3}\right)} =$$

$$= \frac{1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}}{1+\frac{1}{x^3}} \longrightarrow \frac{1}{1} = 1.$$

sono valide in un intorno di $+\infty$

Esempio

$$\lim_{x\to+\infty} \frac{x^2-x}{x^2} = \lim_{x\to+\infty} \frac{x^2}{x^2}$$

VERO MA PERCHE'?

Ricordiamo:

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$$

$$\lim_{x\to0}(1+x)^{\frac{1}{x}} = e$$

$\downarrow \log_a$

$$\frac{1}{x}\cdot\log_a(1+x) \to \log_a e$$

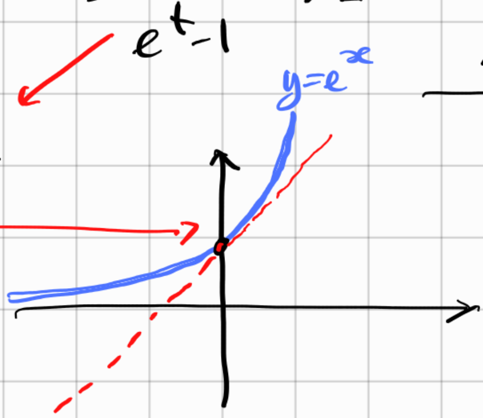
$$\ln = \log_e$$

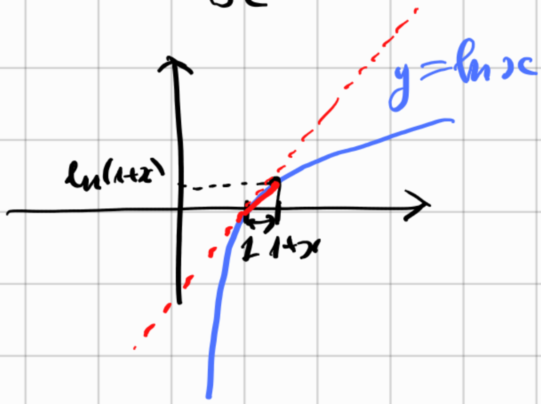
$$\lim_{x\to0} \boxed{\frac{\ln(1+x)}{x}} = 1$$

$x = e^t - 1$

$$\frac{\ln(e^t)}{e^t-1} = \frac{t}{e^t-1} \longrightarrow 1$$

$$\lim_{x\to0} \boxed{\frac{e^x-1}{x}} = 1$$

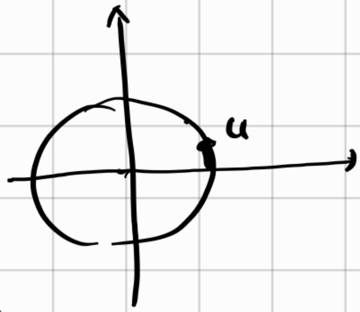




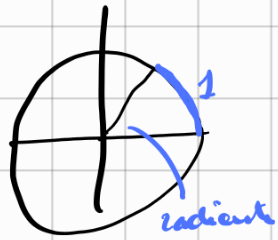
In maniera simile, ma più complicata, si dimostra che

$$\lim_{x\to 0} \boxed{\frac{\sin x}{x}} = c \qquad c\in\mathbb{R}$$

velocità del moto circolare uniforme che avete scelto



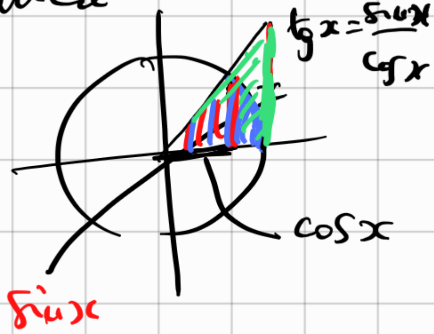
Se vogliamo avere velocità 1 la scelta dell'unità di misura è il radiante.



Con questa scelta il periodo è $\tau = 2\pi$ ← definizione di π.

[2π è la lunghezza della circonferenza unitaria]

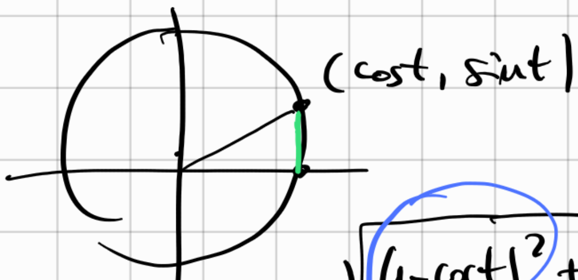
Dando per scontata l'interpretazione geometrica la dimostrazione è più semplice.



$$\sin x \leq x \leq \operatorname{tg} x$$

$$\begin{array}{ccccc} 1 & \leq & \dfrac{x}{\sin x} & \leq & \dfrac{1}{\cos x} \\ \| & & \downarrow & & \downarrow \quad \text{per } x\to 0 \\ 1 & & 1 & & 1 \end{array}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$



$$\frac{\sqrt{(1-\cos t)^2 + \sin^2 t}}{t} = \sqrt{\left(\frac{1-\cos t}{t}\right)^2 + \left(\frac{\sin t}{t}\right)^2} \longrightarrow \sqrt{0^2+1^2} = 1.$$

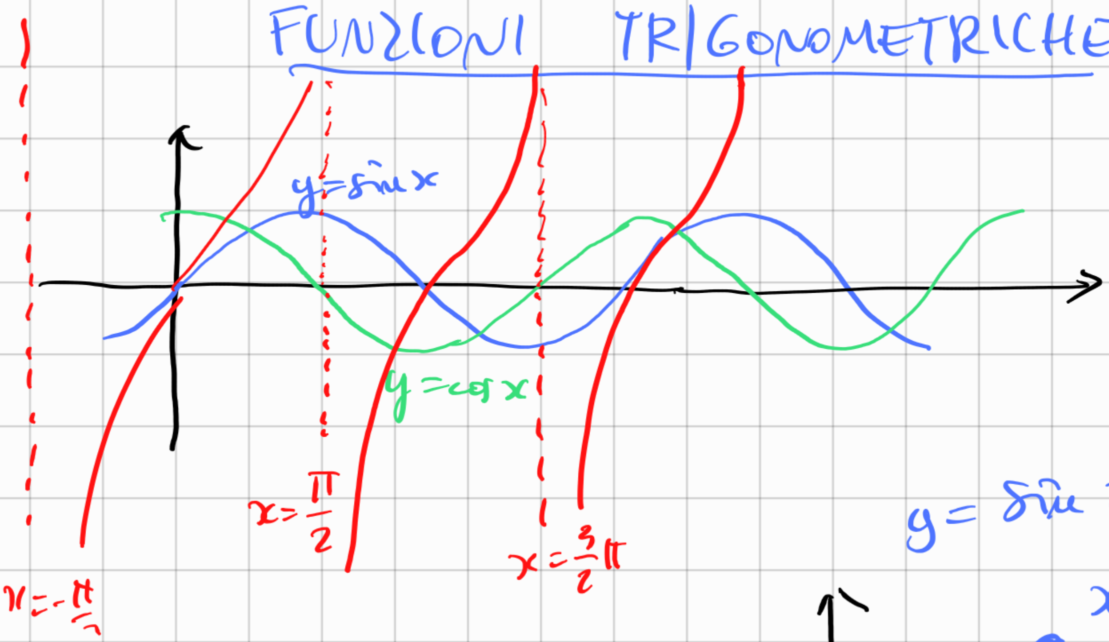
per $t\to 0$

$$\frac{1-\cos x}{x} = \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$\xrightarrow[\text{per } x \to 0]{} 1 \cdot \frac{0}{2} = 0 .$$

FUNZIONI TRIGONOMETRICHE

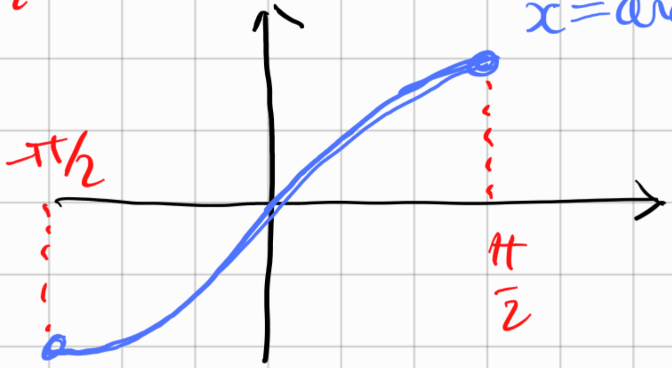
$tg\, x = \frac{\sin x}{\cos x}$



INVERSE

$y = \sin x$

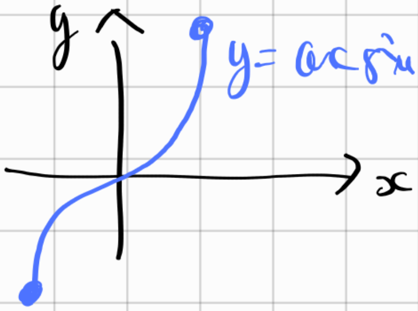
$x = \arcsin y$

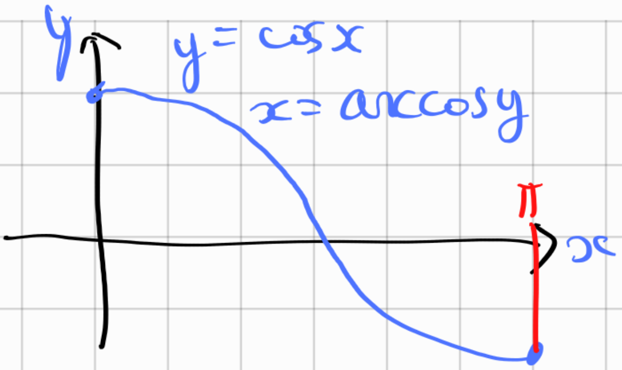


$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ è bigettiva

la funzione inversa si chiama arcsin

$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

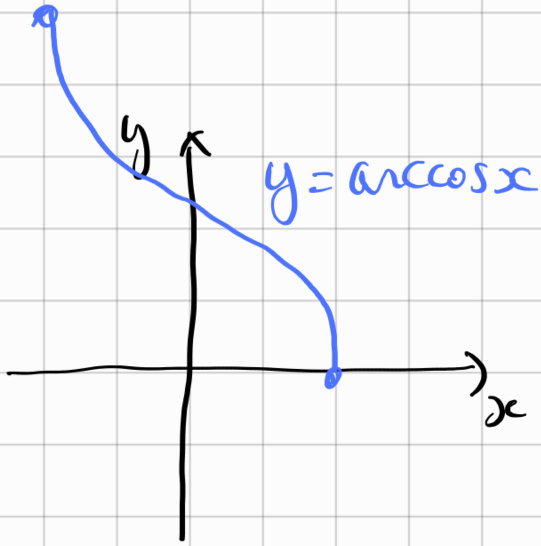




$\cos x : [0, \pi] \to [-1, 1]$

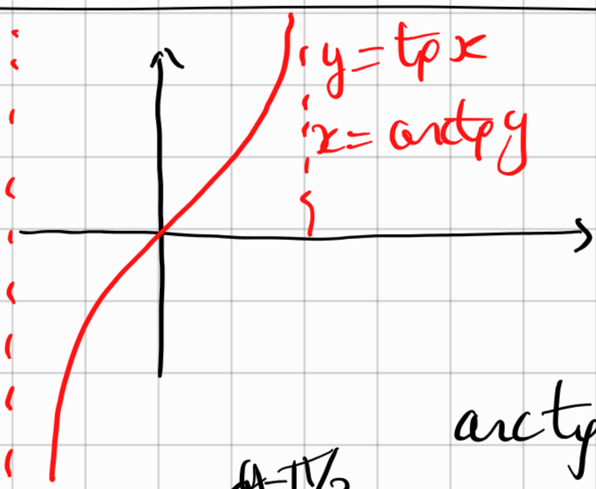
è invertibile. L'inversa si chiama:

$\arccos : [-1, 1] \to [0, \pi]$



$\arccos x = y$

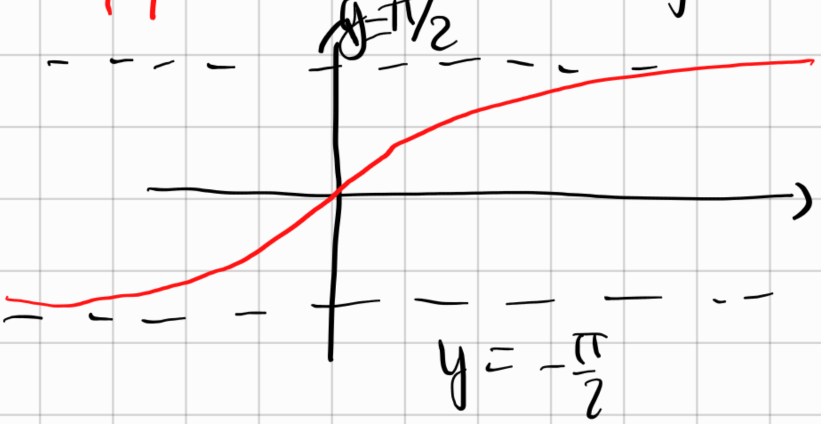
y tale che $\cos y = x$.



$\operatorname{tg} x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

è invertibile.

$\operatorname{arctg} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



ESPONENZIALE COMPLESSO

$z = x + iy$

$x, y \in \mathbb{R}$

$$e^{x+iy} = e^x \cdot \varphi(y) = e^x \cdot (\cos y + i \,\mathrm{sen}\, y)$$

$$\boxed{e^{iy} = \cos y + i \,\mathrm{sen}\, y}$$

$$e^{z+w} = e^z \cdot e^w$$

$$\lim_{z\to 0} \frac{e^z - 1}{z} = 1$$

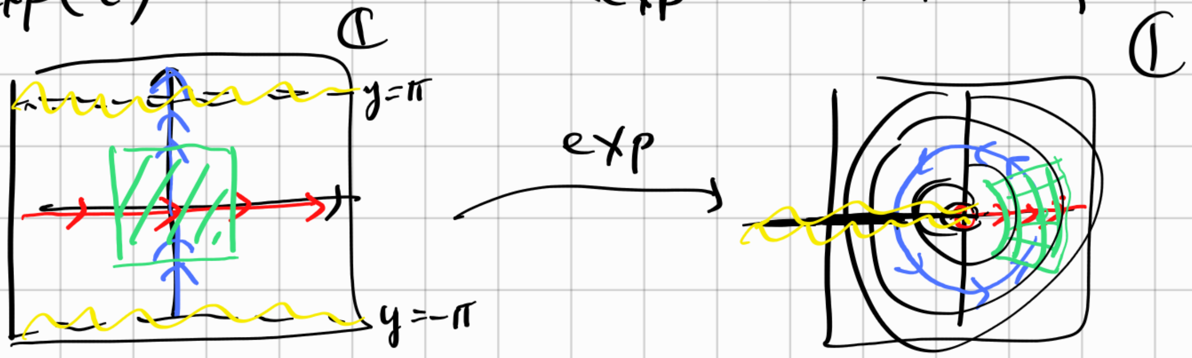
$z = x \in \mathbb{R}$ $\qquad \dfrac{e^x - 1}{x}$

$z = iy,\ y \in \mathbb{R}$ $\qquad \dfrac{e^{iy} - 1}{iy} = \dfrac{\cos y - 1 + i \,\mathrm{sen}\, y}{iy}$

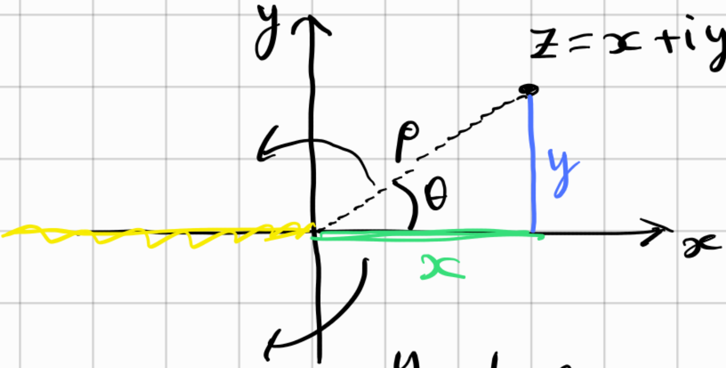
$$= \frac{1}{i}\underbrace{\frac{\cos y - 1}{y}}_{\to 0} + \underbrace{\frac{\mathrm{sen}\, y}{y}}_{\to 1} \to 1.$$

$\boxed{\text{ES } (*) \text{ dimostrare che } \dfrac{e^z - 1}{z} \to 1 \text{ per } z \to 0 \text{ in } \mathbb{C}.}$

$\exp(z) = e^z$ $\qquad \exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$



FORMA TRIGONOMETRICA e ESPONENZIALE



$z = x + iy$ ← rappresentazione cartesiana.

$\rho = |z| = \sqrt{x^2 + y^2}$

$\dfrac{y}{x} = \operatorname{tg}\theta$

$\theta = \arg z = \begin{cases} \operatorname{arctg}\dfrac{y}{x} & \text{se } x > 0 \\ ? & \text{se } x < 0,\ y > 0 \\ ? & \text{se } x < 0,\ y > 0 \end{cases}$

$\boxed{z = \rho \cdot \cos\theta + i\rho\sin\theta}$ ← rappresentazione trigonometrica di z

$(\rho;\theta)$ coordinate polari

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

$= \operatorname{atan2}(x, y)$ ← nei linguaggi di programmazione

$\boxed{z = \rho \cdot e^{i\theta}}$ ← rappresentazione esponenziale.

RADICI n-esime

$n \in \mathbb{N},\ c \in \mathbb{C}$
$c \neq 0$

Problema: risolvere $\boxed{z^n = c}$

c

$z = \rho e^{i\theta}$ $\qquad c = r e^{i\alpha}$

$z^n = (\rho e^{i\theta})^n = \rho^n \cdot (e^{i\theta})^n = \rho^n (\underbrace{e^{i\theta} \cdots e^{i\theta}}_{n}) = \rho^n (e^{\overbrace{i\theta + \dots + i\theta}^{n \text{ volte}}})$

$\downarrow$

$$= \rho^n \cdot e^{in\theta}$$

$$\rho^n e^{in\theta} = r e^{i\alpha}$$

$$(z^n = c)$$

$$|z^n| = |c|$$

$$\rho^n = r \qquad \rho = \sqrt[n]{r} \qquad e^{in\theta} = e^{i\alpha}$$

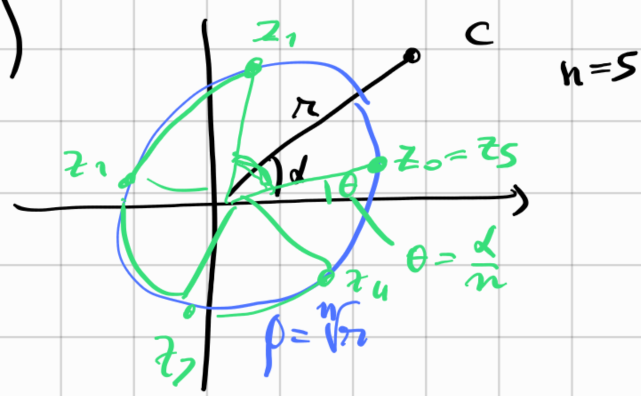
$$\Updownarrow$$

$$n\theta = \alpha + 2k\pi$$

$$k \in \mathbb{Z}$$

$$\theta = \frac{\alpha}{n} + k \cdot \frac{2\pi}{n}$$

$$z_k = \sqrt[n]{r} \cdot e^{i\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right)}$$



ho n soluzioni distinte:

$$\underbrace{z_0, z_1, \dots, z_{n-1}}$$

in quanto $z_{k+n} = z_k$.

$\square$