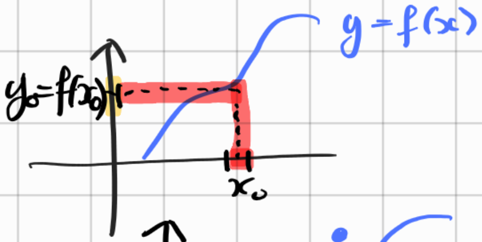
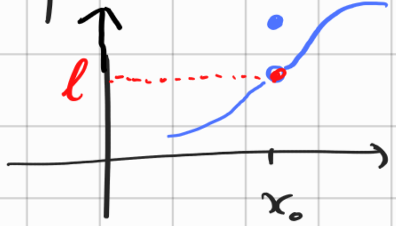
ANALISI MATEMATICA B
LEZIONE 13 — 18.10.2023

f continua:



LIMITE $\quad f: A \to \mathbb{R}$



Diremo che $f(x) \to \ell$
per $x \to x_0$ se

$$\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ \ell & \text{se } x = x_0 \end{cases}$$

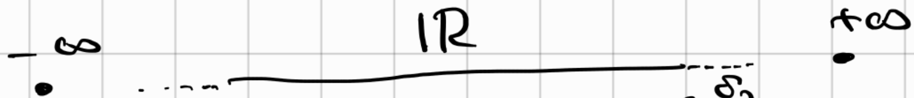
è continua in x_0.

Ovvero

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in A : |x - x_0| < \delta \Rightarrow |\tilde{f}(x) - \tilde{f}(x_0)| < \varepsilon$$
$$x \neq x_0 \qquad\qquad \tilde{f}(x_0) = \ell$$

$$\boxed{\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in A \setminus \{x_0\} : |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon}$$

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

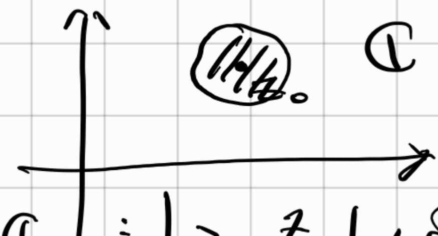


$$|x - x_0| < \delta \iff x \in B_\delta(x_0)$$

valore assoluto

palla
di raggio δ e centro x_0. $\longrightarrow$ $B_\delta(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta\}$

Lo stesso si può fare in $\mathbb{C}$:



$$B_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

modulo

Lo stesso si può fare in $\mathbb{R}^n$:

$$B_\delta(\underline{p}) = \{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{p}\| < \delta\}$$

$$\underline{x} = (x_1, x_2, \dots, x_n)$$

$$\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Continuità:

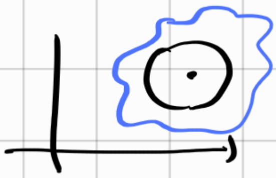
$f: A \to \mathbb{R}$

$\forall x \in A$

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$$

$$\forall \varepsilon > 0\ \exists \delta > 0 : f(A \cap B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$$

Definizione "topologica"



$$\to \mathcal{B}_{x_0} = \{ B_r(x_0) : r > 0 \}$$

intorni basilari di x_0.

$$\left[\ \mathcal{U}_{x_0} = \{ U : \exists B \in \mathcal{B}_{x_0} : U \supseteq B \} \ \right]$$

U intorno.

$f: A \to \mathbb{R}$ è continua in x_0 se e solo se:

$$\forall U \in \mathcal{B}_{f(x_0)}\ \exists V \in \mathcal{B}_{x_0} : f(A \cap V) \subseteq U.$$

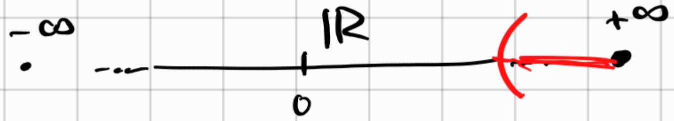
"U intorno di $f(x_0)$" "V intorno di x_0"

$f(x) \to l$ per $x \to x_0$ se e solo se:

$\forall U$ intorno di l $\exists V$ intorno di x_0 tale che $f(A \cap V \setminus \{x_0\}) \subseteq U$.

INTORNI DI ∞

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

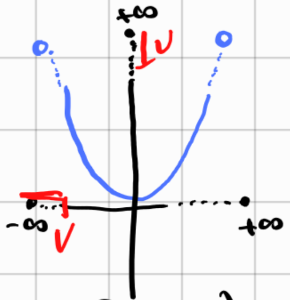


$\mathcal{B}_{+\infty} = \{ (M, +\infty] : M \in \mathbb{R} \}$ ← intorni basilari

$\mathcal{B}_{-\infty} = \{ [-\infty, M) : M \in \mathbb{R} \}$

__Es__ $f(x) = x^2$ $\qquad f(x) \to +\infty$ per $x \to -\infty$

$f : \mathbb{R} \to \mathbb{R}$



Verifica: $\forall U \in \mathcal{B}_{+\infty}\ \exists V \in \mathcal{B}_{-\infty} : f(V) \subseteq U$

$\forall M \in \mathbb{R}\ \exists V \in \mathcal{B}_{-\infty} : f(V) \subseteq (M, +\infty]$

$\forall M \in \mathbb{R}\ \exists N \in \mathbb{R} : f(A \cap [-\infty, N) \setminus \{-\infty\}) \subseteq (M, +\infty]$

$\forall M \in \mathbb{R}\ \exists N \in \mathbb{R} : f((-\infty, N)) \subseteq (M, +\infty]$

$\forall M \in \mathbb{R}\ \exists N \in \mathbb{R} : \forall x \in (-\infty, N) : f(x) \in (M, +\infty]$

$$\boxed{\forall M \in \mathbb{R}\ \exists N \in \mathbb{R} : \forall x : x < N \Rightarrow f(x) > M.}$$

$f(x) \to +\infty$ per $x \to -\infty$

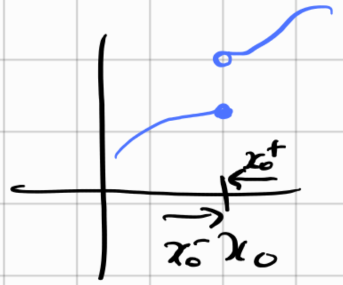
__Esercizio__ Scrivere la definizione di limite:

__$l \in \mathbb{R}$__ $\qquad f(x) \to l$ per $x \to +\infty$

LIMITI UNILATERALI

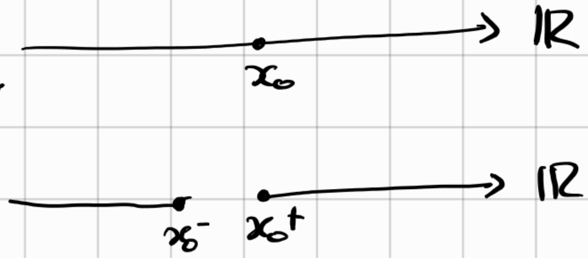
$f(x) \to \ell$ per $x \to x_0^+$

$(x \to x_0^-)$



$\mathcal{B}_{x_0^+} = \{ [x_0, x_0+\delta) : \delta > 0 \}$

$\mathcal{B}_{x_0^-} = \{ (x_0-\delta, x_0] : \delta > 0 \}$



$$\left[f(x) \to \begin{matrix} \ell^+ \\ \ell^- \end{matrix} \quad \text{per } x \to x_0 \right]$$

Esercizio Scrivere la definizione di limite $\ell \in \mathbb{R}$

$f : A \to \mathbb{R}$

$f(x) \to \ell$ per $x \to x_0^-$

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : x_0 - \delta < x < x_0 \Rightarrow |f(x) - \ell| < \varepsilon$$

Esercizio 🏠 Scrivere la definizione di limite:

$f(x) \to -\infty$ per $x \to x_0^+$

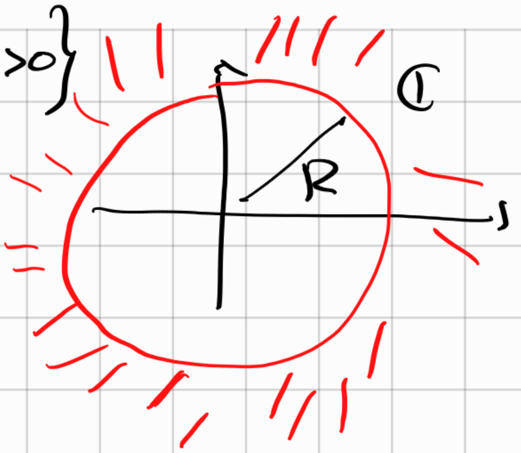
Moralmente: $+\infty = \infty^-$

$-\infty = \infty^+$

Su $\mathbb{C}$ (e su $\mathbb{R}^n$)

$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$\mathcal{B}_\infty = \{ \{ z \in \overline{\mathbb{C}} : |z| > R \} : R > 0 \}$$



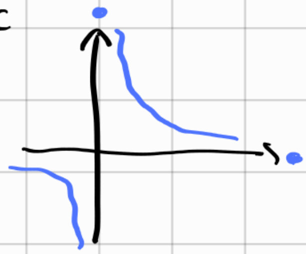
Esercizio $\boxed{1}$ $\quad f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$f(z) = \frac{1}{z} \qquad \frac{1}{0} = \infty$$
$$\frac{1}{\infty} = 0$$

f è continua!

Esercizio $\quad f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \qquad f(x) = \frac{1}{x}$

si estende con continuità su $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$

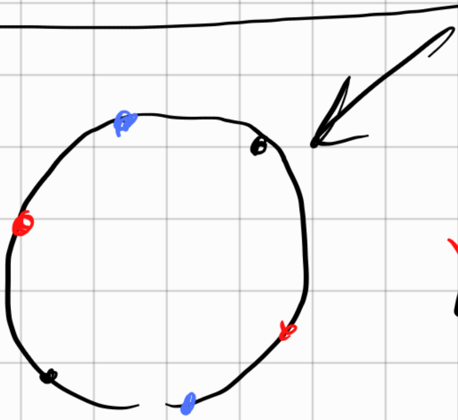


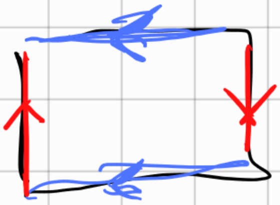
$$\left[\frac{1}{x} \to \infty \quad \text{per } x \to 0 \right]$$

$$\frac{1}{x} \to +\infty \quad \text{per } x \to 0^+$$

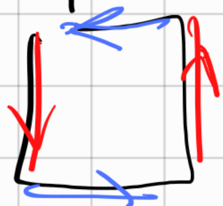
$$\frac{1}{x} \to -\infty \quad \text{per } x \to 0^-$$

Toro





Spazio proiettivo



[VIHART su youtube]

Nastro di Möbius

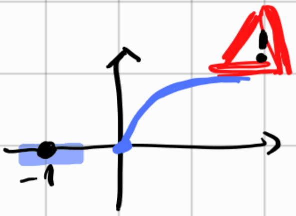
Bottiglia di Klein

Teorema (unicità del limite) Se x_0 è un punto di accumulazione per A, $f: A \to \mathbb{R}$, se $f(x) \to \ell_1$ e $f(x) \to \ell_2$ per $x \to x_0$

allora $\ell_1 = \ell_2$.

Esempio $\sqrt{x} \to 42$ per $x \to -1$

$A = [0, +\infty)$



Def x_0 è punto di accumulazione per A

se ogni intorno di x_0 interseca $A \setminus \{x_0\}$

$\forall U \in \mathcal{B}_{x_0} : A \cap U \setminus \{x_0\} \neq \emptyset$

Operatore di limite se x_0 è punto di accumulazione di A e $f: A \to \mathbb{R}$ definiamo

$$\lim_{x \to x_0} f(x)$$

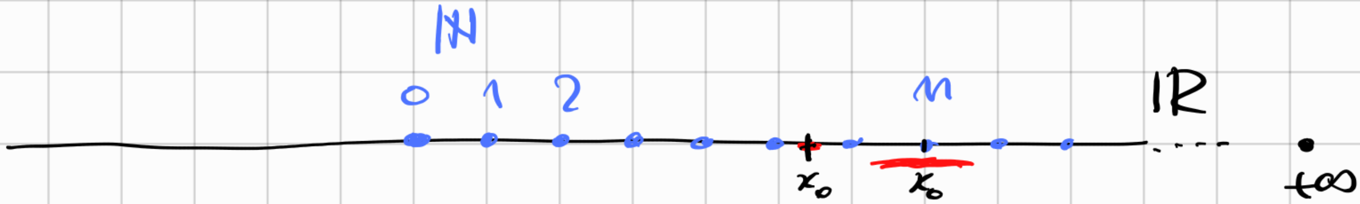
l'unico ℓ, se esiste, tale che $f(x) \to \ell$ per $x \to x_0$.

Osservazione $A = \mathbb{N}$ $\quad f: \mathbb{N} \to \mathbb{R}$

$f(0), f(1), f(2), \ldots\ldots$

si chiama **successione** $\quad a_0, a_1, a_2, \ldots\ldots$

Es $a_n = n^2$, $\quad$ Es: $a_n = (-1)^n$, $\quad$ Es $a_n = \sin n$



$x_0 \in \mathbb{R}$ non è mai punto di accumulazione per $\mathbb{N}$

$+\infty$ è l'unico punto di accumulazione di $\mathbb{N}$.

$$\lim_{n \to +\infty} a_n = \ell, \qquad \ell \in \overline{\mathbb{R}}$$

Esercizio $\quad \lim_{x \to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x \to +\infty} 2^x = +\infty.$