Curve nello spazio $\mathbb{R}^d$

Curva parametrizzata (parametrica)

$$\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d$$

intervallo

$$\gamma(t) = (x_1(t), \dots, x_d(t))$$

$$t \in I$$

$$\gamma \in C^k(I) \iff x_j \in C^k(I)$$

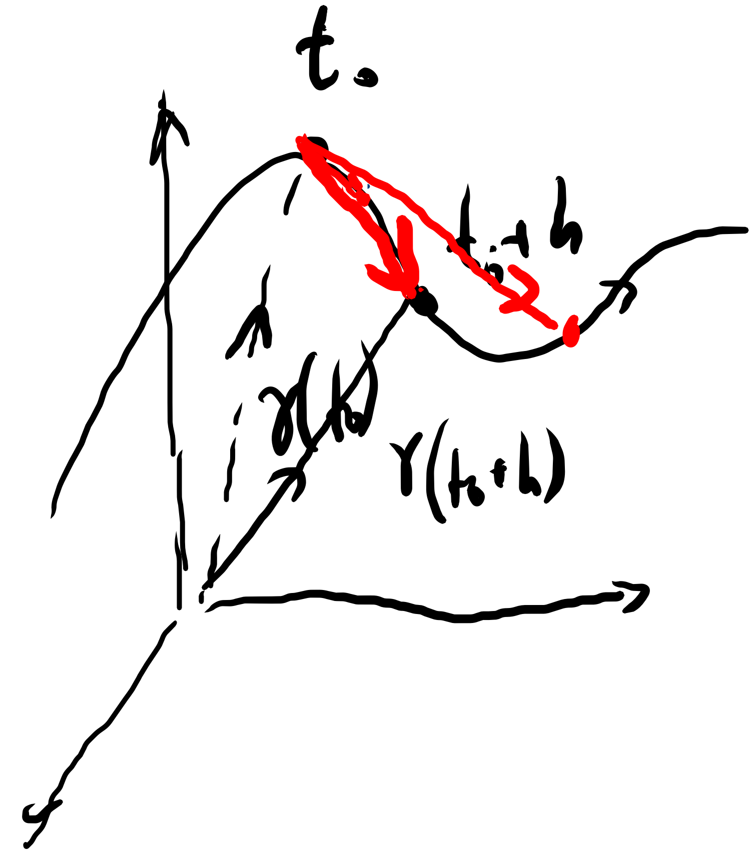
$$\dot\gamma(t_0) = \lim_{h\to 0} \boxed{\frac{\gamma(t_0+h)-\gamma(t_0)}{h}} =$$

velocità media su $[t_0, t_0+h]$ (se $h>0$) o $[t_0+h, t_0]$, se $h<0$

velocità istantanea all'istante t_0

$$= \left(\lim_{h\to 0}\frac{x_1(t_0+h)-x_1(t_0)}{h}, \dots, \lim_{h\to 0}\frac{x_d(t_0+h)-x_d(t_0)}{h}\right)$$

$$= \left(\dot x_1(t_0), \dots, \dot x_d(t_0)\right)$$



(!) $\dot\gamma(t_0)$ è tangente alla curva nel punto $\gamma(t_0)$, ovvero

$$\gamma(t_0+h) = \underbrace{\gamma(t_0)+\dot\gamma(t_0)\cdot h} + o(|h|)$$

per $h\to 0$.

Se $\dot\gamma(t_0) \neq 0$, allora nell'intorno piccolo di $\gamma(t_0)$ la curva $\gamma(\cdot)$ è "quasi" una retta (retta tangente) di equazione

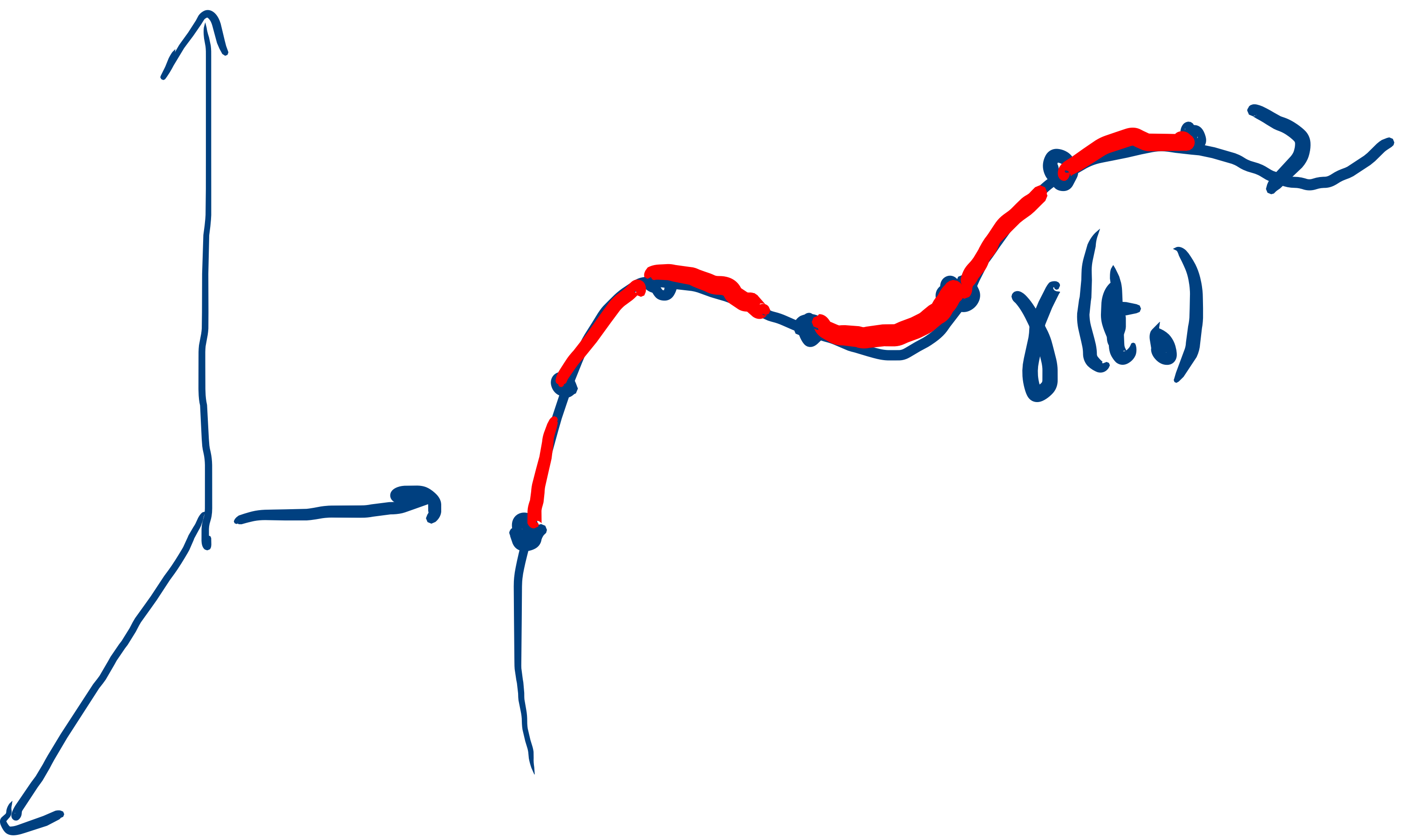
$$\boxed{r(t) = \gamma(t_0) + \dot\gamma(t_0)\cdot(t-t_0)}$$

<u>Def.</u> $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$ si chiama **regolare**, se $\gamma \in C^1(I)$ e $\dot\gamma(t) \neq 0 \quad \forall t \in I$.

$|\dot{\gamma}(t)|$ — "velocità assoluta"

$[t_0, t_0+h]$ $\Delta s = |\dot{\gamma}(t_0)|\, h$ (se $h>0$)

($h \ll 1$).



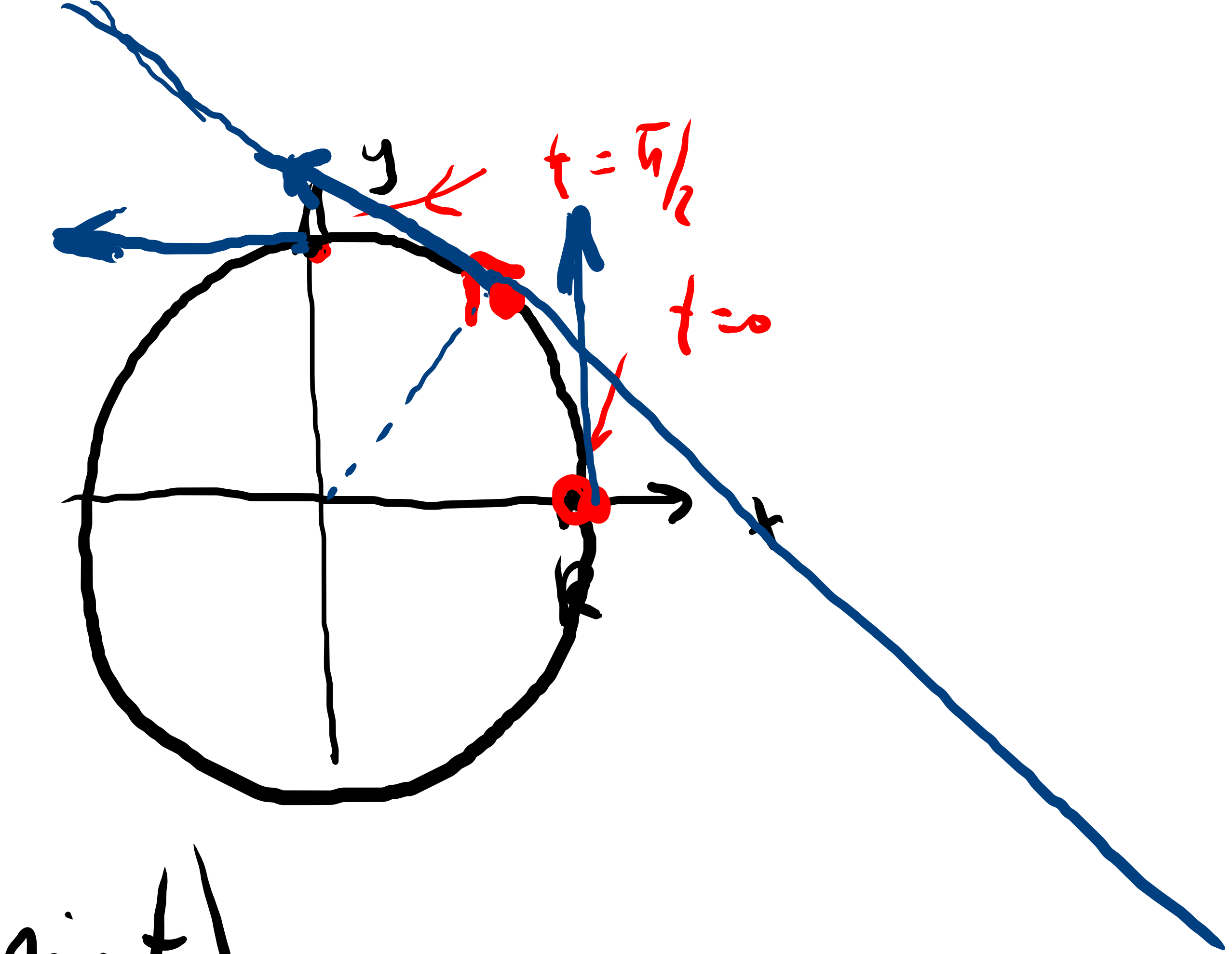
lunghezza d'arco

$$\ell(\gamma) = \int_I |\dot{\gamma}(t)|\, dt$$

Esempi: 1°

$$\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \end{cases}$$

$$t \in [0, 2\pi]$$



$$\gamma(t) = (x(t), y(t)) = (R\cos t, R\sin t)$$

$$\dot\gamma(t) = (-R\sin t, R\cos t)$$

$$|\dot\gamma(t)| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = R \neq 0$$

curva regolare

In particolare $\dot\gamma(0) = (0, R)$

$$\dot\gamma\left(\frac{\pi}{4}\right) = \left(-\frac{R}{\sqrt2}, \frac{R}{\sqrt2}\right)$$

$$\dot\gamma\left(\frac{\pi}{2}\right) = (-R, 0)$$

la retta tangente alla circonferenza nel punto

$$\gamma\left(\frac{\pi}{4}\right) = \left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right)$$

è

$$r(t) = \left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right) + \left(-\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right) \cdot (t - t_0)$$

$$= \left(\frac{R}{\sqrt{2}} - \frac{R}{\sqrt{2}}(t-t_0), \frac{R}{\sqrt{2}} + \frac{R}{\sqrt{2}}(t-t_0)\right)$$

ovvero
$$\begin{cases} x(t) = \frac{R}{\sqrt{2}}(1 - t + t_0) \\ y(t) = \frac{R}{\sqrt{2}}(1 + t - t_0) \end{cases}$$

lunghezza d'arco (di tutta la circonferenza)

$$\dot{\gamma}(t) = (-R\sin t, R\cos t)$$

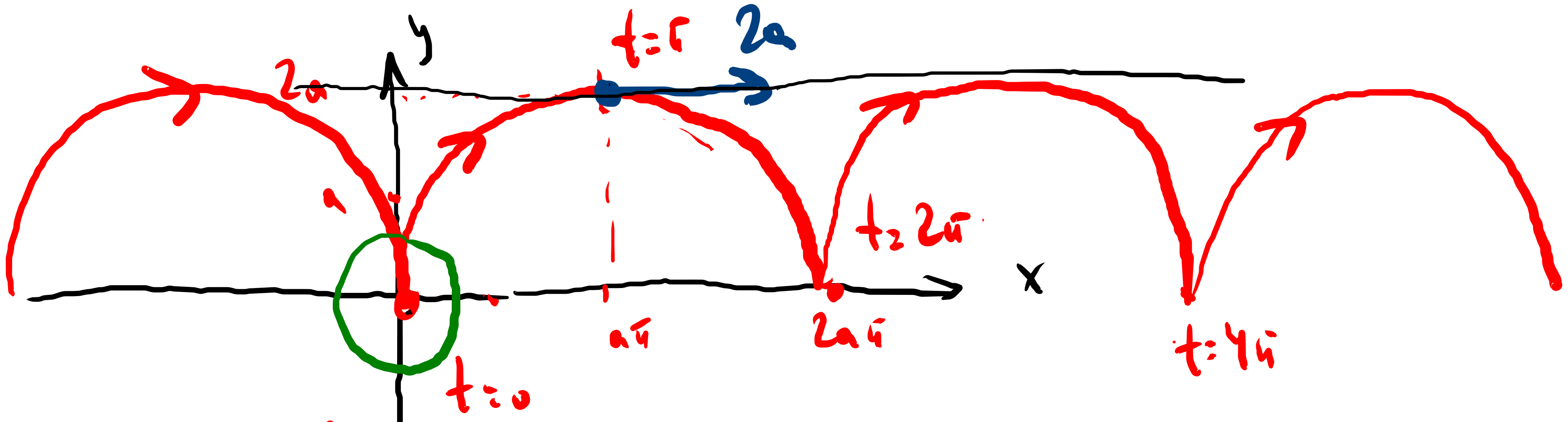
$$|\dot{\gamma}(t)| = R$$

$$\ell = \int_0^{2\pi} |\dot{\gamma}(t)|\, dt = \int_0^{2\pi} R\, dt = R\int_0^{2\pi} dt =$$

$$= 2\pi R$$

2°) $\gamma(t) = (x(t), y(t)) \qquad t \in \mathbb{R}$

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t) \qquad \underline{a > 0}$$



Cicloide

$$\dot\gamma(t) = (\dot x(t), \dot y(t)) = (a(1 - \cos t), a \sin t)$$

$$\dot\gamma(0) = 0 = \dot\gamma(2\pi) = \dot\gamma(4\pi) = \dot\gamma(2k\pi)$$

$$\dot\gamma(\pi) = (a(1 - \underbrace{\cos \pi}_{-1}), a \cancel{\sin \pi}) = (2a, 0)$$

retta tangente alla cicloide nel punto $\gamma(\pi)$

$$\gamma(\pi) = (a\pi, a) \qquad \dot\gamma(\pi) = (2a, 0)$$

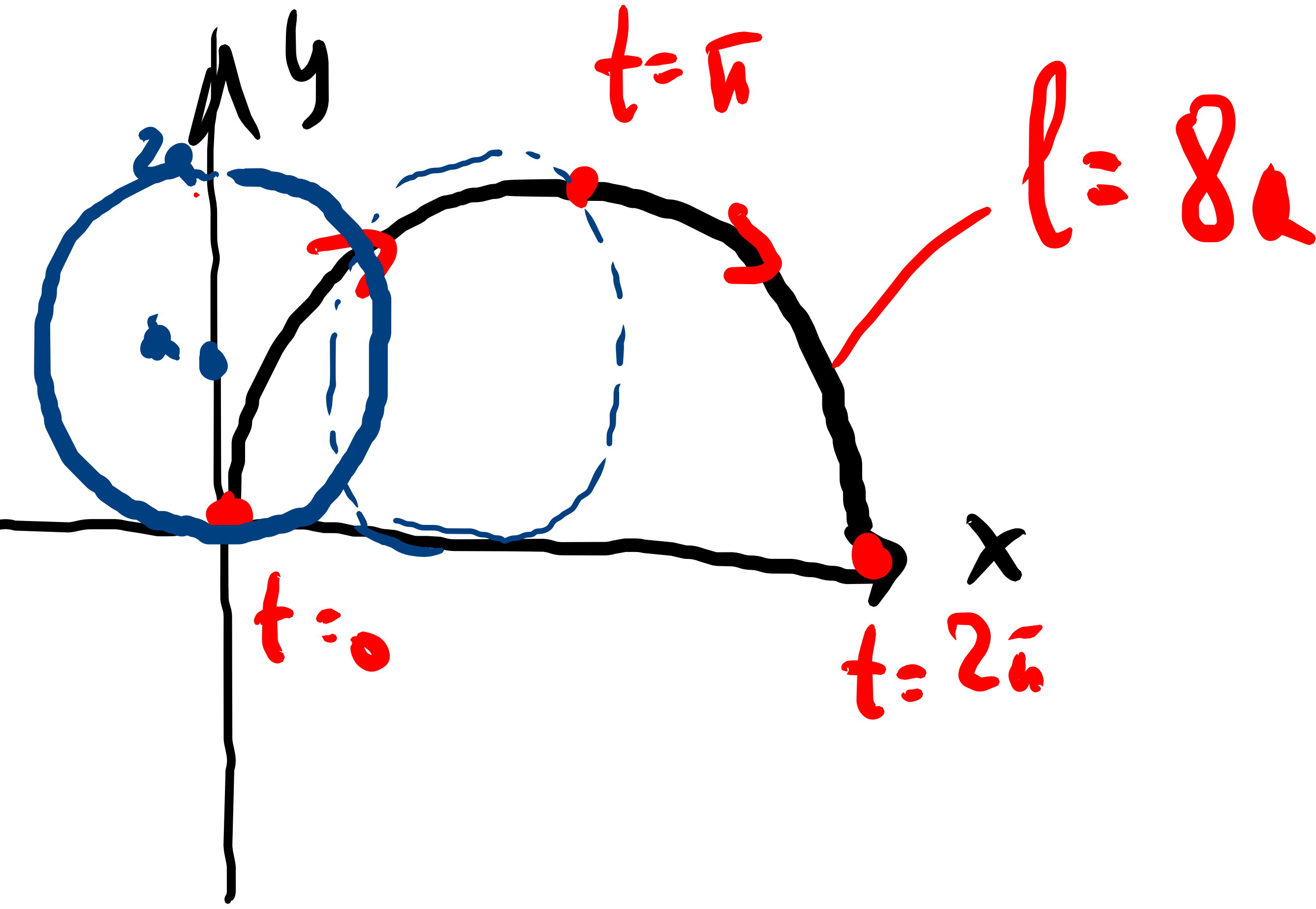
$$r(t) = (a\pi, a) + (2a, 0)(t - t_0) = (a\pi + 2a(t - t_0), a)$$

oppure
$$\begin{cases} x(t) = a\pi + 2a(t - t_0) \\ y(t) = a \end{cases}$$

Lunghezza di un arco della cicloide

$$\gamma(t) = \big(a(t - \sin t),\ a(1-\cos t)\big)$$

$$t \in [0, 2\pi]$$



$$\dot\gamma(t) = \big(a(1-\cos t),\ a\sin t\big)$$

$$|\dot\gamma(t)| = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t} =$$

$$= \sqrt{a^2 - 2a^2\cos t + \underline{a^2\cos^2 t} + \underline{a^2\sin^2 t}} =$$

$$= \sqrt{2a^2 - 2a^2\cos t} = \sqrt{2}\,a\sqrt{1-\cos t} =$$

$$= \sqrt{2}\, a \sqrt{1 - \cos 2\,{}^{t}\!/_{2}} = \sqrt{2}\, a \sqrt{1 - \left(1 - 2\sin^2 {}^{t}\!/_{2}\right)} =$$

$$= \sqrt{2}\, a \sqrt{2 \sin^2 {}^{t}\!/_{2}} =$$

$$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha}$$

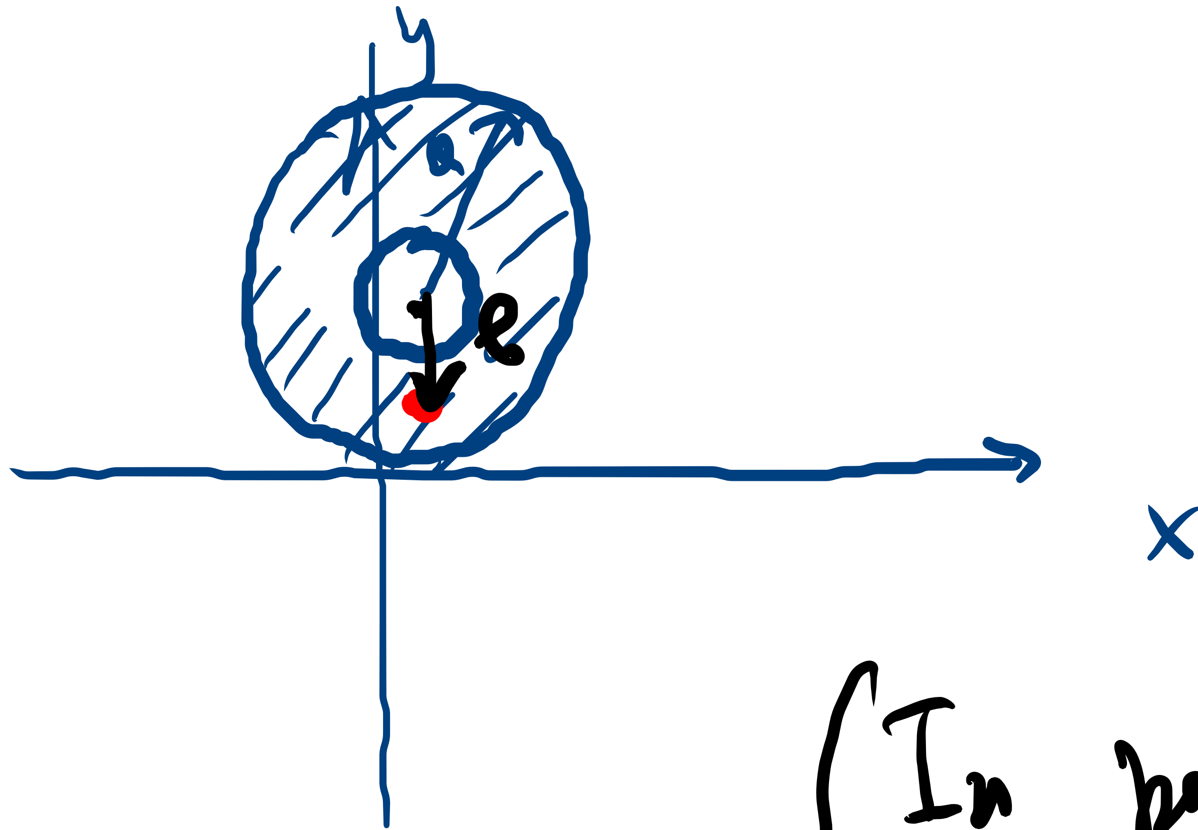
$$= 2a \sqrt{\sin^2 {}^{t}\!/_{2}} =$$

$$= 2a \left|\sin {}^{t}\!/_{2}\right|$$

$$\ell = \int_0^{2\pi} |\dot{\gamma}(t)|\, dt = 2a \int_0^{2\pi} \left|\sin \underbrace{{}^{t}\!/_{2}}_{\tau}\right| dt = 2a \int_0^{\pi} |\sin \tau|\, 2\, d\tau$$

$$= 4a \int_0^{\pi} \sin \tau\, d\tau =$$

$$= 4a \left(-\cos \tau\right)\Big|_0^{\pi} = 4a(1+1) = 8a$$

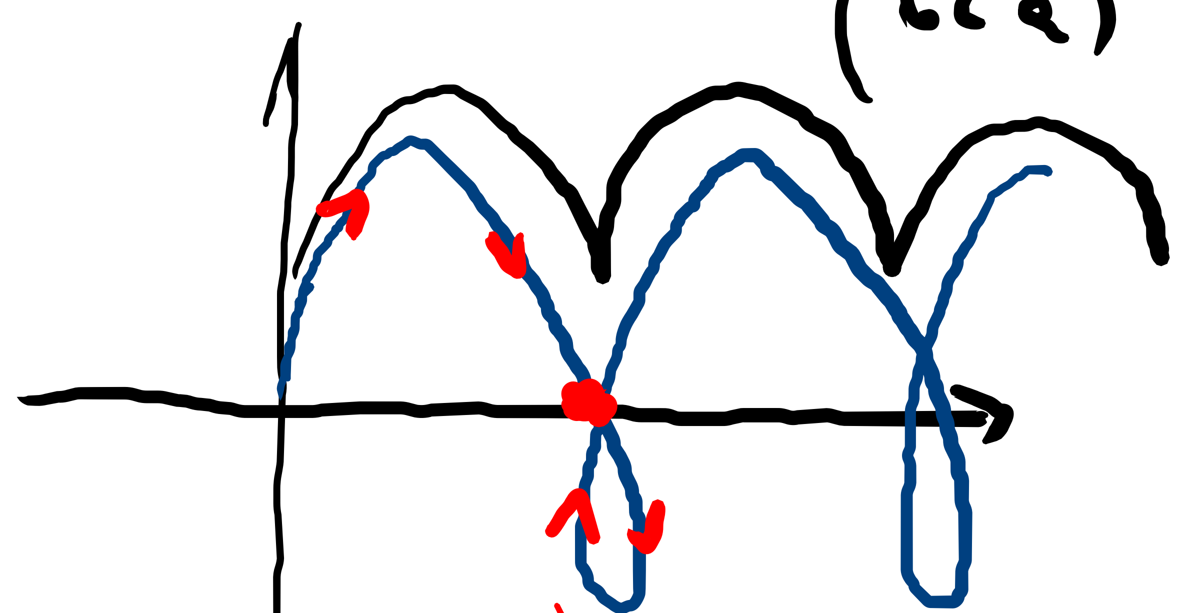


$$x(t) = at - b\sin t$$
$$y(t) = a - b\cos t$$

(In particolare, se $b = a$, è una cicloide)

Esercizio: Disegnare queste curve: i) per $0<b<a$

ii) $b>a$.



$(b<a)$

Calcolo (differenziale) per funzioni di d variabili reali.

$$f : A \subset \mathbb{R}^d \to \mathbb{R}$$

aperto

$x_0 \in A$ $\qquad$ f differenziabile in x_0

$$f(x_0 + h) = f(x_0) + \nabla f(x_0) \cdot h + o(\|h\|) \qquad h \in \mathbb{R}^d, \quad \|h\| \to 0.$$

$$\nabla f(x_0) = \left(f_{x_1}(x_0), f_{x_2}(x_0), \dots, f_{x_d}(x_0) \right)$$

<u>Def.</u> Derivata direzionale di f in direzione di un vettore unitario $v \in R$ nel
("versore")

punto $x_0 \in A$

$$f'(x_0)v\,,\ (D_v f)(x_0)\,,\ \frac{\partial f}{\partial v}(x_0)$$

$$\boxed{\frac{\partial f}{\partial v}(x_0) := \frac{d}{dt} f(x_0 + tv)\Big|_{t=0}}$$

(!)

"formula del gradiente"

Th. Se f è diff. in x_0, allora è derivabile lungo ogni direzione v in x_0 ed inoltre

$$\frac{\partial f}{\partial v}(x_0) = \nabla f(x_0) \cdot v$$

Dim. $f(x_0+h) = f(x_0) + \nabla f(x_0) \cdot h + o(\|h\|)$ $\quad \|h\| \to 0$

$$\frac{\partial f}{\partial v}(x_0) := \frac{d}{dt} f(x_0+tv)\Big|_{t=0} =$$

$$= \frac{d}{dt}\left(f(x_0) + \nabla f(x_0) \cdot t v + o(|t| \cdot \cancel{\|v\|}^{1}) \right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(f(x_0) + t\, \underbrace{\nabla f(x_0) \cdot v} + o(t) \right)\Big|_{t=0} =$$

$$= \lim_{t \to 0} \frac{\left(\cancel{f(x_0)} + \cancel{t}\, \nabla f(x_0) \cdot v + o(t) - \cancel{f(x_0)} \right)}{\cancel{t}} =$$

$$= \lim_{t \to 0} \left(\underbrace{\nabla f(x_0) \cdot v} + \frac{o(t)}{t} \right) = \nabla f(x_0) \cdot v$$

q.e.d.

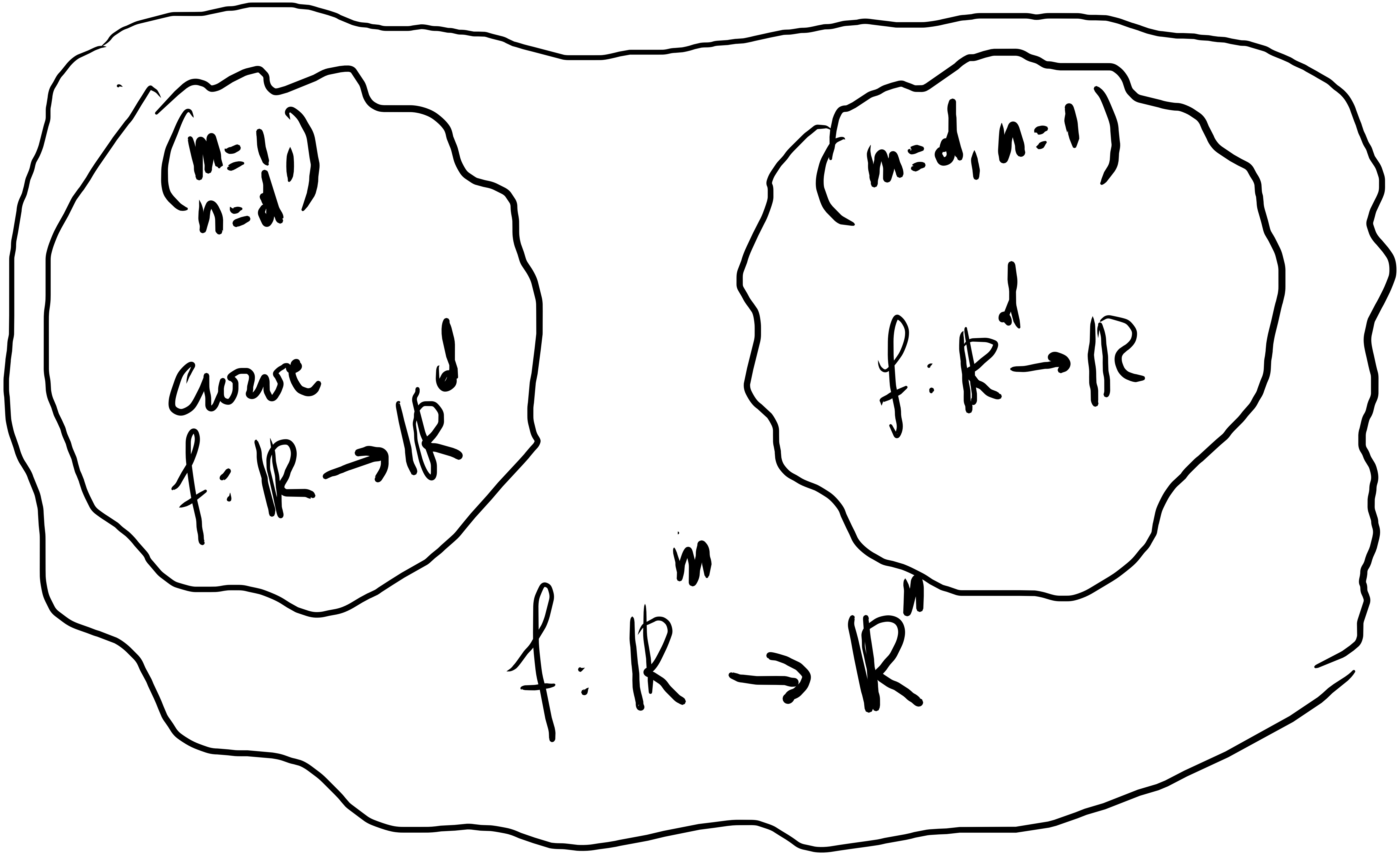
Corollario

Se $f : A \subset \mathbb{R}^d \to \mathbb{R}$ è diff in
aperto

$x_0 \in A$, e $\nabla f(x_0) \neq 0$;

allora il versore $v^+ := \dfrac{\nabla f(x_0)}{|\nabla f(x_0)|}$

indica la direzione di "<u>massima crescita</u>"
di f in punto x_0


(m=1, n=d)
curve
f: ℝ → ℝ^d
(m=d, n=1)
f: ℝ^d → ℝ
f: ℝ^m → ℝ^n

Funzioni $f: \mathbb{R}^m \to \mathbb{R}^n$

Esempi. 1°) Curve $m=1$, $n \geq 1$.

2°) $f: \mathbb{R}^m \to \mathbb{R}$ $\quad n=1, \quad m \geq 1$

"Campi scalari", ad "campo di temperatura"

$\Omega \subset \mathbb{R}^m$ $\quad x \in \Omega$ $\quad u(x)$ - temperatura nel pt. $x \in \Omega$.

$$u: \Omega \subset \mathbb{R}^m \to \mathbb{R}$$

3°) Superfici parametrizzate (parametriche) in $\mathbb{R}^n$

[bidimensionali, di solito in $\mathbb{R}^3$]

$A \subset \mathbb{R}^2$ $\quad u : A \subset \mathbb{R}^2 \to \mathbb{R}^n \quad (n \geq 2)$

(di solito $n = 3$)

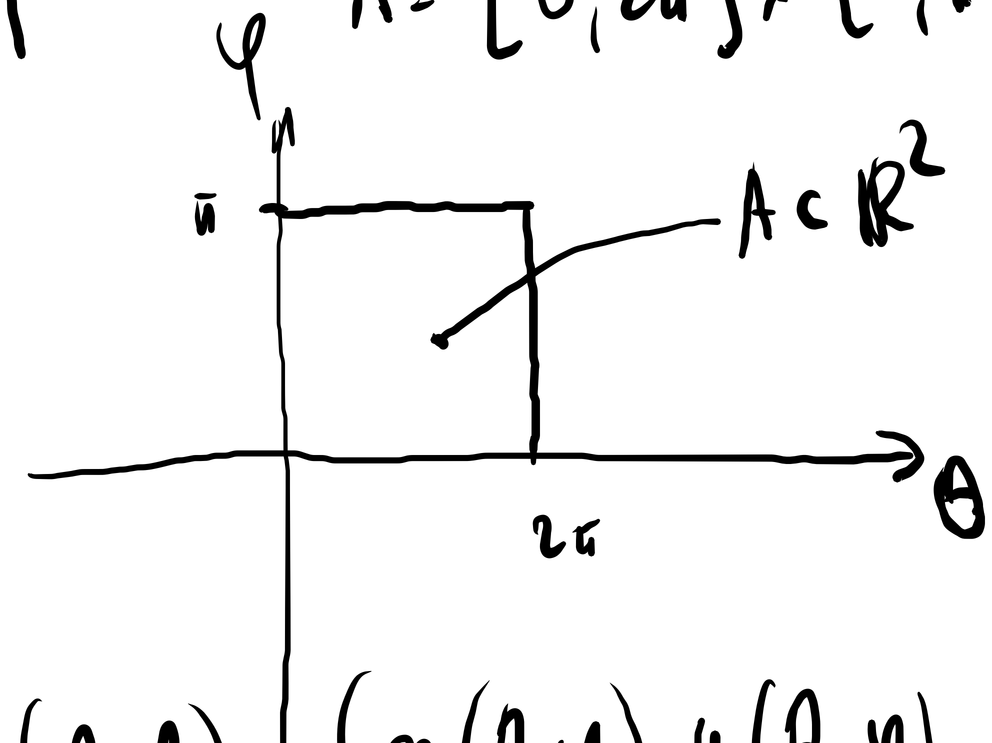
$u = u(x_1, x_2) \in \mathbb{R}^n$

$(x_1, x_2) \in A$

$u(x_1, x_2) = (u_1(x_1, x_2), \dots, u_n(x_1, x_2))$

3.1°)

$A = [0, 2\pi] \times [0, \pi]$



$$u(\theta, \varphi) = \big(x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi)\big)$$

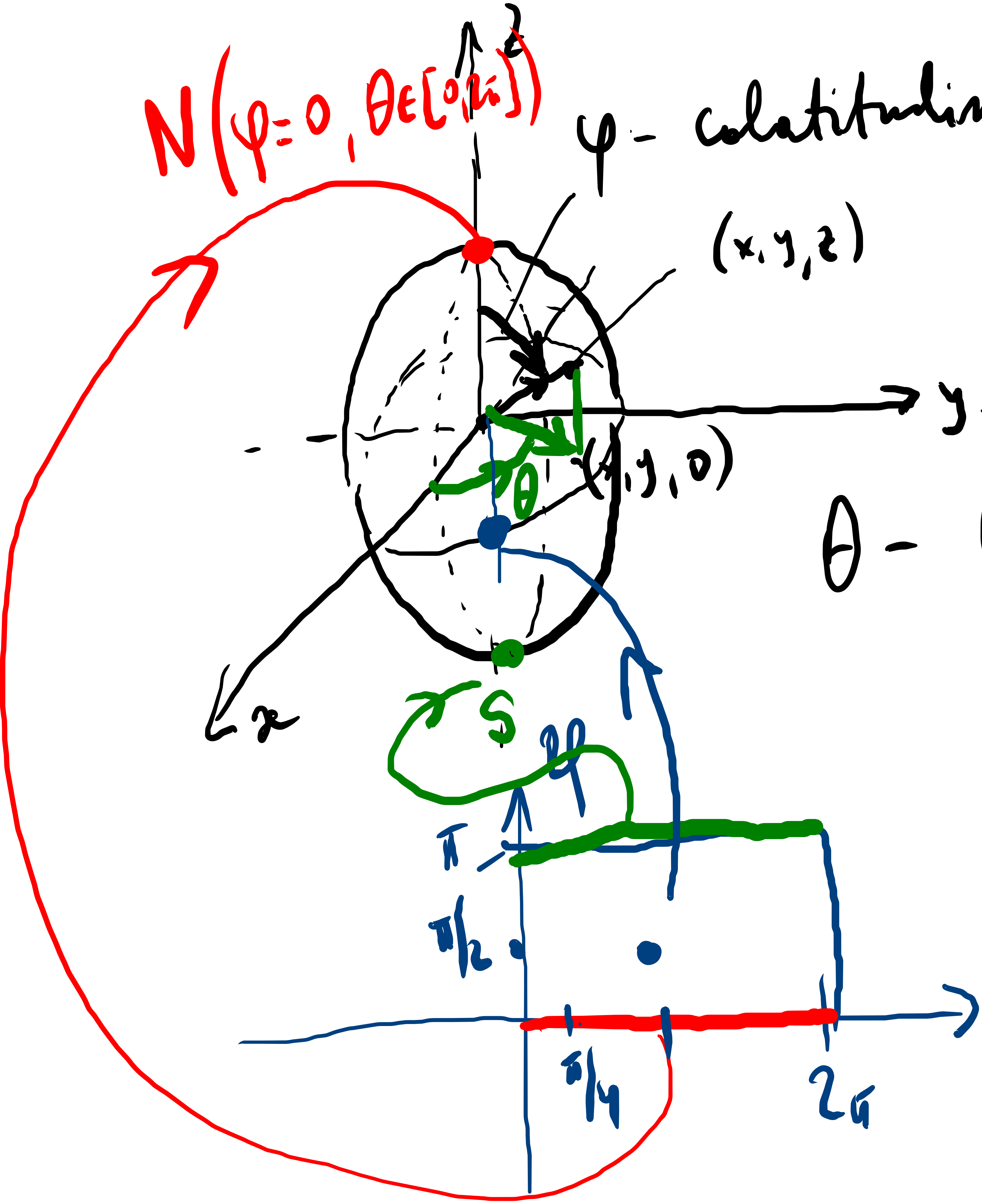
$$\begin{cases} x(\theta,\varphi) := R\cos\theta\sin\varphi \\ y(\theta,\varphi) := R\sin\theta\sin\varphi \\ z(\theta,\varphi) := R\cos\varphi \end{cases}$$

1) $\varphi = 0$, $\theta \in [0, 2\pi]$: $x = 0$, $y = 0$, $z = R$

$$x^2 + y^2 = R^2\cos^2\theta\sin^2\varphi + R^2\sin^2\theta\sin^2\varphi =$$
$$= R^2\sin^2\varphi\,(\cos^2\theta + \sin^2\theta) = R^2\sin^2\varphi$$

$$x^2 + y^2 + z^2 = R^2\sin^2\varphi + R^2\cos^2\varphi = R^2$$

$$|(x,y,z)|^2 = R^2$$



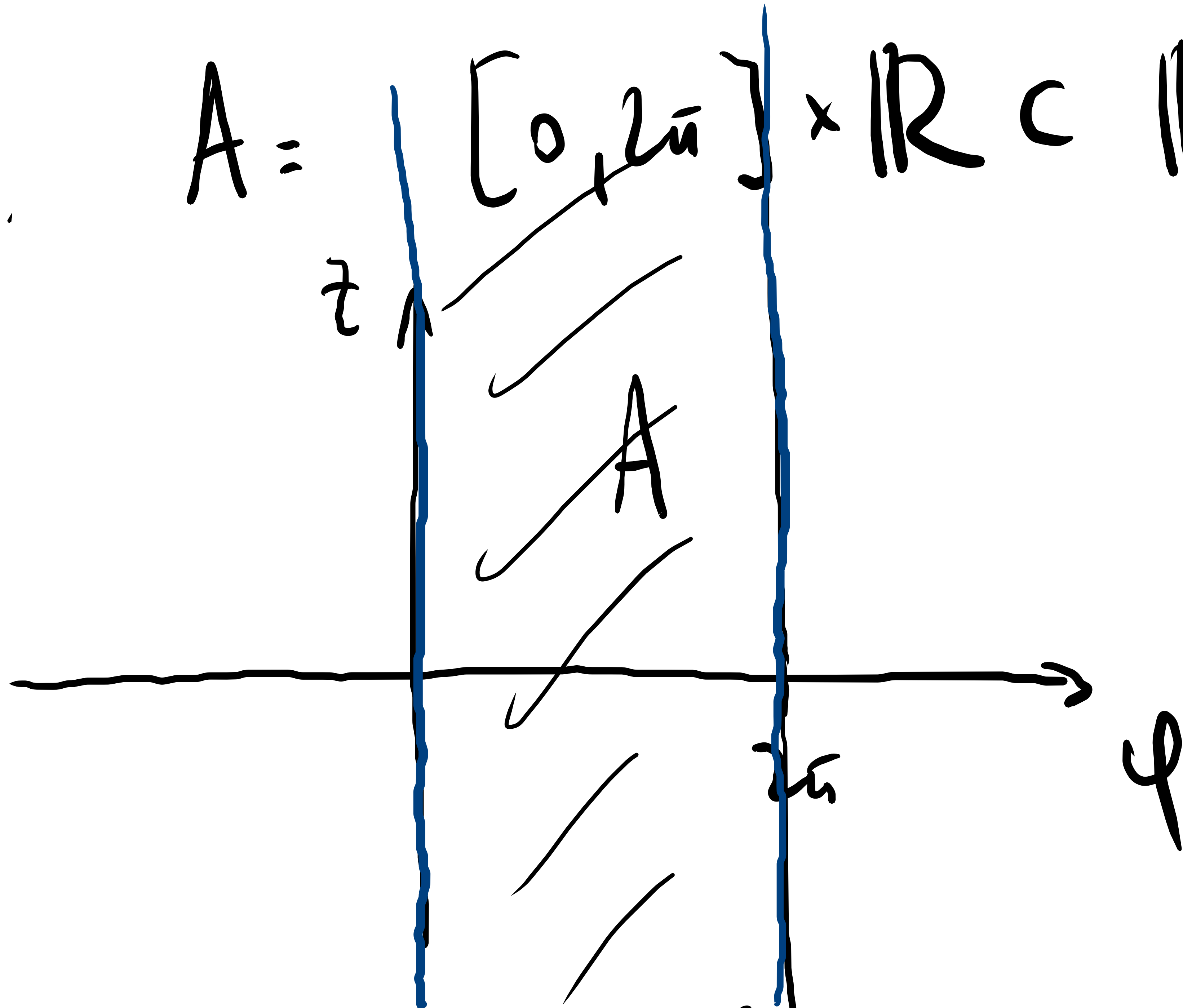
φ – colatitudine

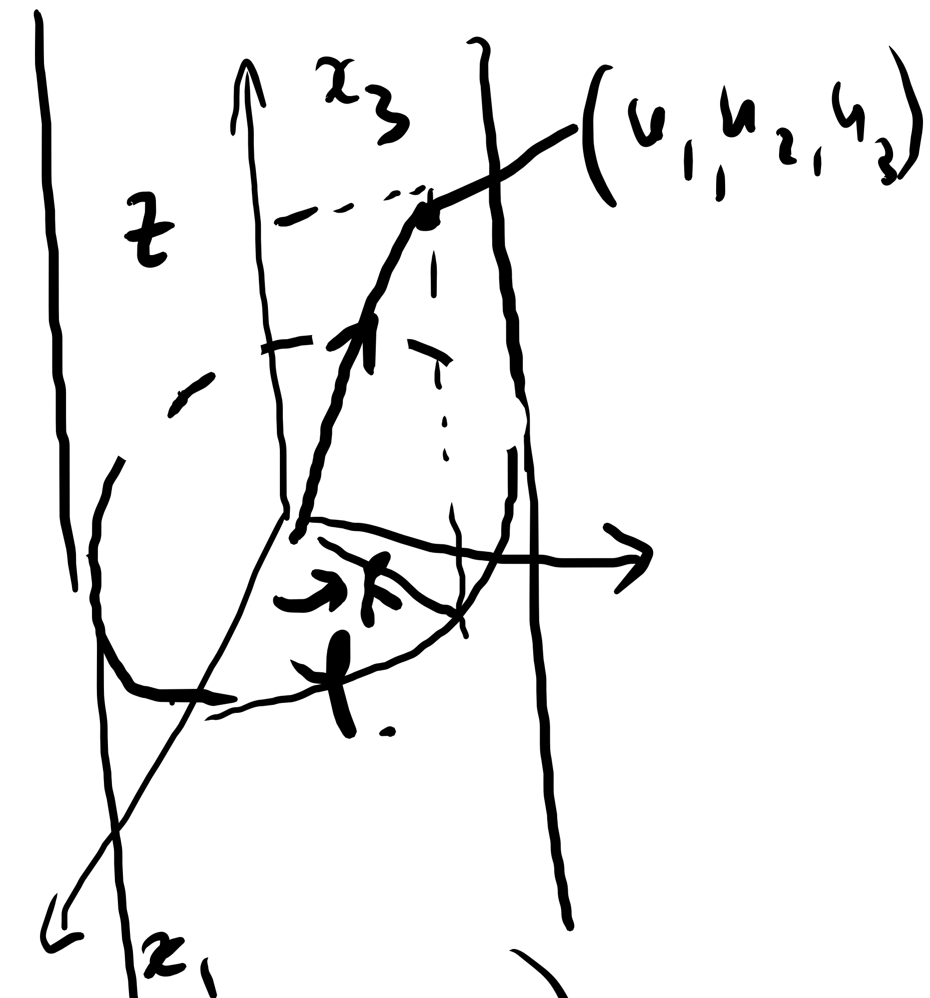
coordinate sferiche

θ – longitudine

$$(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$$

$\underline{2^\circ}$) $A = [0, 2\pi] \times \mathbb{R} \subset \mathbb{R}^2$





$$u(\varphi, z) = (u_1(\varphi, z), u_2(\varphi, z), u_3(\varphi, z))$$

$$\begin{cases} u_1(\varphi, z) = R\cos\varphi \\ u_2(\varphi, z) = R\sin\varphi \\ u_3(\varphi, z) = z \end{cases}$$

Esempi: 1°) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = e^{xy} y + \cos x$$

$$\mathbf{0} = (0,0)$$

$$\nabla f(\mathbf{0}) = \nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) =$$

$$= \left(y^2 e^{xy} - \sin x, \; e^{xy} + xye^{xy} \right)\Big|_{\substack{x=0\\y=0}} =$$

$$= (0, 1).$$

$v = (v_x, v_y) \qquad |v| = 1.$

$$\frac{\partial f}{\partial v}(0,0) = \frac{d}{dt} f(0 + tv)\Big|_{t=0} =$$

$$\boxed{f(x,y) = y e^{xy} + \cos x}$$

$$= \frac{d}{dt} f(tv_x, tv_y)\Big|_{t=0} =$$

$$= \frac{d}{dt}\Big|_{t=0} \left(t v_y e^{t^2 v_x v_y} + \cos t v_x \right) =$$

$$= \left(v_y e^{t^2 v_x v_y} + t v_y v_x v_y e^{t^2 v_x v_y} \cdot 2t + v_x(-\sin t v_x) \right)\Big|_{t=0}$$

$$= v_y$$

$$\frac{\partial f}{\partial v}(0,0) = v_y$$

Verifichiamo la formula del gradiente.

$$\frac{\partial f}{\partial v}(0,0) = \nabla f(0,0) \cdot v =$$

$$= (0,1) \cdot (v_x, v_y) =$$

$$= 0 \cdot v_x + 1 \cdot v_y = \underline{v_y}.$$

$\underline{2^\circ}|$ $f(x,y) = x^{2/3} y^{1/3}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial v}(0,0) = \frac{d}{dt} f(0+tv_x, 0+tv_y)\Big|_{t=0} = \frac{d}{dt}\left(t^{2/3} v_x^{2/3} \underline{t}^{1/3} v_y^{1/3}\right)\Big|_{t=0} =$$

$$= \frac{d}{dt}\left(t\, v_x^{2/3} v_y^{1/3}\right)\Big|_{t=0} =$$

$$= v_x^{2/3} v_y^{1/3}$$

$$\boxed{v = (v_x, v_y) \qquad |v| = 1.}$$

$$\frac{\partial f}{\partial v}\Big|(0,0) = v_x^{2/3} v_y^{1/3}$$

$$v_x^{2/3} v_y^{1/3} = \frac{\partial f}{\partial v}(0,0) \neq \nabla f(0,0)\cdot v = 0.$$

Quindi, f non è differenziabile in $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = f'_x(x,0)\Big|_{x=0} = 0.$$

$$f(x,0) = x^{2/3}\cdot 0^{1/3} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = f'_y(0,y)\Big|_{y=0} = 0$$

$$f(0,y) = 0^{2/3}\cdot y^{1/3} = 0.$$

$$\nabla f(0,0) = (0,0)$$