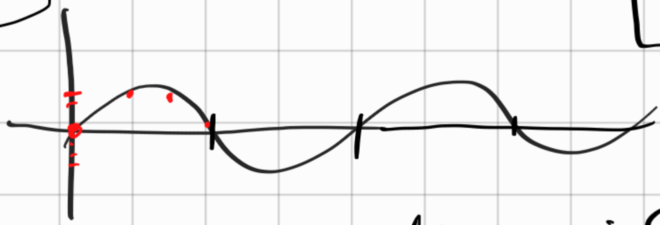
ANALISI MATEMATICA B
LEZIONE 32 - 7.12.2022

$\boxed{\sum_k \sin k}$ $\qquad a_k = \sin k \qquad$ $\boxed{\lim_{k\to+\infty} \sin k \text{ non esiste}}$



$\lim_{x\to+\infty} \sin x$ non esiste

$\lim_{k\to+\infty} \sin(\pi k) = 0$

$\lim_{k\to+\infty} \underbrace{\sin\left(k\frac{\pi}{2}\right)}$ non esiste

è periodica $0, 1, 0, -1, 0, 1, \dots$

freq. $\sin n \geq \sin\left(\frac{\pi}{2}-\frac{1}{2}\right) > \frac{1}{2}$ $\qquad \frac{\pi}{2}-\frac{1}{2} > \frac{\pi}{6}$

freq $\sin n \leq \sin\left(-\frac{\pi}{2}+\frac{1}{2}\right) < -\frac{1}{2}$ $\qquad 3\pi - 3 > \pi$

$2\pi > 3$

Se $\sin n \to \ell$ $\qquad \ell \geq \frac{1}{2}$ e $\ell \leq -\frac{1}{2}$

assurdo

Si potrebbe dimostrare che $L = \{\text{punti limite di } \sin n\} = [-1, 1]$

(serve sapere che π è irrazionale)

$\boxed{\sum_{k=1}^{+\infty} \frac{\sin k}{k} \text{ è convergente?}}$ $\qquad |\sin k| \leq 1$

$\left|\frac{\sin k}{k}\right| \leq \frac{1}{k} \qquad \sum \frac{1}{k} = +\infty.$

ESPONENZIALE COMPLESSO

Unitario

$$\boxed{e^z = e^{x+iy} := e^x \cdot \underbrace{(\cos y + i \sin y)}}$$

$z = x+iy$

$\varphi : \mathbb{R} \to U$

$$\boxed{e^{i\theta} := \cos\theta + i \sin\theta}$$ FORMULA EULERO

$\cos y + i \sin y := \varphi\left(\frac{y}{2\pi}\right)$

$$e^{z+w} = e^{x+iy+a+ib} = e^{x+a}\cdot\left(\cos(y+b) + i\,\text{sen}(y+b)\right)$$

$z = x+iy$

$w = a+ib$

$$= e^{x}\cdot e^{a}\cdot \varphi\left(\frac{y+b}{2\pi}\right) = e^{x}e^{a}\varphi\left(\frac{y}{2\pi}\right)\varphi\left(\frac{b}{2\pi}\right)$$

$$= e^{z}\cdot e^{w}$$

Senza usare φ: $\quad \cos(y+b) + i\,\text{sen}(y+b) = (\cos y + i\,\text{sen}\,y)\cdot(\cos b + i\,\text{sen}\,b)$

$$e^{x+iy} = w \qquad |w| = e^{x}, \quad \text{"}\operatorname{Arg} w = y\text{"}$$

e^{z} è $2\pi i$ - periodico

$$z\in\mathbb{C} \xrightarrow{e^{z}} w\in\mathbb{C}$$

$$e^{iy} = \cos y + i\,\text{sen}\,y$$

$$\cos y = \operatorname{Re} e^{iy}$$
$$\sin y = \operatorname{Im} e^{iy}$$

$$\boxed{\begin{cases} \cos y = \dfrac{e^{iy}+e^{-iy}}{2} \\[2ex] \sin y = \dfrac{e^{iy}-e^{-iy}}{2i} \end{cases}}$$

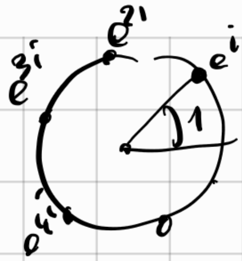
$$\left[\begin{array}{l} \cosh x = \dfrac{e^{x}+e^{-x}}{2} \\[2ex] \sinh x = \dfrac{e^{x}-e^{-x}}{2} \end{array}\right]$$

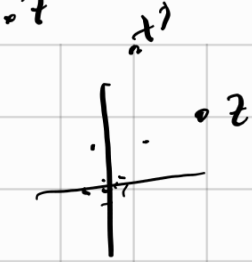
$$\left[\begin{array}{l} \overline{e^{iy}} = \overline{\cos y + i\,\text{sen}\,y} \\ \quad = \cos y - i\,\text{sen}\,y \\ \quad = \cos(-y) + i\,\text{sen}(-y) \\ \quad = e^{-iy} \end{array}\right]$$

$$\left[\begin{array}{l} e^{\bar z} = \overline{e^{z}} \\ e^{\overline{x+iy}} = e^{x-iy} = e^{x}\cdot e^{-iy} \\ \quad = e^{x}\,\overline{e^{iy}} \\ \quad = \overline{e^{x+iy}} \end{array}\right]$$

$$\sin n = \operatorname{Im} e^{in} = \operatorname{Im}\left(e^{i}\right)^{n}$$

$$e^{in} = e^{i(1+1+1+\dots+1)}$$
$$= e^{i+i+i+\dots+i} = e^{i}\cdot e^{i}\cdots e^{i} = \left(e^{i}\right)^{n}$$





$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

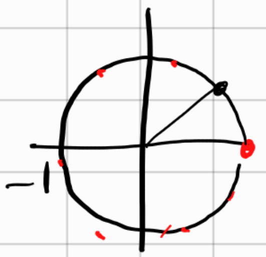
$q \in \mathbb{C}$, $q \neq 1$.

Se $|q|<1$ $\displaystyle\sum_{k=0}^{+\infty} q^k = \frac{1}{1-q}$

Se $|q|>1$ $\sum q^k$ non converge.

$|q^k| = |q|^k \to +\infty$

Se $|q|=1$?



manca primite se $|k|=1$

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$

la serie $\sum q^k$ è indeterminata se $|q|=1$ e $q \neq 1$.

$\sum \frac{\sin k}{k}$ assomiglia $\sum \frac{(-1)^k}{k}$

L'idea è di generalizzare il criterio di Leibniz

SOMMA PER PARTI

" $\displaystyle\sum^{n} a_k \cdot B_k = A_n B_n - \sum^{n} A_k b_k$ "

$\int f \cdot G = F \cdot G - \int F \cdot g$

$F = \int f \quad G = \int g$

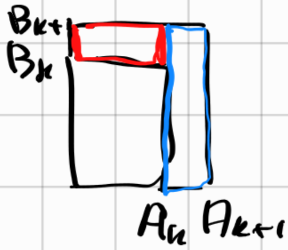
Se a_k, b_k sono successioni $A_n = \displaystyle\sum_{k=0}^{n-1} a_k$, $B_n = \displaystyle\sum_{k=0}^{n-1} b_k$

$A_0 = 0 \qquad B_0 = 0$

dim

$$\underbrace{A_n\cdot B_n - A_0\cdot B_0}_{A_0\cdot B_0 = 0} = \sum_{k=0}^{n-1}\left(A_{k+1}B_{k+1} - A_k B_k\right)$$

$$= \sum_{k=0}^{n-1}\left(A_{k+1}B_{k+1} - A_k B_{k+1} + A_k B_{k+1} - A_k B_k\right)$$



$$= \sum_{k=0}^{n-1}\left(A_{k+1}-A_k\right)B_{k+1} + \sum_{k=0}^{n-1}A_k\left(B_{k+1}-B_k\right)$$

$$= \sum_{k=0}^{n-1} a_k\cdot B_{k+1} + \sum_{k=0}^{n-1} A_k\cdot b_k$$

Teo a_k, b_k qualunque $A_n = \sum_{k=0}^{n-1} a_k$, $B_n = \sum_{k=0}^{n-1} b_k$

SOMMA PER PARTI

$$\sum_{k=0}^{n-1} A_k\cdot b_k = A_n\cdot B_n - \sum_{k=0}^{n-1} a_k\cdot B_{k+1}$$

Teo (criterio di Dirichlet)

a_k, b_k qualunque, A_k, B_k come prima

Hyp $A_k \to 0$, $\sum |a_k|$ convergente, B_k limitata $\Rightarrow |B_k| \le L < +\infty$

Tesi: $\sum A_k\cdot b_k$ convergente

Generalità Leibniz

A_k decrescente infinitesima.

$\sum \underbrace{(-1)^k}_{b_k} A_k$

dim

per parti

$$\sum_{k=0}^{n-1} A_k\cdot b_k = \underbrace{A_n\cdot B_n}_{\downarrow\, 0} - \sum_{k=0}^{n-1} a_k\cdot B_{k+1}$$

$$\left[\; 0 \le |A_n\cdot B_n| \le |A_n|\cdot L \longrightarrow 0 \qquad A_n B_n \to 0 \;\right]$$

$$|a_k \cdot B_{k+1}| \le L \cdot |a_k|$$

$$\sum |a_k B_{k+1}| \le L \cdot \sum |a_k|$$

↳ convergente.

quindi $\sum a_k \cdot B_{k+1}$ è convergente

$\triangle$

$$\left[\sum_{k=0}^{+\infty} A_k \cdot b_k = -\sum_{k=0}^{+\infty} a_k \cdot B_{k+1} \right]$$

Esercizio $\left(\sum \frac{\sin n}{n} \right)$ è convergente $\quad A_n = \frac{1}{n} \quad b_n = \sin n$

Verifichiamo le ipotesi di Dirichlet $\quad A_n = \frac{1}{n} \to 0$

$$a_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n^2+n} \qquad \sum |a_n| \text{ è convergente}$$

$$b_n = \sin n \qquad B_n = \sum_{k=0}^{n-1} \sin k$$

$$\sum_{k=0}^{n-1} \sin k = \sum_{k=0}^{n-1} \operatorname{Im} e^{ik} = \operatorname{Im} \sum_{k=0}^{n-1} e^{ik} = \operatorname{Im} \sum_{k=0}^{n-1} (e^i)^k$$

$$= \operatorname{Im} \frac{1 - e^{ni}}{1 - e^i} \qquad \frac{1}{1-e^i} = \bullet$$

è limitata

$$\left| \frac{1 - e^{ni}}{1 - e^i} \right| = \frac{|1 - e^{ni}|}{|1 - e^i|} \le \frac{1 + |e^{ni}|}{|1 - e^i|} = \frac{2}{|1 - e^i|} < +\infty$$