ANALISI MATEMATICA B
LEZIONE 72 - 28.3.2022

Eq. differenziali (ordinarie)

- Eq. lineari del I ordine in forma normale
- Eq. a variabili separabili (I ordine)

Eq. diff. ordinarie lineari di ordine n a coefficienti costanti (forma normale)

$$L[u] = u^{(n)} + \underbrace{a_{n-1} u^{(n-1)} + \ldots \ldots + a_1 \cdot u' + a_0 \cdot u}_{\text{coefficienti}} = g(x)$$

$u = u(x)$

coefficienti costanti : $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$.

Equazione omogenea : $g(x) = 0$ $\forall x$

$$L[u](x) = u^{(n)}(x) + a_{n-1} \cdot u^{(n-1)}(x) + \ldots + a_1 u'(x) + a_0(x) u(x) = 0 \quad (*)$$

Eq. lineare: $L[u] = 0$

L operatore lineare

u_1, u_2 soluzioni $\Rightarrow$ $c_1 \cdot u_1 + c_2 \cdot u_2$ è sol.

Es

$$\Rightarrow \quad u''(x) - 3u'(x) = 0$$

$$u''(x)\, e^{-3x} - 3e^{-3x}\, u'(x) = 0$$

$$\left(u'(x) \cdot e^{-3x}\right)' = 0$$

$$u' e^{-3x} = c$$

$$u' = c \cdot e^{3x}$$

$$u = c\left(\frac{1}{3} e^{3x} + d\right)$$

$$= c_1 \cdot e^{3x} + c_2 . \qquad c_1, c_2 \in \mathbb{R}$$

$$= \operatorname{span}\left\{ e^{3x}, 1 \right\}$$

$u_1 = e^{3x}$ $\quad u_1' = 3e^{3x}$ $\quad u_1'' = 9e^{3x}$

$$9e^{3x} - 3 \cdot 3e^{3x} = 0 \quad \text{ok!}$$

Es $\quad \boxed{u''(x) - 3u'(x) + 2u(x) = 0}$

$u(x) = \boxed{e^{\lambda x}}$ $\quad u'(x) = \lambda e^{\lambda x}$ $\quad u''(x) = \lambda^2 e^{\lambda x}$

$$u'' - 3u' + 2u = \lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$

$$= \underline{\left(\lambda^2 - 3\lambda + 2\right)} e^{\lambda x} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{12} = \frac{3 \pm \sqrt{9-8}}{2} = \begin{cases} 2 \\ 1 \end{cases} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \end{aligned}$$

$$u_1(x) = e^{x} \qquad u_2(x) = e^{2x}.$$

u_1 e u_2 sono lin. indipendenti.

Tutte le sol. sono della forma:

$$u(x) = c_1 \cdot e^{x} + c_2 e^{2x}.$$

Polinomio associato alla eq. diff. lineare a coefficienti costanti: $(*)$

$$\underline{P(\lambda) = \lambda^{n} + \overset{\downarrow}{a_{n-1}} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + \overset{\downarrow}{a_0}}$$

Se $\lambda_1, \dots, \lambda_n$ sono le radici del polinomio

$$(\#) \quad P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$

L'eq. differenziale si può scrivere così: $\qquad D^0 = I$

$$\underbrace{D^{n} u + a_{n-1} D^{n-1} u + \dots + D^{1} u + D^{0} u}_{} = 0$$

$$(D^{n} + a_{n-1} D^{n-1} + \dots + D + I)[u] = 0 \qquad (A+B)[u] = A[u] + B[u]$$

$$P(D)[u] = 0 \qquad (c \cdot A)[u] = c \cdot A[u]$$

$$A^{n}[u] = A[A \dots A[u] \dots] \qquad (A \cdot B)[u] = A[B[u]]$$

$$P(D)\cdot[u]=0$$

$$(D-\lambda_1\cdot I)(D-\lambda_2 I)\cdots\boxed{(D-\lambda_n I)[u]=0}.$$

$$(D-\lambda_n I)[u]=0$$

$$\begin{cases} Du=\lambda_n u & u_n=e^{\lambda_n\cdot x} \\ Du=\lambda_{n-1}u & u_{n-1}=e^{\lambda_{n-1}\cdot x} \\ \vdots & \vdots \\ Du=\lambda_1 u & u_1=e^{\lambda_1\cdot x} \end{cases}$$

$$D^2u-3Du+2u=0$$

$$(D-2)\cdot(D-1)[u]=0$$

Verifica

$$D^2u-3Du+2u\overset{?}{=}(D-2)\cdot(D-1)[u]$$

$$(D-2)\left[(D-1)[u]\right]=(D-2)[Du-u]$$

$$=(D-2)[u'-u]$$

$$=D[u'-u]-2(u'-u)=u''-u'-2u'+2u$$

$$=u''-3u'+2u.\quad\checkmark$$

$$(D-2)(D-1)[u] = 0$$

$(D-1)u = 0$ $\quad u' = u \quad u = c_1 e^x$

$(D-2)u = 0$ $\quad u' = 2u \quad u = c_2 e^{2x}$

$$u_1 = e^x \quad , \quad u_2 = e^{2x}$$

$u(x) = c_1 e^x + c_2 e^{2x}$ sono tutte sol. al variare $c_1, c_2 \in \mathbb{R}$.

Teoria: lo spazio delle sol. ha dim $= 2$.
Quindi queste sono tutte le soluzioni.

Metodo risolutivo

1. Scrivo il polinomio associato alla eq. lineare omogenea. $P(\lambda)$

2. Fattorizzo $p(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$

Se $\lambda_1 \dots \lambda_n$ sono tutte reali e distinte una base dello sp. delle soluzioni è:

$$u_1 = e^{\lambda_1 x}, \dots, u_n = e^{\lambda_n x}.$$

Tutte le sol. si scrivono nella forma:

$$u(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots + c_n e^{\lambda_n x}.$$

Es $\quad u'''(x) - u'(x) = 0$

Polinomio associato: $P(\lambda) = \lambda^3 - \lambda = \lambda(\lambda^2 - 1)$
$= \lambda(\lambda-1)(\lambda+1)$

$\lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = -1.$

Le soluzioni sono:

$$u(x) = c_1 + c_2 e^{x} + c_3 e^{-x} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Verifica
$$\left.\begin{aligned} u'(x) &= c_2 e^{x} - c_3 e^{-x} \\ u''(x) &= c_2 e^{x} + c_3 e^{-x} \\ u'''(x) &= c_2 e^{x} - c_3 e^{-x} \end{aligned}\right]$$

$u''' - u' = 0$

Verifichiamo che $\{1, e^{x}, e^{-x}\}$ sono indipendenti?

$$\boxed{c_1 \cdot 1 + c_2 \cdot e^{x} + c_3 e^{-x} = 0} \overset{?}{\Rightarrow} c_1 = c_2 = c_3 = 0$$

$\forall x \in \mathbb{R}$

$$\begin{cases} x = 0 \\ x = 1 \\ x = 2 \end{cases} \qquad \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 \cdot e + c_3 e^{-1} = 0 \\ c_1 + c_2 e^{2} + c_3 e^{-2} = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & e & e^{-1} \\ 1 & e^2 & e^{-2} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & e & e^{-1} \\ 1 & e^2 & e^{-2} \end{pmatrix} = \left(\frac{1}{e} - e\right) - 1\left(e^{-2} - e^{-1}\right) + 1\left(e^2 - e\right)$$

$$= \left(\frac{1}{e} - e\right) - \frac{1}{e}\left(\frac{1}{e} - 1\right) + e(e-1)$$

$A \underline{x} = 0$

$$= \frac{1}{e}\left(1 - e^2\right) - \frac{1}{e^2}\left(1 - e\right) + e(e-1)$$

$\underline{x} = A^{-1}(0) = 0$

$$= \frac{1}{e}(1-e)(1+e) + \left(\frac{1}{e^2} + e\right)(e-1)$$

$$= (e-1)\left[-\frac{1}{e}(1+e) + \frac{1}{e^2}\left(1+e^3\right)\right]$$

$$= \frac{e-1}{e^2}\left[-e - e^2 + 1 + e^3\right]$$

$$\neq 0.$$

$f(x) = c_1 + c_2 e^{x} + c_3 e^{-x} = 0 \quad \forall x$

$$\lim_{x \to +\infty} f(x) = \begin{cases} +\infty & \text{se } c_2 > 0 \\ -\infty & \text{se } c_2 < 0 \\ 0 & \text{se } c_2 = 0 \end{cases} \implies c_2 = 0.$$

$$\lim_{x \to -\infty} f(x) = 0 \iff c_3 = 0$$

$$\implies c_1 = 0 \qquad \square$$

<u>Oppure</u> $0 = f'(x) = c_2 e^{x} - c_3 e^{-x} = 0$

$$0 = e^x f'(x) = C_2 e^{2x} - C_3$$

$$0 = (e^x f'(x))' = 2C_2 e^{2x}$$

$$\underline{C_2 = 0} \Rightarrow C_3 = 0 \Rightarrow C_1 = 0.$$

<u>Es</u>

$$u'' - 2u' + u = 0$$

$$(D^2 - 2D + \cancel{1}^{I})[u] = 0$$

$$(D-1)^2 [u] = 0 \qquad \lambda_1 = \lambda_2 = 1$$

$$(D-1)[u] = 0$$

$$Du - u = 0 \qquad u' = u \qquad u_1 = e^x$$

$$A = D - 1 \qquad A^2 [u] = 0$$

$$u : \quad (D-1)^2 [u] = 0$$

$$(D-1)[u] = c e^x$$

$$u' - u = c e^x$$

$$u' e^{-x} - e^{-x} u = c$$

$$(u \cdot e^{-x})' = c \qquad u \cdot e^{-x} = cx + d$$

$$u = (cx+d)e^x$$

$\uparrow$ tutte le soluzioni

$$u = c\,xe^x + d\cdot e^x = c\cdot u_2 + d\cdot u_1$$

$$u_1(x) = e^x \qquad u_2(x) = x\cdot e^x.$$

<u>Regola generale</u> 1. Trovo $P(\lambda)$ pol. associato

2. Fattorizzo $P(\lambda) = (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_k)^{m_k}$

$\uparrow$

$\lambda_1, \dots, \lambda_k$ reali distinte. $m_1 + \dots + m_k = n$

$n = \deg P.$

Una base di sol. indipendenti è:

$$\left[\begin{array}{l} e^{\lambda_1 x}, \; xe^{\lambda_1 x}, \; x^2 e^{\lambda_1 x}, \dots, x^{m_1-1}e^{\lambda_1 x}, \\ \vdots \\ e^{\lambda_k x}, \; xe^{\lambda_k x}, \dots\dots, x^{m_k-1}e^{\lambda_k x} \end{array}\right. \qquad [m_1 + m_2 + \dots + m_k]$$

<u>Idea</u> $(D-\lambda)\cdot[q(x)\cdot e^{\lambda x}] = [q'(x)e^{\lambda x} + \cancel{q(x)\cdot\lambda e^{\lambda x}} - \cancel{\lambda q(x)e^{\lambda x}}]$

$$= q'(x)e^{\lambda x}$$

$\swarrow$

$$(D-\lambda)^k[q(x)e^{\lambda x}] = q^{(k)}(x)e^{\lambda x} = 0 \quad \text{se } k > \deg q$$

Esempio

$$u^{IV} - 2u''' + 2u' - u = 0$$

$$P(\lambda) = \lambda^4 - 2\lambda^3 + 2\lambda - 1$$

$$P(1) = 1 - 2 + 2 - 1 = 0$$

...

$$P(\lambda) = (\lambda-1)^3(\lambda+1)$$

$$(\lambda-1)^3(\lambda+1)$$
$$= (\lambda^3 - 3\lambda^2 + 3\lambda - 1)(\lambda+1)$$
$$= \lambda^4 - 3\lambda^3 + \cancel{3\lambda^2} - \lambda$$
$$+ \lambda^3 - \cancel{3\lambda^2} + 3\lambda - 1$$
$$= \lambda^4 - 2\lambda^3 + 2\lambda - 1$$

$\lambda_1 = 1 \qquad m_1 = 3$

$\lambda_2 = -1 \qquad m_2 = 1$

Tutte le soluzioni sono:

$$u(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{-x}$$

$$u'' + u = 0$$

$$\lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i$$

$$u(x) = c_1 e^{ix} + c_2 e^{-ix}$$

$$\text{span}_{\mathbb{C}}\{e^{ix}, e^{-ix}\} = \text{span}_{\mathbb{C}}\{\sin x, \cos x\}$$

$$u(x) = c_1 \cos x + c_2 \sin x$$