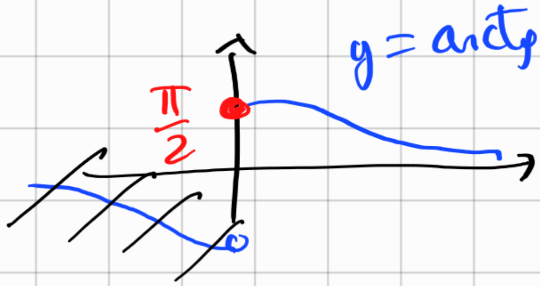
ANALISI MATEMATICA B
LEZIONE 55 — 14.2.2022

Polinomi di Taylor per $x \to 0$

$\cos\left(\frac{1}{2}\operatorname{arctg}\frac{1}{x^2}\right)$ $\qquad\qquad$ $\sqrt{\cos x + \cos(3x)}$

$x > 0$



- $\operatorname{arctg}\frac{1}{x}$

$\operatorname{arctg} y$ per $y \to +\infty$

① estendo $\operatorname{arctg}\frac{1}{x}$ in $x = 0$
e faccio le derivate

② $\operatorname{arctg}\left(\frac{1}{x}\right) = \ldots\ldots \operatorname{arctg} x$

$$f(x) = \begin{cases} \operatorname{arctg}\frac{1}{x} & \text{se } x > 0 \\ \frac{\pi}{2} & \text{se } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{-1/x^2}{1+\left(\frac{1}{x}\right)^2} = -\dfrac{1}{1+x^2} & \text{se } x > 0 \\ -1 & \text{se } x = 0 \end{cases}$$

$$\boxed{f'(x) \overset{x>0}{=} -\frac{1}{x^2+1}} = -(\operatorname{arctg} x)'$$

$$\left(\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x}\right)' = 0$$

$$\boxed{\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = c}$$

per $x \to +\infty$: $\quad \operatorname{arctg} x \to \frac{\pi}{2}$, $\quad \operatorname{arctg}\frac{1}{x} \to 0$ $\qquad c = \frac{\pi}{2}$

(2) $\underbrace{\operatorname{arctg}\frac{1}{x} = \frac{\pi}{2} - \operatorname{arctg} x}$ $\qquad \forall x > 0$

$$\operatorname{tg}\left(\frac{\pi}{2} - y\right) = \frac{1}{\operatorname{tg} y}$$

$$\frac{\pi}{2} - y = \operatorname{arctg}\frac{1}{\operatorname{tg} y}$$

$$y = \operatorname{arctg} x$$

$$\operatorname{arctg}\frac{1}{x} = \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)\right)$$

$$\operatorname{arctg}\frac{1}{x^2} = \frac{\pi}{2} - x^2 + \frac{x^6}{3} + o(x^6) = \frac{\pi}{2} - x^2 + o(x^4)$$

$$\cos\left(\frac{1}{2}\operatorname{arctg}\frac{1}{x^2}\right) = \cos\left(\frac{\pi}{4} - \frac{x^2}{2} + o(x^4)\right) \qquad x \to 0$$

$$= \cos\frac{\pi}{4}\cdot\cos\left(\frac{x^2}{2} + o(x^4)\right) + \sin\frac{\pi}{4}\sin\left(\frac{x^2}{2} + o(x^4)\right)$$

$$= \frac{\sqrt{2}}{2}\left[1 - \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + o(x^4)\right] + \frac{\sqrt{2}}{2}\left[\frac{x^2}{2} + o(x^4)\right]$$

$$= \dots\dots$$

$$\sqrt{\cos x + \cos(3x)}$$

$$\sqrt{1+y} = (1+y)^{\frac{1}{2}}$$

$$\sqrt{2+y} = \sqrt{2\left(1+\frac{y}{2}\right)}$$

$$= \sqrt{2} \cdot \sqrt{1+\frac{y}{2}}$$

$$\|$$

$$\sqrt{1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)+1-\frac{(3x)^2}{2}+\frac{(3x)^4}{24}+o(x^4)}$$

$$= \sqrt{2-5x^2+\frac{41}{12}x^4+o(x^4)}$$

$$\boxed{(1+y)^{1/2} = 1+\frac{1}{2}y-\frac{1}{8}y^2+o(y^2)}$$

$$= \sqrt{2} \cdot \left(1-\frac{5}{2}x^2+\frac{41}{24}x^4+o(x^4)\right)^{\frac{1}{2}}$$

$$= \sqrt{2}\left(1+\frac{1}{2}\left[-\frac{5}{2}x^2+\frac{41}{24}x^4+o(x^4)\right]-\frac{1}{8}\left(-\frac{5}{2}x^2+o(x^2)\right)^2 + o(x^4)\right)$$

$$= \sqrt{2}\left(1-\frac{5}{4}x^2+\frac{41}{6\cdot 8}x^4-\frac{1}{8}\frac{25}{4}x^4+o(x^4)\right)$$

$$= \sqrt{2}\left(1-\frac{5}{4}x^2+\frac{7}{12\cdot 8}x^4+o(x^4)\right)$$

...

$$\sum_k \left(e - \left(1+\frac{1}{k}\right)^k\right)^\alpha$$

Per quali $\alpha \in \mathbb{R}$ converge?

$\left(1+\frac{1}{k}\right)^k \to e \qquad k \to +\infty$

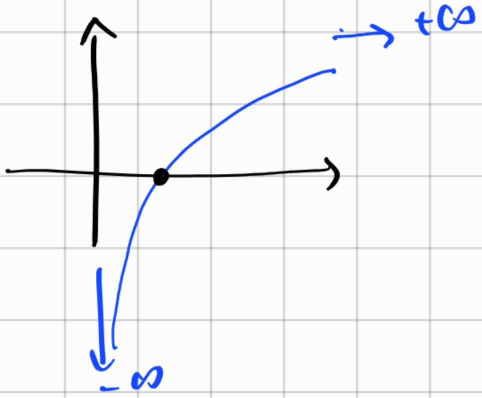
$\alpha > 0$ è necessario per la convergenza.

$x = \frac{1}{k} \qquad x \to 0^+ \quad$ se $k \to +\infty$

$$f(x) = e - (1+x)^{\frac{1}{x}}$$

$$= e - e^{\frac{1}{x}\ln(1+x)}$$

$$a_n = \left(f\left(\tfrac{1}{n}\right)\right)^\alpha$$



$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + o(x)$$

$$f(x) = e - e^{1 - \frac{x}{2} + o(x)}$$

$$e^t = 1 + t + o(t)$$

$$= e - e \cdot e^{-\frac{x}{2} + o(x)}$$

$$= e\left(1 - e^{-\frac{x}{2} + o(x)}\right) = e\left(\cancel{1} - \left(\cancel{1} - \frac{x}{2} + o(x)\right)\right)$$

$$= \frac{e}{2}x + o(x) = \frac{e}{2}x\underbrace{\left(1 + o(1)\right)}_{\downarrow\, 1} \sim \frac{e}{2}x$$

per $x \to 0$

$$a_k = \left(f\left(\tfrac{1}{k}\right)\right)^\alpha \sim \left(\frac{e}{2}\,\frac{1}{k}\right)^\alpha \qquad f\left(\tfrac{1}{k}\right) \sim \frac{e}{2}\,\frac{1}{k}$$

per $k \to +\infty$

$$\sum \boxed{\left(\frac{e}{2}\right)^\alpha} \frac{1}{k^\alpha} = \left(\frac{e}{2}\right)^\alpha \sum \frac{1}{k^\alpha}$$

converge $\Leftrightarrow$ $\alpha > 1$.

$$\ln(1+x) = x + O(x^2) = x + c \cdot \frac{x^2}{2} + o(x^2)$$

con $c \in \mathbb{R}$

[a me serve $c \neq 0$.

$$\ln(1+x) = x + \Omega(x^2)$$

$$f(x) \in \Omega(x^2) \qquad 0 < c \leq \left|\frac{f(x)}{x^2}\right| \leq C_1$$

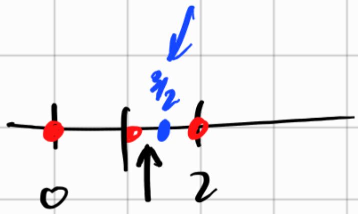
ad esempio $f(x) \sim k \cdot x^2$ con $k \neq 0$.]

$$f(x) = \ln(1+x) + x\sqrt{x} - x \qquad \text{per } x \to 0^+$$

$$\left[= o(x) + \cancel{x^{3/2}} \right] \qquad x^{3/2} = o(x)$$

$$= -\frac{x^2}{2} + o(x^2) + x^{3/2}$$

$$= x^{3/2} - \frac{x^2}{2} + o(x^2)$$



$$= x^{3/2} + o(x^{3/2})$$

Potrei rispondere fatica:

$$\ln(1+x) = x + O(x^2) \qquad \text{per } x \to 0$$

$$f(x) = \cancel{x} + O(x^2) + x\sqrt{x} - \cancel{x}$$

$$= x\sqrt{x} + O(x^2) = x\sqrt{x} + o(x\sqrt{x})$$
$$\sim x\sqrt{x}$$

$$g(x) = x\sqrt{x} = x^{3/2}, \quad g'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$g \in C^1$.

$$g(x) = g(0) + g'(0)\cdot x + o(x)$$

ma non posso fare lo sviluppo al II ordine.

FUNZIONI ANALITICHE

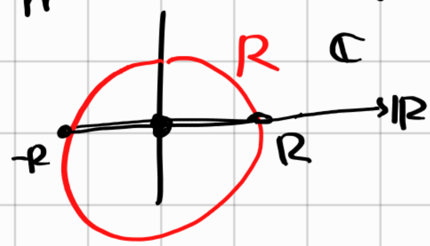
$x \in \mathbb{R}$

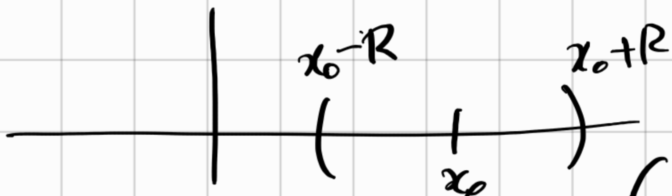
$a_k \in \mathbb{R}$

R = raggio di convergenza

$$g(x) = \sum_{k=0}^{+\infty} a_k \cdot x^k$$

$$f(x) = \underbrace{\sum_{k=0}^{+\infty} a_k (x-x_0)^k}_{} = g(x-x_0)$$

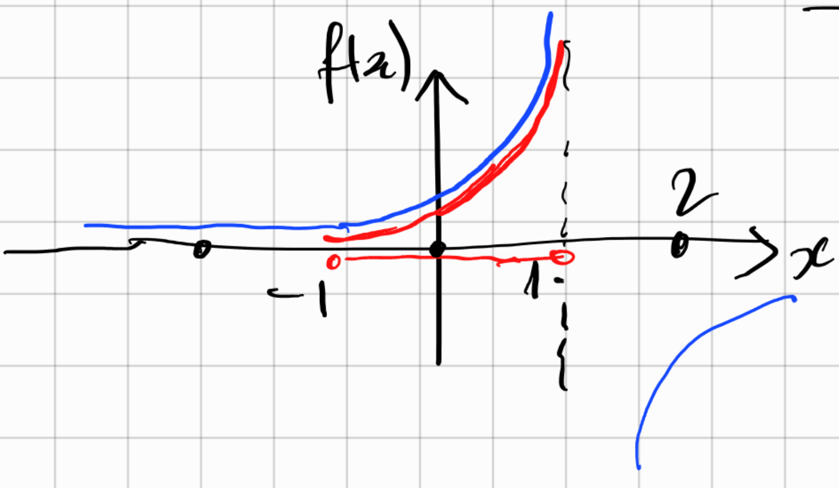




$|x|<1$

$$f(x) = \frac{1}{1-x} = \underbrace{\sum_{k=0}^{+\infty}}_{} x^k$$

$a_k = 1$



Def $f : A \subseteq \mathbb{R} \to \mathbb{R}$ si dice essere analitica se $\forall x_0 \in A$ $\exists \rho > 0$ $\exists a_k \in \mathbb{R}$: $\forall x \in A \cap (x_0 - \rho, x_0 + \rho)$ si ha:

$$f(x) = \sum_{k=0}^{+\infty} a_k (x - x_0)^k$$

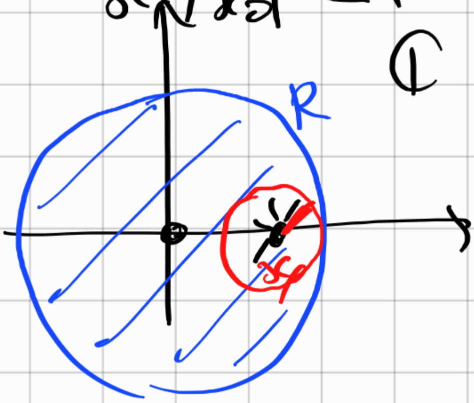
Domanda:

Se $f(x) = \sum_{k=0}^{+\infty} a_k x^k$

per $|x| < R$ se $|x_0| < R$, $|x| < R$

esistono b_k tc

$$f(x) = \sum_{k=0}^{+\infty} b_k (x - x_0)^k$$



Teorema Sì se $|x_0| < R$

Posto $\rho = R - |x_0|$ esistono b_k

b_k

$$f(x) = \sum_{k=0}^{+\infty} b_k (x - x_0)^k$$

$\uparrow$

ha raggio ρ

In particolare se $R = +\infty$ $\quad f: \mathbb{R} \to \mathbb{R}$

è analitica.

<u>Es</u> e^x, $\sin x$, $\cos x$ sono

analitiche.

<u>idea</u> dimostrazione.

$x_0 \qquad x = x_0 + h$

$$f(x) = \sum_{k=0}^{+\infty} a_k x^k = \sum_{k=0}^{+\infty} a_k (x_0 + h)^k \qquad h = x - x_0$$

$$= \sum_{k=0}^{+\infty} \sum_{j=0}^{k} a_k \binom{k}{j} x_0^{k-j} h^j$$

$$\overset{?}{=} \sum_{j=0}^{+\infty} \underbrace{\left[\sum_{k=j}^{+\infty} a_k \binom{k}{j} x_0^{k-j} \right]}_{b_j} h^j$$

$$= \sum_{k=0}^{+\infty} b_k h^k$$