ANALISI MATEMATICA B
LEZIONE 35 - 13.12.2021

Es 3 test

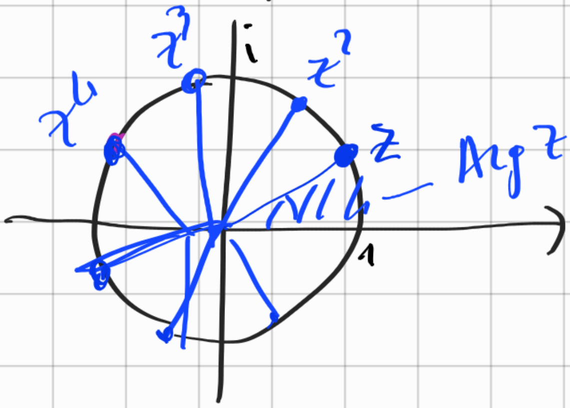
$$z = \frac{1+i\sqrt{3}}{2}$$

$\mathrm{Re}\, z = \frac{1}{2}$

$\mathrm{Im}\, z = \frac{\sqrt{3}}{2}$

$$|z| = \sqrt{\mathrm{Re}^2 z + \mathrm{Im}^2 z}$$

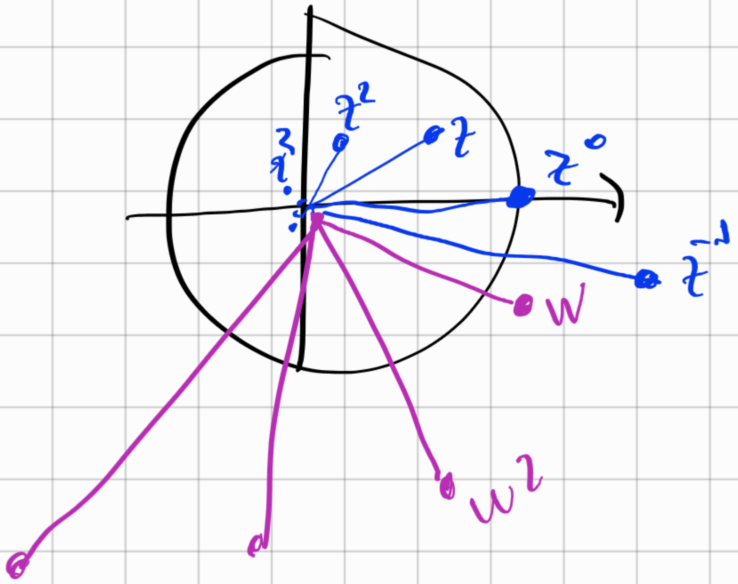
$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$



$z^2 = ?$

$|z^2| = |z|^2 = 1$

$\mathrm{Arg}(z^2) = 2\,\mathrm{Arg}\, z$



$$|z^n| = |z|^n$$

$$z^{-1} \cdot z = 1$$

$$|z^{-1}| \cdot |z| = 1$$

$$|z^{-1}| = \frac{1}{|z|}$$

$$\mathrm{Arg}\, z^{-1} = -\mathrm{Arg}\, z$$

(i) $|z_n - z| \to 0$ è equivalente a

(iii) $\forall \varepsilon > 0 \ \exists N : \ n > N \Rightarrow |z_n - z| < \varepsilon$

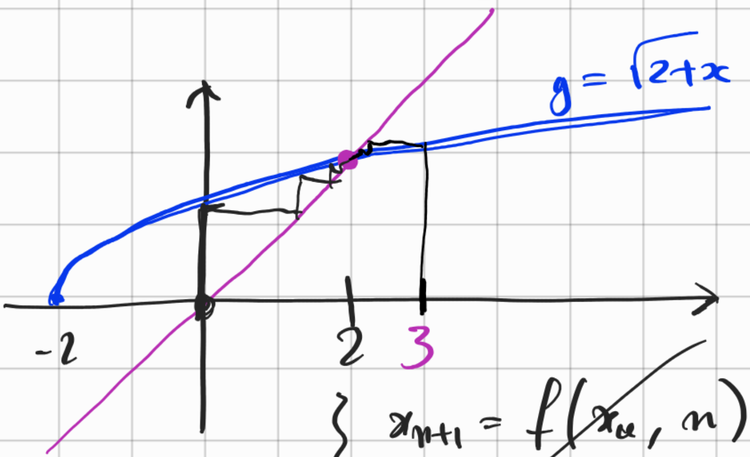
(i) per definizione di limite:

modulo ← valore assoluto

$\forall \varepsilon > 0 \ \exists N : \ n > N \Rightarrow \big|\, |z_n - z| \,\big| < \varepsilon$

ES

$$\begin{cases} x_{n+1} = \sqrt{2 + x_n} \\ x_0 = 0 \ \ 3 \end{cases}$$



$$\boxed{x_{n+1} = f(x_n)}$$

$$\begin{cases} x_{n+1} = f(x_n, n) \\ x_{n+2} = f(x_n, x_{n+1}) \end{cases}$$

$f(x) = \sqrt{2+x}$

2 è l'unico punto fisso $\qquad \boxed{\left(\sqrt{2+x} = x\right)}$

$I = [-2, 2]$ è invariante?

$-2 \le x \le 2$ $\qquad$ $\underline{f \text{ è crescente}}$

$\Downarrow \qquad \Downarrow$

$f(-2) \le f(x) \le f(2)$ $\qquad$ $a \le b \Rightarrow f(a) \le f(b)$

$\ \ \| \qquad\qquad\qquad \|$

$\ \ 0 \qquad\qquad\qquad 2$ $\qquad$ $[-2,2] \xrightarrow{f} [0,2] \subseteq [-2,2]$

$f(x) \ge x$ su I $\qquad$ $\boxed{\sqrt{2+x} \ge x} \iff -2 \le x \le 2$

$x_{n+1} = f(x_n) \geq x_n \Rightarrow x_n$ è crescente

$\Downarrow$

$\lim_{n \to +\infty} x_n = \ell$ esiste

$x_n \in [-2, 2] \Rightarrow \ell \in [-2, 2]$

$$\begin{array}{ccc} x_{n+1} & = & f(x_n) \\ \downarrow & & \downarrow \\ \ell & = & f(\ell) \end{array}$$

ℓ è un punto fisso

$\Downarrow$

$\ell = 2$.

ESPONENZIALE COMPLESSO

Dato $z \in \mathbb{C}$

$$\exp(z) = \boxed{\sum_{k=0}^{+\infty} \frac{z^k}{k!}} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots = \lim_{N \to +\infty} \sum_{k=0}^{N} \frac{z^k}{k!}$$

(1) converge? OK

(2) Proprietà:

(a) $\exp(0) = \sum_{k=0}^{+\infty} \frac{0^k}{k!} = 1 + 0 + 0 \dots = 1$

(b) $\boxed{\exp(z+w) \overset{?}{=} \exp(z) \cdot \exp(w)}$

$$\exp(z+w) = \sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{+\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}}{n!}$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \sum_{n=0}^{+\infty} \underbrace{\sum_{\substack{j+k=n \\ j \geq 0 \\ k \geq 0}}}_{} \frac{z^k}{k!} \frac{w^j}{j!}$$

$j = n-k$

$$\frac{\binom{n}{k}}{n!} = \frac{1}{k!\,(n-k)!}$$

$$\overset{?}{=} \sum_{j,k \in \mathbb{N}} \frac{z^k}{k!} \frac{w^j}{j!} \overset{?}{=}$$

$$(\exp z)(\exp w) = \left(\sum_{k=0}^{+\infty} \frac{z^k}{k!}\right)\left(\sum_{j=0}^{+\infty} \frac{j^k}{j!}\right) =$$

(c) $$\lim_{\substack{z \to 0 \\ z \neq 0}} \frac{\exp(z) - 1}{z} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\frac{\exp(z)-1}{z} = \frac{\displaystyle\sum_{k=0}^{+\infty} \frac{z^k}{k!} - 1}{z} = \frac{\displaystyle\sum_{k=1}^{+\infty} \frac{z^k}{k!}}{z}$$

$$= \sum_{k=1}^{+\infty} \frac{z^{k-1}}{k!} = 1 + \frac{z}{2} + \frac{z^2}{3!} + \dots \xrightarrow[z \to 0]{} 1.$$

??

$\exp(x) = e^x$ se $x \in \mathbb{R}$.

($\exp(x+y) = \exp(x) \cdot \exp(y) \to$ omomorfismo di $(\mathbb{R}, +) \to (\mathbb{R}_+, \cdot)$

$\exp(x) \in \mathbb{R}$ se $x \in \mathbb{R}$ perché tutti gli addendi sono reali

$\exp(x)$ è crescente ?

$x \leq y \quad y = x + t, \ t \geq 0 \qquad \exp(y) = \exp(x+t)$

$= \exp(x) \cdot \exp(t) \geq \exp(x) \cdot 1$

$\exp(x) > 0$ se $x \geq 0$ $\forall k$! $\frac{x^k}{k!} \geq 0$

$\exp(x) \geq 1 = \frac{x^0}{0!}$

se $\exp(z - z) = \exp(z) \cdot \exp(-z)$

$\exp(0) = 1$

$\exp(-z) = \frac{1}{\exp(z)}$

se $x \geq 0$ $\quad \exp(-x) = \frac{1}{\exp(x)} > 0 \ (\leq 1)$

$\exp : \mathbb{R} \to \mathbb{R}_+$ è un omomorfismo crescente
$\quad + \qquad \cdot$

$\Downarrow$

$\exp(x) = a^x$

con $a = \exp(1)$

$1 = \lim_{x \to 0} \frac{\exp(x) - 1}{x} = \boxed{\lim_{x \to 0} \frac{a^x - 1}{x}} =$

$$= \lim_{x\to 0} \frac{e^{x\ln a} - 1}{x \cdot \ln a} \ln a = \ln a$$

1

$$\ln a = 1$$

$$a = e \quad \square$$

$$\exp(x) = e^x$$

In particolare $\quad e = e^1 = \exp 1 = \sum_{k=0}^{+\infty} \frac{1}{k!}$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$$

$$= 2 + 0{,}5 + 0{,}166\bar{6} + 0{,}041\bar{6} + 0{,}008\bar{3} + \dots$$

$$\sim 2{,}71\bar{6}$$

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(1) converge?

Se $z \in \mathbb{R}$ $\quad z = x \quad \sum_{k=0}^{+\infty} \frac{x^k}{k!}$ converge assolutamente

$$\left|\frac{x^k}{k!}\right| = \frac{|x|^k}{k!}$$

criterio del rapporto

$$\frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k!}} = \frac{|x|}{k+1} \to 0$$

Teorema $a_k \in \mathbb{C}$, Se $\sum_{k=0}^{+\infty} |a_k|$ converge allora

la serie $\sum a_k$ converge.

(def $\sum a_k$ è assolutamente convergente se $\sum |a_n|$ è convergente $a_n \in \mathbb{C}$.)

dim $a_k = x_k + i \cdot y_k$ $\qquad \sum |a_k| < +\infty$

$$|a_k| = \sqrt{|x_k|^2 + |y_k|^2}$$

$\geq |x_k|$

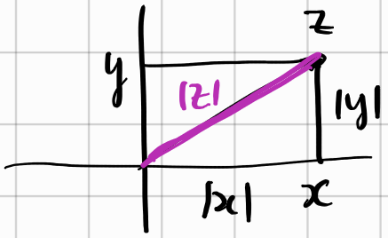
$\geq |y_k|$

$$\Longrightarrow \begin{cases} \sum |x_k| < +\infty \\ \sum |y_k| < +\infty \end{cases}$$

$\Downarrow$

$\sum x_k = x$ converge

$\sum y_k = y$ converge



$$\sum_{k=0}^{+\infty} a_k = \sum_{k=0}^{+\infty} (x_k + i y_k) = \sum_{k=0}^{+\infty} x_k + i \sum_{k=0}^{+\infty} y_k = x + iy.$$

limite del prodotto, limite della somma. $\square$

Osservazione:

$$\left| \sum_{k=0}^{+\infty} a_k \right| \leq \sum_{k=0}^{+\infty} |a_k| \quad \text{se quest'ultima converge}$$

$N \to +\infty$ $\qquad$ $N \to +\infty$

$$|z+w| \leq |z| + |w|$$

$$\left| \sum_{k=0}^{N} a_k \right| \leq \sum_{k=0}^{N} |a_k|$$

$z_n \to z$

$|z_n| \to |z|$

$\square$

$\sum \frac{z^k}{k!}$ è assolutamente convergente $\forall z \in \mathbb{C}$

$\frac{|z|^k}{k!} \longrightarrow$ criterio del rapporto $\longrightarrow$ ok

$\frac{|z|}{k+1} \to 0$

Quand'è che posso cambiare l'ordine degli addendi in una somma infinita?

Teorema Se $\sum |a_n|$ è convergente allora

$$\sum_{n=0}^{+\infty} a_n = \sum_{n=0}^{+\infty} a_{\sigma(n)}$$

dove σ è una qualunque "permutazione" di $\mathbb{N}$

cioè $\sigma : \mathbb{N} \to \mathbb{N}$ bigettiva.

Controesempio $a_n = \frac{(-1)^n}{n}$

$\sum \frac{(-1)^n}{n}$ è convergente

$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n} = +\infty$

non assolutamente convergente

Addirittura $\forall x \in \mathbb{R}$ $\exists \sigma : \mathbb{N} \to \mathbb{N}$ bigettiva

t.c. $\sum_{n=0}^{+\infty} a_{\sigma(n)} = x$.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots+\frac{1}{N_1} > 42$$

$$-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}$$

< 42

> 42

< 42

$\square$