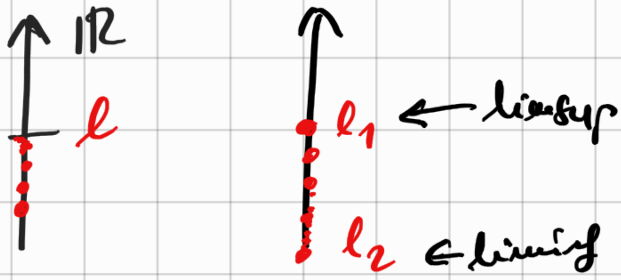
ANALISI MATEMATICA B
LEZIONE 25 – 17.11.2021

PUNTI LIMITE

a_n successione reale



$$L = \{ l \in \overline{\mathbb{R}} : \exists\, a_{n_k} \to l \}$$

$$\lim a_n = l \iff L = \{l\}$$

B.W : $L \neq \emptyset$ $\quad \# L \neq 0 \quad \# L = 1$

a_n è indeterminata $\iff \# L \geq 2$

— o —

limsup e liminf

ES $\quad \lim \sqrt[n]{n!} = +\infty \qquad (n!)^{\frac{1}{n}} = e^{\frac{1}{n}\ln n!} = e^{\frac{\ln n + \ln(n-1) + \dots + \ln 1}{n}}$

$$a_n = \frac{\ln 1 + \ln 2 + \dots + \ln k + \dots + \ln n}{n} = \underbrace{\frac{\ln 1}{n} + \frac{\ln 2}{n} + \dots + \frac{\ln n}{n}}_{n \text{ addendi}} \to 0 \quad \text{NO!}$$

$\forall k \leq n$

(*) $\quad a_n \geq \underbrace{(\ln k) \cdot \frac{(n-k+1)}{n}}_{b_n} \longrightarrow \ln k \xrightarrow[k \to +\infty]{} +\infty$

$$\frac{(n-k+1)}{n} \to 1$$

$\longrightarrow \forall M > 0 \ \exists N > 0 : n > N \Rightarrow a_n > M$

Prendo k t.c. $(\ln k) - 1 > M$

Siccome $a_n \geq b_n \longrightarrow \ln k$

$a_n > \ln k - 1 > M$ definitivamente

$\Rightarrow \quad \ln(n!) \gg n$

(*)

$$a_n \overset{k}{\geq} b_n$$

$$\lim a_n \geq \lim b_n = \underline{\underline{\ln k}} \quad \forall k$$

$$\limsup a_n = \sup L \qquad L = \{\text{punti limite di } a_n\}$$

$$\liminf a_n = \inf L$$

$\sup L \geq \inf L$

$$\lim a_n = \ell \iff \limsup a_n = \liminf a_n = \ell$$

Nell'esempio precedente:

$$\liminf a_n \overset{a_n \geq b_n}{\geq} \liminf b_n = \ln k \qquad \forall k$$

$$\liminf a_n = +\infty \qquad \limsup a_n \geq \liminf a_n = +\infty$$

$$\lim a_n = +\infty$$

Teorema (somma alla Cesáro) $\quad \ell \in \overline{\mathbb{R}}$

Se $a_n \to \ell$ allora $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k \to \ell$.

dim Prendiamo $q \underset{(>)}{<} \ell$ (dimostriamo che $\liminf b_n \geq q$) $\quad$ sup $\quad \leq q$

q ℓ

$a_n \to \ell \; : \exists N: \; n > N \Rightarrow a_n \underset{(\leq)}{\geq} q.$

N fissato (dipende da q)

$$b_n = \frac{a_1 + a_2 + \dots + a_N + \boxed{a_{N+1} + \dots + a_n}}{n}$$

$n - N$

$$\frac{n-N}{n} \xrightarrow[n\to+\infty]{} 1$$

$$\underset{(\leq)}{\geq} \frac{a_1 + a_2 + \dots + a_N}{n} + \frac{q \cdot (n-N)}{n} \to q$$

$\geq q$

$\to 0$

$$\liminf_{n\to+\infty} \text{(sup)}\; b_n \underset{(\leq)}{\geq} q \qquad \forall q < \ell$$

$$\Rightarrow \liminf_{n\to+\infty} b_n \geq \ell \qquad \left(\limsup_{n\to+\infty} b_n \leq \ell\right)$$

$$\underbrace{\ell \leq \liminf b_n \leq \limsup b_n \leq \ell}_{=}$$

$$\lim b_n = \ell$$

$\square$

Corollario (criterio del rapporto/radice) $\boxed{a_n > 0}$

Se $\boxed{\dfrac{a_{n+1}}{a_n}} \to \ell$ allora $\sqrt[n]{a_n} \to \ell$ $\quad \square$

dim

$$\ln \sqrt[n]{a_n} = \frac{1}{n}\ln a_n$$

$$= \frac{1}{n}\ln\left(\frac{a_n}{a_{n-1}}\cdot\frac{a_{n-1}}{a_{n-2}}\cdot\frac{a_{n-2}}{a_{n-3}}\cdots\cdots\frac{a_2}{a_1}\cdot\frac{a_1}{a_0}\cdot a_0\right)$$

$$= \frac{1}{n}\left[\ln\frac{a_n}{a_{n-1}} + \ln\frac{a_{n-1}}{a_{n-2}} + \dots + \ln\frac{a_1}{a_0} + \ln a_0\right]$$

$$= \boxed{\frac{\ln\frac{a_n}{a_{n-1}} + \dots + \ln\frac{a_1}{a_0}}{n}} + \cancel{\frac{\ln a_0}{n}} \to 0$$

$$\to \ln l \Rightarrow \sqrt[n]{a_n} \to l.$$

$$\ln\frac{a_k}{a_{k-1}} \to \ln l \qquad (\ln 0 = -\infty, \quad \ln +\infty = +\infty)$$

□

Es $\sqrt[n]{n!}$

$$\frac{(n+1)!}{n!} = n+1 \to +\infty$$

Es $\displaystyle\lim_{n\to+\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

SERIE

Se a_n è una successione

SOMMA DELLA SERIE

$$S_n = \sum_{k=0}^{n-1} a_k \qquad \lim_{n\to+\infty} S_n = \sum_{k=0}^{+\infty} a_k \in \overline{\mathbb{R}}$$

Serie degli a_k o serie con addendo generico a_k

Esempi

(1) $a_0 \quad a_1 \quad a_2$

$1+1+1+1+\dots = +\infty$? $\quad \Big| \quad \sum_{k=0}^{+\infty} 1 = +\infty$

$a_n = 1$.

$$S_n = \sum_{k=0}^{n-1} 1 = n \xrightarrow[n\to+\infty]{} +\infty$$

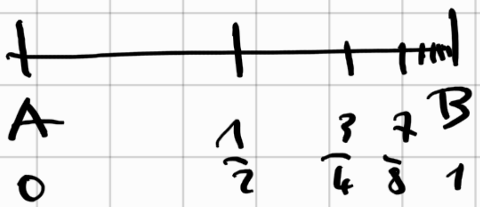
GEOMETRICA

(2) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \qquad \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$

$\left(\frac{1}{2}\right)^0 \quad \left(\frac{1}{2}\right)^1 \quad \left(\frac{1}{2}\right)^2 + \dots$

$|q| < 1$

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \longrightarrow \frac{1}{1-q}$$



Def Diremo che la serie $\sum a_k$ converge (diverge / è indeterminata)

se $S_n = \sum_{k=0}^{n-1} a_k \longrightarrow \ell$, ℓ finito.

(condizione NECESSARIA per la convergenza)

Teorema Se $\sum a_k$ converge allora $a_k \to 0$

dim $S_n = \sum_{k=0}^{n-1} a_k \qquad$ ipotesi $S_n \to \ell$, $\ell \in \mathbb{R}$.

$$\underset{\substack{\downarrow \\ \ell}}{S_{n+1}} - \underset{\substack{\downarrow \\ \ell}}{S_n} = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k = a_n$$

$$l - l = 0 \implies a_n \to 0 \qquad \square.$$

Esempio (SERIE ARMONICA)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \qquad \sum_{k=1}^{+\infty} \frac{1}{k} \overset{?}{=} +\infty$$