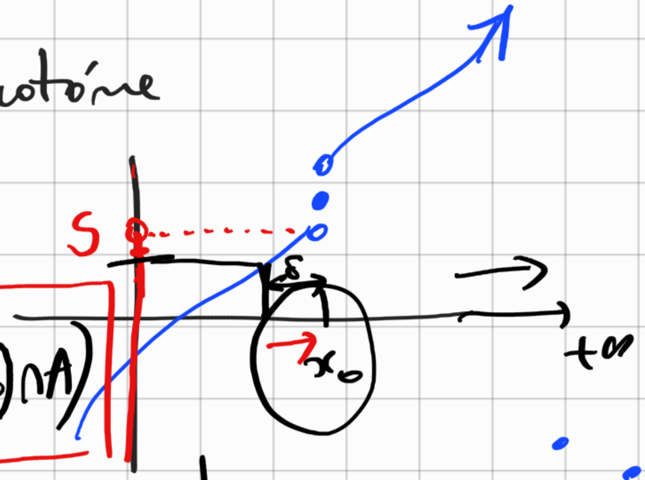
ANALISI MATEMATICA B
LEZIONE 20 5.11.2021

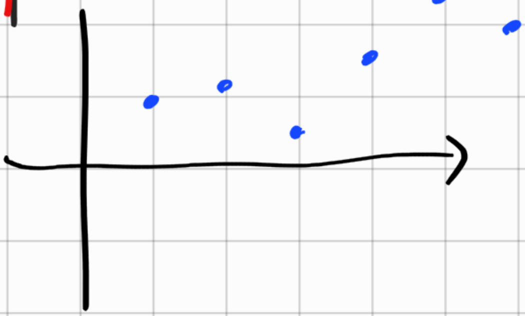
Limite di funzioni monotone



f crescente $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$

$$\boxed{\lim_{x\to x_0^-} f(x) = \sup f((-\infty,x_0)\cap A)}$$

$$\lim_{x\to+\infty} f(x) = \sup f(A)$$



dim

$$S = \sup\{f(x) : x < x_0\} \qquad x_0\in\mathbb{R}$$

certamente (1) $S \geq f(x) \quad \forall x < x_0$ ← sup è un maggiorante

(2) se $m<S$ m non è un maggiorante

$\to \exists\, \delta>0 : \underline{x_0-\delta} < x_0 : f(x_0-\delta) > m$

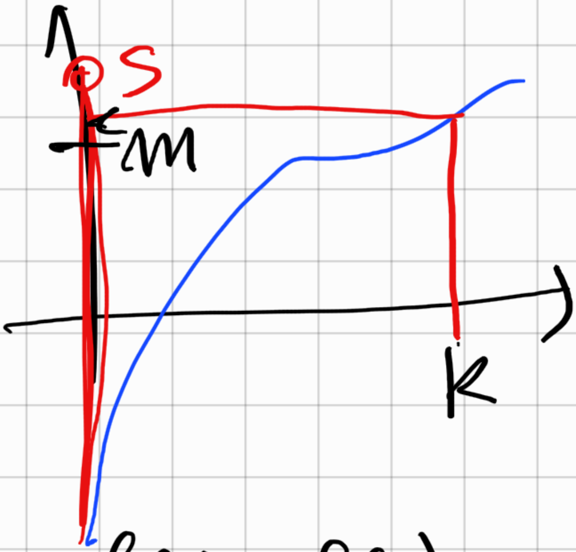
$\forall m<S\ \exists\,\delta>0\ \ m < f(x_0-\delta) \leq f(x) < S$

Se $x > x_0-\delta \Rightarrow f(x) \geq \underline{f(x_0-\delta)} > m$

Se $S\in\mathbb{R}$ $\forall\varepsilon>0$ (posto $m = S-\varepsilon$) $\exists\,\delta>0 : x_0-\delta<x<x_0$

$S-\varepsilon = m < f(x_0-\delta) \leq f(x) < S < S+\varepsilon$

Se $S = +\infty$ ok



Se $x_0 = +\infty$

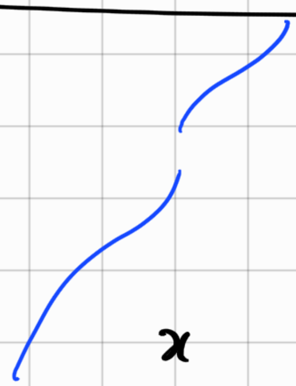
Se $m < S = \sup f(x)$

$\exists\, k : f(k) > m$

$\forall m < S \quad \exists\, k : \forall x > k : f(x) \geq f(k) > m$

e $f(x) < S$

f crescente

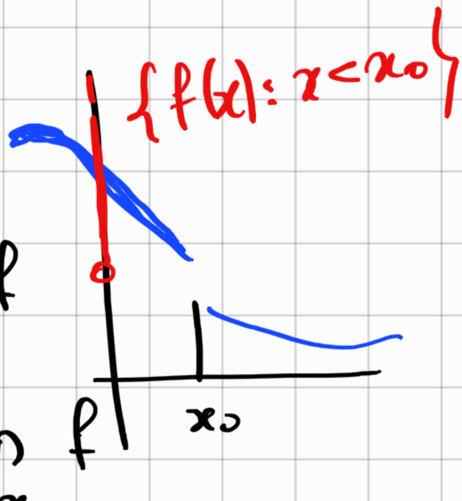


$$\lim_{x\to x_0^-} f = \sup_{x<x_0} f$$

$$\lim_{x\to x_0^+} f = \inf_{x>x_0} f$$

$$\lim_{n\to+\infty} a_n = \sup_{n\in\mathbb{N}} a_n$$

f decrescente



$$\lim_{x\to x_0^-} f = \inf_{x<x_0} f$$

$$\lim_{x\to x_0^+} f = \sup_{x>x_0} f$$

$$\lim_{n\to+\infty} a_n = \inf f$$

$$\left[\ \inf_{x<x_0} f(x) = \inf\{f(x) : x<x_0\} = \inf f((-\infty, x_0))\ \right]$$

Teo (operazioni con i limiti) per $x \to x_0 \in \overline{\mathbb{R}}$

Se $f(x) \to \ell$ e $g(x) \to m$ $\qquad \ell, m \in \overline{\mathbb{R}}$

Allora

$$f(x)+g(x) \to \ell+m$$

$$f(x)\cdot g(x) \to \ell\cdot m$$

$$f(x)-g(x) \to \ell-m$$

$$\frac{f(x)}{g(x)} \longrightarrow \frac{\ell}{m}$$

Se queste operazioni sono definite.

Ovvero le seguenti sono "forme indeterminate"

$+\infty+(-\infty)$ $\quad (-\infty)+(+\infty)$ $\quad (+\infty)-(+\infty)$ $\quad (-\infty)-(-\infty)$

$0\cdot+\infty$ $\quad 0\cdot-\infty$ $\quad \frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $\frac{+\infty}{0}$, $\frac{-\infty}{0}$

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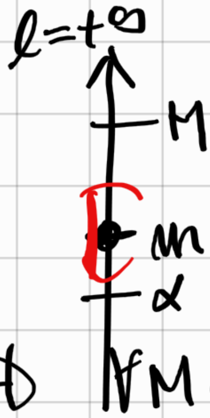
dim (parziale) $\quad \underline{f(x)\to\ell}$, $g(x)\to m$

$$f(x)+g(x) \longrightarrow \ell+m$$

Casi $\ell = +\infty$, $m > -\infty$



$\ell + m = +\infty$

Devo dimostrare: $\lim_{x \to x_0} f(x) + g(x) = +\infty$

$\Leftrightarrow \forall M \in \mathbb{R} \; \exists U \in \mathcal{U}_{x_0} : x \in V \Rightarrow f(x) + g(x) > M$

Se $g(x) \to m$, $\exists V \in \mathcal{U}_{x_0} : x \in V \Rightarrow g(x) > \alpha$

scelto $\alpha < m$

$f(x) + g(x) > f(x) + \alpha$

$g(x) \to m > -\infty$

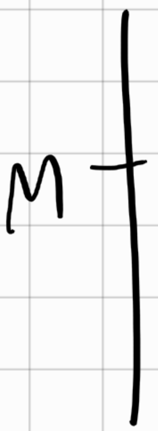
Scelgo $\alpha < m$ $\exists U$ intorno di m

t.c. $\alpha < U$

$\exists V \in \mathcal{U}_{x_0} : x \in V \Rightarrow g(x) \in U \Rightarrow g(x) > \alpha$

$\underbrace{f(x) + g(x)} > \underbrace{f(x) + \alpha}$ (se $x \in V$)

Dato $M \in \mathbb{R}$



Visto che $f(x) \to +\infty$

$\exists W \in \mathcal{U}_{x_0}$ t.c.

$x \in W \Rightarrow f(x) > M - \alpha$

$\Downarrow$

$f(x) + g(x) > M$.

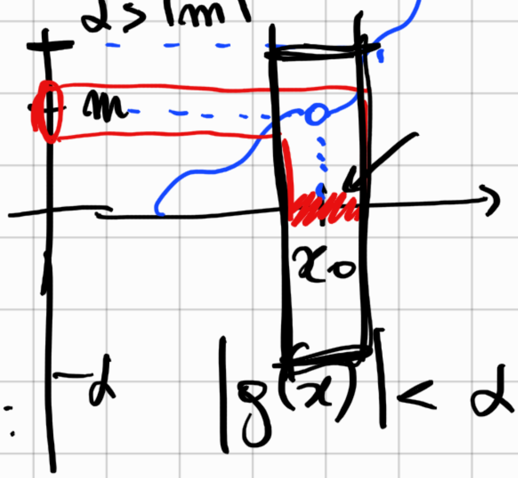
Caso $\ell = 0$, $m \in \mathbb{R}$, $m \cdot \ell = 0$

$$\begin{cases} f(x) \to 0 \\ g(x) \to m \end{cases} \Rightarrow \boxed{f(x) \cdot g(x) \to 0}$$

Se $g(x) \to m \in \mathbb{R}$

$\exists\ \alpha > |m|$ t.c.

un intorno di x_0 : $|g(x)| < \alpha$



$$|f(x) \cdot g(x)| \leq |f(x)| \cdot \alpha \overset{?}{<} \varepsilon$$

$\forall \varepsilon > 0\ \exists U \in \mathcal{U}_{x_0} : x \in U : \quad |f(x) - 0| < \frac{\varepsilon}{\alpha}$

$(f(x) \to 0)$

$$\Downarrow$$

$$|f(x)g(x) - 0| = |f(x)\,g(x)| \leq |f(x)| \cdot \alpha < \varepsilon$$

[$f(x) \cdot g(x) \to 0$ per $x \to x_0$ se

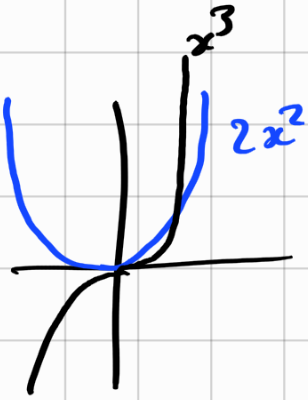
$\forall \varepsilon > 0\ \exists U \in \mathcal{U}_{x_0} : x \in U \Rightarrow |f(x) \cdot g(x)| < \varepsilon$]

Se $f(x) \to \ell \in \mathbb{R}$, $g(x) \to m \in \mathbb{R}$

allora $f(x) \cdot g(x) \to \ell \cdot m$.

Esercizio

$$\boxed{\lim_{x\to+\infty} \frac{x^3-2x^2+3}{(x-7)^2}}$$



$x^3 \to +\infty$ per $x \to +\infty$

$x^2 \to +\infty$

$2\cdot x^2 \to 2\cdot(+\infty) = +\infty$

$x^3 - 2x^2 + 3 \to +\infty \cancel{-(+\infty)}$

$\uparrow$ $+\infty$

NON SI APPLICA IL TEOREMA

$$x^3-2x^2+3 = \overset{+\infty}{x^3}\cdot\underbrace{\left(1-\frac{2x^2}{x^3}+\frac{3}{x^3}\right)}_{\to\, 1-0+0=1} \longrightarrow +\infty\cdot 1 = +\infty$$

$$\frac{2x^2}{x^3} = \frac{2}{x} = 2\cdot\frac{1}{x} \to 2\cdot 0 = 0$$

$$\frac{3}{x^3} = 3\cdot\frac{1}{x^3} \to 0$$

$$\frac{x^3-2x^2+3}{(x-7)^2} = \frac{x^3\cdot\left(1-\frac{2}{x}+\frac{3}{x^3}\right)}{x^2\left(1-\frac{7}{x}\right)^2} = \overset{+\infty}{x}\cdot\frac{\overset{1}{\left(1-\frac{2}{x}+\frac{3}{x^3}\right)}}{\underset{1}{\left(1-\frac{7}{x}\right)^2}} = (*)$$

$$(*) \longrightarrow +\infty\cdot\frac{1}{1} = +\infty$$

Procedimento sbagliato:

$$\lim_{x\to+\infty} \frac{x^3-2x^2+3}{(x-7)^2} = \lim_{x\to+\infty} \frac{x^3\left(1-\cancel{\frac{2}{x}}+\cancel{\frac{3}{x^3}}\right)}{(x-7)^2}$$

0 0

$$= \lim_{x\to+\infty} \frac{x^3}{(x-7)^2} = \dots =$$

VERO

MA INGIUSTIFICATO

$$\frac{x^3-2x^2+3}{x^2-14x+49} = \frac{x^{\cancel{3}}}{\cancel{x^2}} \frac{\left(1-\cancel{\frac{2}{x}}+\cancel{\frac{3}{x^3}}\right)}{\left(1-\cancel{\frac{14}{x}}+\cancel{\frac{49}{x^2}}\right)}$$

0 0 0 0

POTENZE

$$a^{g(x)} \qquad f(x)^{\alpha}$$

Cosa faccio se ho: $= \boxed{f(x)^{g(x)}}$

$$= 2^{g(x)\cdot\log_2 f(x)}$$

se $f(x)>0$

$$f(x) = 2^{\log_2 f(x)}$$

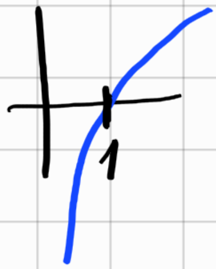
$$f(x)^{g(x)} = \left(2^{\log_2 f(x)}\right)^{g(x)} = 2^{g(x)\cdot\log_2 f(x)}.$$

"0^0"

$$f(x)^{g(x)}$$
$$\parallel$$
$$2^{\underline{g(x)\log_2 f(x)}}$$
$$"\ 2^{0\cdot(-\infty)}\ "$$

$f(x) \to 0^+$

$g(x) \to 0$



"$1^{+\infty}$"

$$f(x)^{g(x)}$$
$$\parallel$$
$$2^{g(x)\cdot\log_2 f(x)} = "\ 2^{+\infty\cdot 0}\ "$$

$f(x) \to 1$

$g(x) \to +\infty$

ES

$$a_n = \left(1+\frac{1}{n}\right)^n$$

NO

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to+\infty} 1^n = 1$$

SBAGLIATO

$$\frac{N(x)}{D(x)}$$

$N(x) \to l$

$D(x) \to m$

NO

$$\lim_{x\to+\infty}\frac{1}{x}\cdot x = \lim_{x\to+\infty} 0\cdot x = 0$$