ANALISI MATEMATICA B
LEZIONE 76 — 19.4.2021

test settimanale

Es 4

$$\begin{cases} u'' + u = \sin(x+1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases} \leftarrow$$ eq. diff. lineare coefficienti costanti non omogenea.

Polinomio: $P(\lambda) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$

soluzioni della omogenea: $u'' + u = 0$

$$u(x) = A \cdot \cos x + B \cdot \sin x$$

Trovo una sol. particolare della non omogenea.

$$Lu = f \qquad f(x) = \sin(x+1) = \alpha e^{ix} + \beta e^{-ix}$$

Cerco una soluzione della forma:

$$u_*(x) = a \cdot x \sin(x+1) + b \cdot x \cdot \cos(x+1)$$

$$u_*'(x) = a \sin(x+1) + a x \cos(x+1) + b \cos(x+1) - b x \sin(x+1)$$

$$u_*''(x) = 2a \cos(x+1) - a x \sin(x+1) - 2b \sin(x+1) - b x \cos(x+1)$$

$$u_\gamma'' + u_\gamma = 2a\cos(x+1) - 2b\sin(x+1)$$

$$\overset{!}{=} \sin(x+1) \qquad \begin{cases} a = 0 \\ -2b = 1 \end{cases} \begin{cases} b = -\frac{1}{2} \end{cases}$$

$$u_\gamma(x) = -\frac{1}{2}x\cos(x+1)$$

Tutte le sol: $u(x) = A\cos x + B\sin x - \frac{1}{2}x\cos(x+1)$

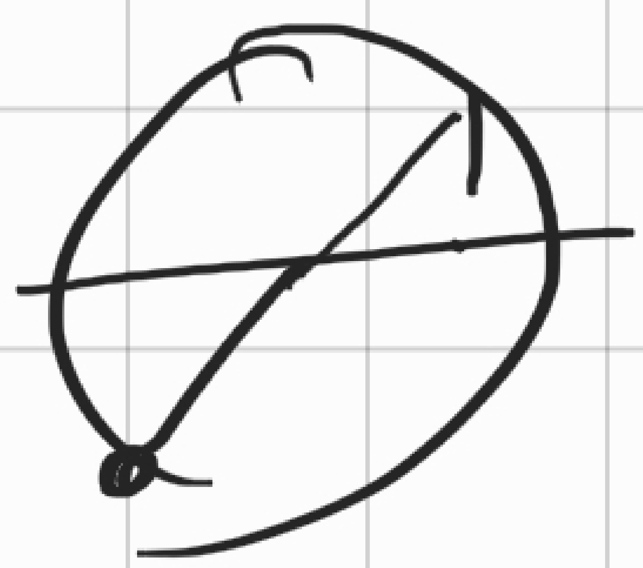
$$u(0) = A \overset{!}{=} 0$$

$$u'(x) = -A\sin x + B\cos x - \frac{1}{2}\cos(x+1) + \frac{1}{2}x\sin(x+1)$$

$$u'(0) = B - \frac{1}{2}\cos 1 \overset{!}{=} 0$$

$$\begin{cases} A = 0 \\ B = \frac{1}{2}\cos 1 \end{cases}, \quad u(x) = \frac{1}{2}\cos 1\sin x - \frac{1}{2}x\cos(x+1)$$

$$u(\pi) = -\frac{\pi}{2}\cos(\pi+1)$$



$$= \frac{\pi}{2}\cos 1 \qquad \square$$

$$\alpha e^{ix} + \beta e^{-ix} = a\cos x + b\sin x = f(x)$$

se f è reale

allora a e b sono reali.

$a = \mathrm{Re}\, a + i\, \mathrm{Im}\, a$

$b = \dots\dots$

$$\begin{cases} \mathrm{Re}\, a \cos x + \mathrm{Re}\, b \sin x = f(x) \\ \mathrm{Im}\, a \cos x + \mathrm{Im}\, b \sin x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \mathrm{Im}\, a = 0 \\ \mathrm{Im}\, b = 0 \end{cases}$$

Es 3

$$u^{IV} - 2u'' + u = e^{x}$$

$$P(\lambda) = \lambda^4 - 2\lambda^2 + 1$$

$$= (\lambda^2 - 1)^2 = (\lambda - 1)^2 (\lambda + 1)^2$$

$\lambda_1 = 1 \quad m_1 = 2$

$\lambda_2 = -1 \quad m_1 = 2 \qquad e^{x} = e^{\mu x} \quad \mu = 2$

$$u_p = C \cdot x^2 e^{x}$$

$$u(x) = (A + Bx)e^{x} + (C + Dx)e^{-x} + u_p(x). \ \square$$

Metodo della variazione delle costanti
(eq. diff. lineari a coefficienti costanti)

$$u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_1 u' + a_0 u = b(x)$$

Le sol. della omogenea sono: $u_k = x^{\ell} e^{\lambda_j x}$

$$u(x) = A_1 u_1 + \dots + A_n u_n$$

Cerco una soluzione particolare

$$
\begin{aligned}
a_0 \quad u_p(x) &= A_1(x) \cdot u_1 + \dots + A_n(x) \cdot u_n \\
a_1 \cdot \quad u_p'(x) &= A_1(x) \cdot u_1' + \dots + A_n(x) \cdot u_n' + \underbrace{A_1' u_1 + \dots + A_n' u_n}_{=0} \\
u_p''(x) &= A_1 \cdot u_1'' + \dots + A_n \, u_n'' + \underbrace{A_1' u_1' + \dots + A_n' u_n'}_{=0} \\
&\vdots \\
a_{n-1} \quad & \\
1 \cdot \quad u_p^{(n)} &= A_1 u_1^{(n)} + \dots + A_n u_n^{(n)} + \underbrace{A_1' u_1^{(n-1)} + \dots + A_n' u_n^{(n-1)}}_{= b(x)}
\end{aligned}
$$

$$\underbrace{A_1(x) \cdot Lu_1}_{=0} + \dots + \underbrace{A_n(x) \cdot Lu_n}_{=0} + b(x)$$

$$\underbrace{u_p^{(n)} + a_{n-1} u_p^{(n-1)} + \dots + a_1 u' + a_0 u}_{} = Lu = b(x)$$

$$\begin{cases} A_1'(x)\, u_1^{(2)} + \dots + A_n'(x)\, u_n(x) = 0 \\ A_1'(x)\, u_1' + \dots + A_n'\, u_n' = 0 \\ \vdots \\ A_1'(x) \cdot u_1^{(n-2)} + \dots + A_n'\, u_n^{(n-2)} = 0 \\ A_1'(x) \cdot u_1^{(n-1)} + \dots + A_n'\, u_n^{(n-1)} = b(x) \end{cases}$$

$$\underbrace{\begin{bmatrix} u_1, & u_2, & \dots & u_n \\ u_1', & u_2', & \dots & u_n' \\ \vdots & & & \\ u_1^{(n-1)}, & \dots & \dots & u_n^{(n-1)} \end{bmatrix}}_{W(x)} \cdot \begin{bmatrix} A_1'(x) \\ \vdots \\ A_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b(x) \end{bmatrix}$$

Teo Se $u_1 \dots u_n$ sono soluzioni indipendenti di una eq. lineare omogenea di ordine n Allora $\det W(x) \neq 0 \quad \forall x$.

Esempio $u''(x) + u(x) = x$

$P(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$

$\rightarrow u_1(x) = \cos x$

$\rightarrow u_2(x) = \sin x$

soluzioni indipendenti della omogenea.

1. $u_p(x) = \underline{A(x) \cdot \cos x + B(x) \sin x}$

$$0\cdot\ u_\gamma'(x) = -A(x)\sin x + B(x)\cos x + \left\{\begin{array}{l} A'(x)\cos x + B'(x)\sin x = 0 \\ -A'(x)\sin x + B'(x)\cos x = x \end{array}\right.$$

$$1\cdot\ u_\gamma''(x) = -A(x)\cos x - B(x)\sin x + \ldots$$

$$u_\gamma''(x) + u_\gamma(x) = 0 + 0 + x \quad \underline{\text{ok!}}$$

$$\begin{cases} A'(x)\cos x + B'(x)\sin x = 0 \\ -A'(x)\sin x + B'(x)\cos x = x \end{cases}$$

$$\det\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\begin{cases} B'(x) = -A'(x)\dfrac{\cos x}{\sin x} \\ -A'(x)\sin x - A'(x)\dfrac{\cos^2 x}{\sin x} = x \end{cases}$$

$$-A'(x)\sin^2 x - A'(x)\cos^2 x = x\sin x$$

$$-A'(x) = x\sin x$$

$$A(x) = \int x(-\sin x)\,dx = x\cos x - \int 1\cdot\cos x\,dx$$

$$= x\cos x - \sin x$$

$$B'(x) = x \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} = x\cos x$$

$$B(x) = \int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$
$$\uparrow \quad \uparrow$$
$$= x \sin x + \cos x$$

$$u_p(x) = (x\cos x - \sin x)\cos x + (x \sin x + \cos x)\sin x$$

$$= \underline{x \cos^2 x} - \cancel{\sin x \cos x} + \underline{x \sin^2 x} + \cancel{\cos x \sin x}$$

$$= x \qquad\qquad u_p'(x) = 1$$
$$u_p''(x) = 0$$

$$u_p'' + u_p = 0 + x = x.$$

Tutte le soluzioni: $u(x) = A \cdot \cos x + B \sin x + x$

Potevamo usare il metodo di similarità:

$$u'' + u = x = x \cdot e^{\mu x} \qquad \mu = 0$$
$$P(\lambda) = (\lambda + i)(\lambda - i)$$

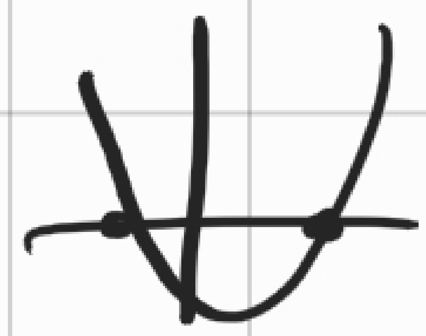
Cerco $u_p(x) = ax + b$
$$u_p'(x) = a$$
$$u_p''(x) = 0$$

$$u_\gamma'' + u_\gamma = 0 + ax + b \overset{!}{=} x$$

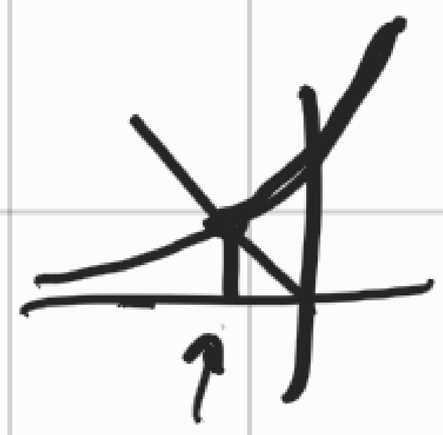
$b = 0$ $\qquad u_\gamma(x) = x.$

$a = 1$

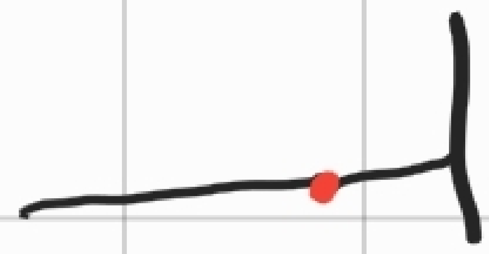
$x^2 = 2 \rightsquigarrow x = \pm\sqrt{2}$



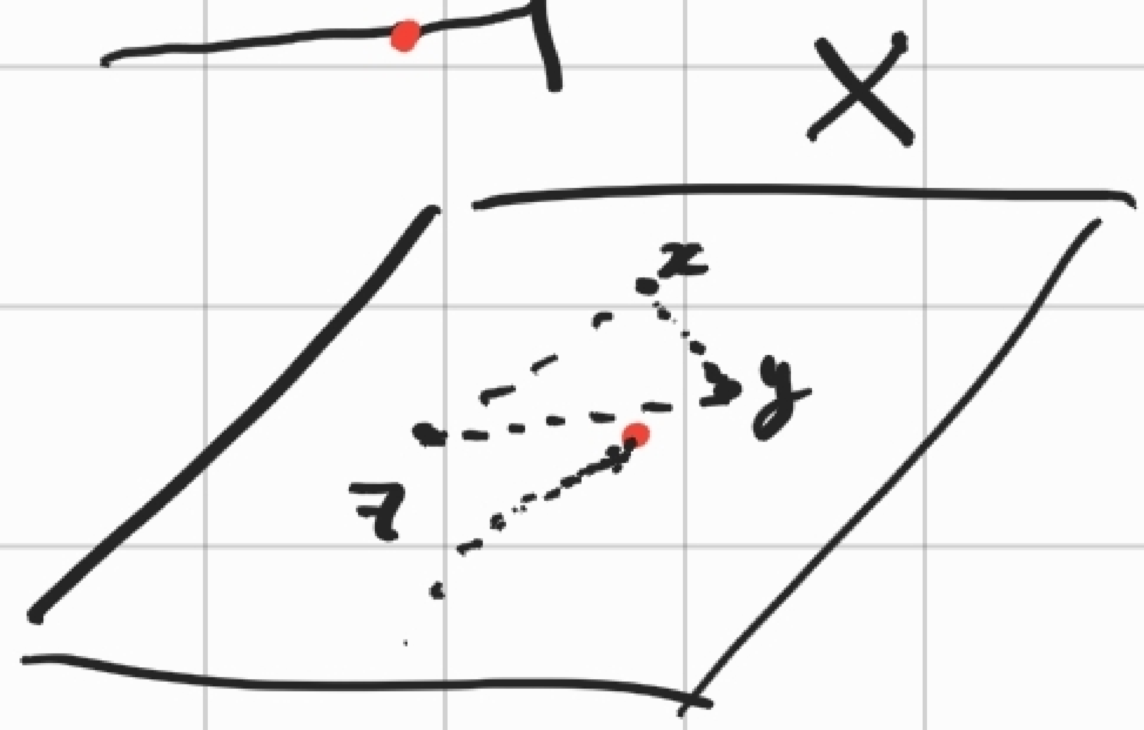
$e^x = -x \rightsquigarrow ?$



$\uparrow$ continuità di $\mathbb{R}$



Completezza



Spazio metrico X insieme, $d : X \times X \to \mathbb{R}$

d si dice una distanza su X se:

(i) $d(x,y) \geq 0$, $d(x,y) = 0 \iff x = y$.

(ii) $d(x,z) \leq d(x,y) + d(y,z)$

(iii) $d(x,y) = d(y,x)$

X con la distanza d si dice essere uno spazio metrico

$f: X \to Y$ continua (X, Y sp. metrici)

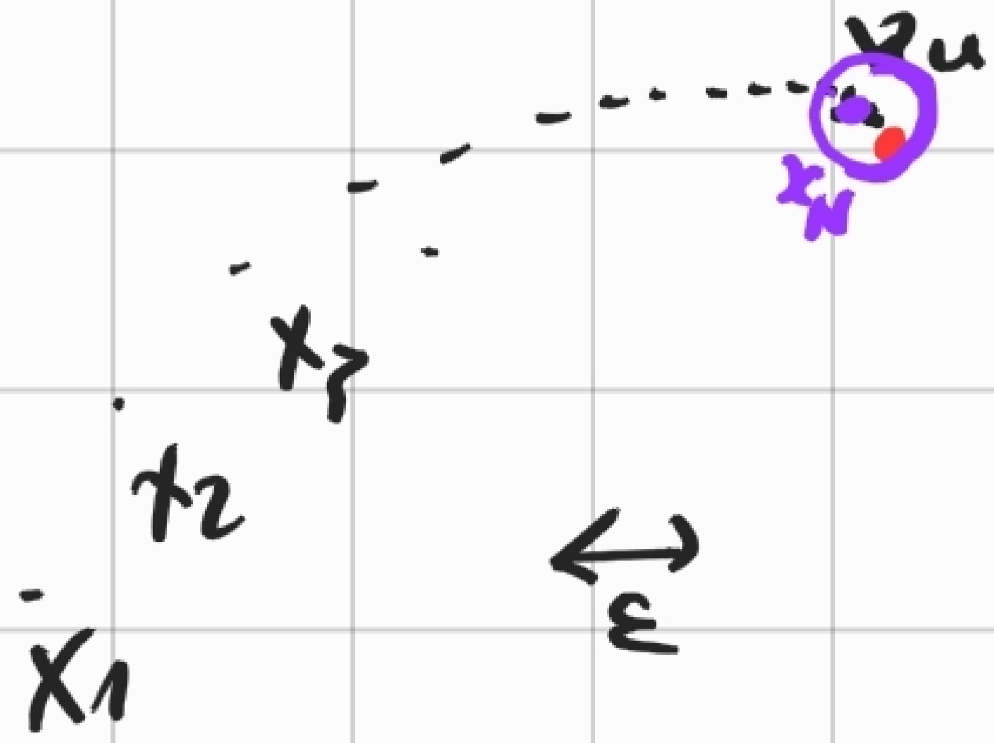
se $\forall x_0 \in X$: $\forall \varepsilon > 0\ \exists \delta > 0$: $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

$\uparrow$ X $\qquad \uparrow$ Y

Successione di Cauchy:

Diremo che $x_n \in X$ (X sp. metrico) è di Cauchy se:

$$\forall \varepsilon > 0\ \exists N : \forall i, j \geq N : d(x_i, x_j) < \varepsilon$$



Def. Lo spazio metrico X si dice essere <u>Completo</u> se ogni successione di Cauchy converge.