ANALISI MATEMATICA B
LEZIONE 73 - 13.4.2021

Eq. differenziali primo ordine

$$F(x, u(x), u'(x)) = 0$$

Caso 0: (F non dipende da u, in forma normale)

$$u'(x) = f(x)$$

Sol: $u \in \int f$

$$\begin{cases} u'(x) = f(x) \\ u(x_0) = y_0 \end{cases} \leftarrow$$

$$\begin{cases} u \in \int f \\ u(x_0) = y_0 \end{cases}$$

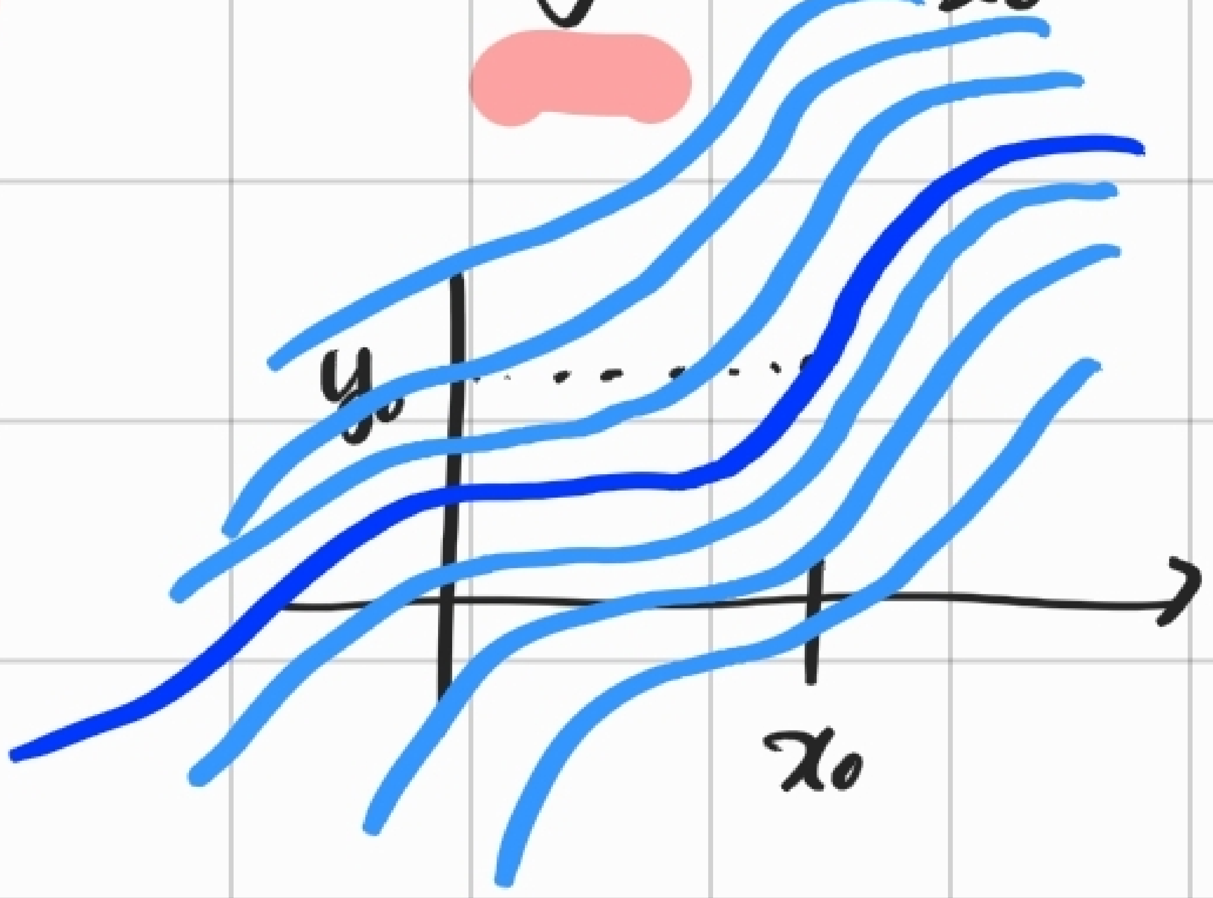
Se u è definita su un intervallo I

$$u : I \to \mathbb{R}$$

$$x_0 \in I$$

$$\begin{cases} u(x) = c + \int_{x_0}^{x} f(t)\,dt \\ u(x_0) = y_0 \end{cases}$$

$$u(x) = y_0 + \int_{x_0}^{x} f(t)\,dt.$$



Equazione del primo ordine lineare in forma normale

$$u'(x) = p(x)\cdot u(x)$$ ← forma normale

$L[u]=0$

$$\underbrace{u'(x) + a(x)\cdot u(x)} = 0$$ ← lineare omogenea

spazi vettoriali

$u \in C^1(I)$

$$L : C^1(I) \to C^0(I)$$
$$u \longmapsto L[u]$$

$$L[u](x) = u'(x) + a(x)\cdot u(x).$$

<u>Esercizio</u> L è lineare

$L[u]=b$

$$\boxed{u'(x) + a(x)u(x) = b(x)}$$ non omogenea

<u>Es</u> $u'(x) + x\cdot u(x) = x^2$

Metodo risolutivo

$$(u \cdot c)' = \underline{c'u + cu'}$$

$$u'(x) + a(x)\,u(x)$$

fattore integrante

$$= u'(x)e^{A(x)} + a(x)\,e^{A(x)}u(x)$$

Se scelgo $A \in \int a$ $\quad \left(e^{A(x)}\right)' = A'(x)e^{A(x)}$
$= a(x)e^{A(x)}$

$$= \left(u(x) \cdot e^{A(x)}\right)'$$

$$\longrightarrow \quad u'(x) + a(x)\,u(x) = b(x)$$

moltiplico ambo i membri per $\underline{e^{A(x)}}$ $\neq 0$ con $A \in \int a$

$$u'(x)\,e^{A(x)} + a(x)u(x)e^{A(x)} = b(x)e^{A(x)}$$

$$\left(u(x) \cdot e^{A(x)}\right)' = b(x) \cdot e^{A(x)}$$

$$u(x) \cdot e^{A(x)} \in \int b(x)\, e^{A(x)}\, dx$$

$$u(x) \in e^{-A(x)} \int b(x)\, e^{A(x)}\, dx.$$

ES $\quad u'(x) + x\,u(x) = x^2$

$e^{\frac{x^2}{2}} \leftarrow$ fattore integrante

$$u' \cdot e^{\frac{x^2}{2}} + x\, e^{\frac{x^2}{2}} u = x^2 e^{\frac{x^2}{2}}$$

$$\left(u \cdot e^{\frac{x^2}{2}}\right)' = x^2 e^{x^2/2}$$

$$u \cdot e^{\frac{x^2}{2}} \in \int x^2 e^{x^2/2}\, dx$$

$$u \in e^{-x^2/2} \int x^2 e^{x^2/2}\, dx$$

$$\begin{cases} u'(x) + x\,u(x) = x^2 \\ u(0) = 1 \end{cases} \qquad u: \mathbb{R} \to \mathbb{R}$$

$$u(x) \cdot e^{\frac{x^2}{2}} \in \int x^2 e^{x^2/2}\, dx$$

$$u(x) \cdot e^{x^2/2} = c + \int_0^x t^2 e^{t^2/2}\, dt$$

$$u(x) = c \cdot e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \int_0^x t^2 e^{t^2/2}\, dt$$

$$= c \cdot e^{-\frac{x^2}{2}} + \int_0^x t^2\, e^{t^2/2 - \frac{x^2}{2}}\, dt$$

$$u(0) = 1$$

$$1 = u(0) = c \cdot e^{0} + \int_0^0 \text{—}\, dt$$
$$= c$$

$$\boxed{u(x) = e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \int_0^x t^2 e^{t^2/2}\, dt}$$

ES
$$\begin{cases} u' - \dfrac{u}{x} = x^2 \\ u(-1) = 0 \end{cases}$$

$x \neq 0$

$u : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

Mi interesso alle soluzioni definite per $x < 0$

$$u' - \frac{u}{x} = x^2 \qquad a(x) = -\frac{1}{x}$$

$$\left[\; e^{A(x)} = e^{-\ln(-x)} = -\frac{1}{x} \qquad A(x) = -\ln|x| = -\ln(-x) \;\right]$$

$$-\frac{1}{x} u' + \frac{u}{x^2} = -x$$

$$\left(-\frac{1}{x} u\right)' = -x$$

$$-\frac{1}{x}\cdot u(x) \in \int -x \, dx$$

$$-\frac{1}{x} u(x) = -\frac{x^2}{2} + c$$

$$\boxed{u(x) = \frac{x^3}{2} - cx} = x\left(\frac{x^2}{2} - c\right)$$

$$u(-1) = 0$$

$$0 = u(-1) = \frac{(-1)^3}{2} - c(-1)$$

$$0 = -\frac{1}{2} + c \qquad c = \frac{1}{2}$$

$$\boxed{u(x) = \frac{x^3}{2} - \frac{x}{2}} = \frac{x}{2}\left(x^2 - 1\right)$$

Verifica:

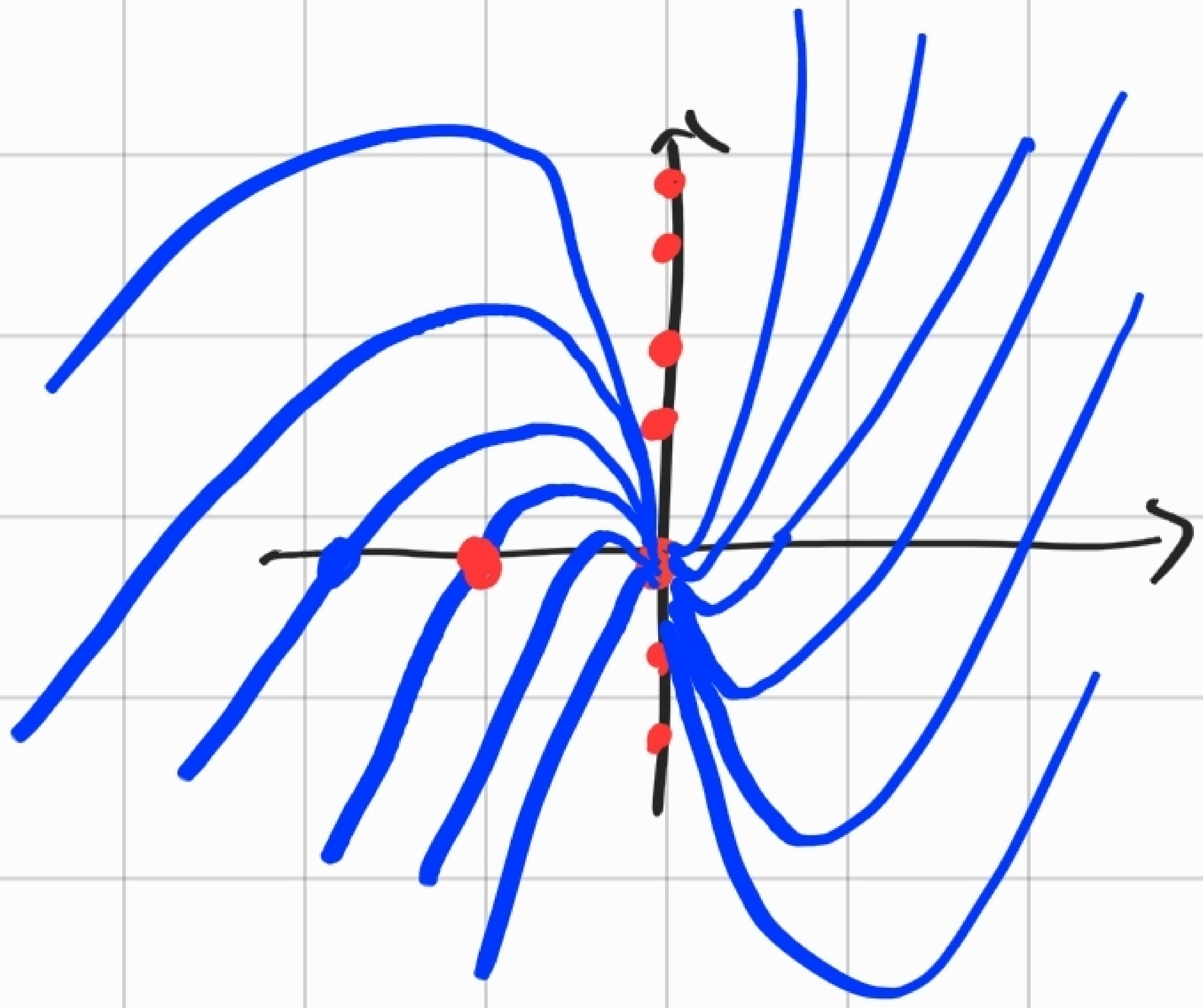
$$u'(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$u' - \frac{u}{x} = \left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \left(\frac{x^2}{2} - \frac{1}{2}\right)$$

$$= x^2 \quad \checkmark$$

$$u(-1) = \frac{(-1)^3}{2} - \frac{(-1)}{2} = -\frac{1}{2} + \frac{1}{2} = 0. \checkmark$$

Tutte le soluzioni: $\quad u(x) = \frac{x^3}{2} - cx$



Le soluzioni di $L[u] = b$ si ottengono prendendo una qualunque soluzione u_*

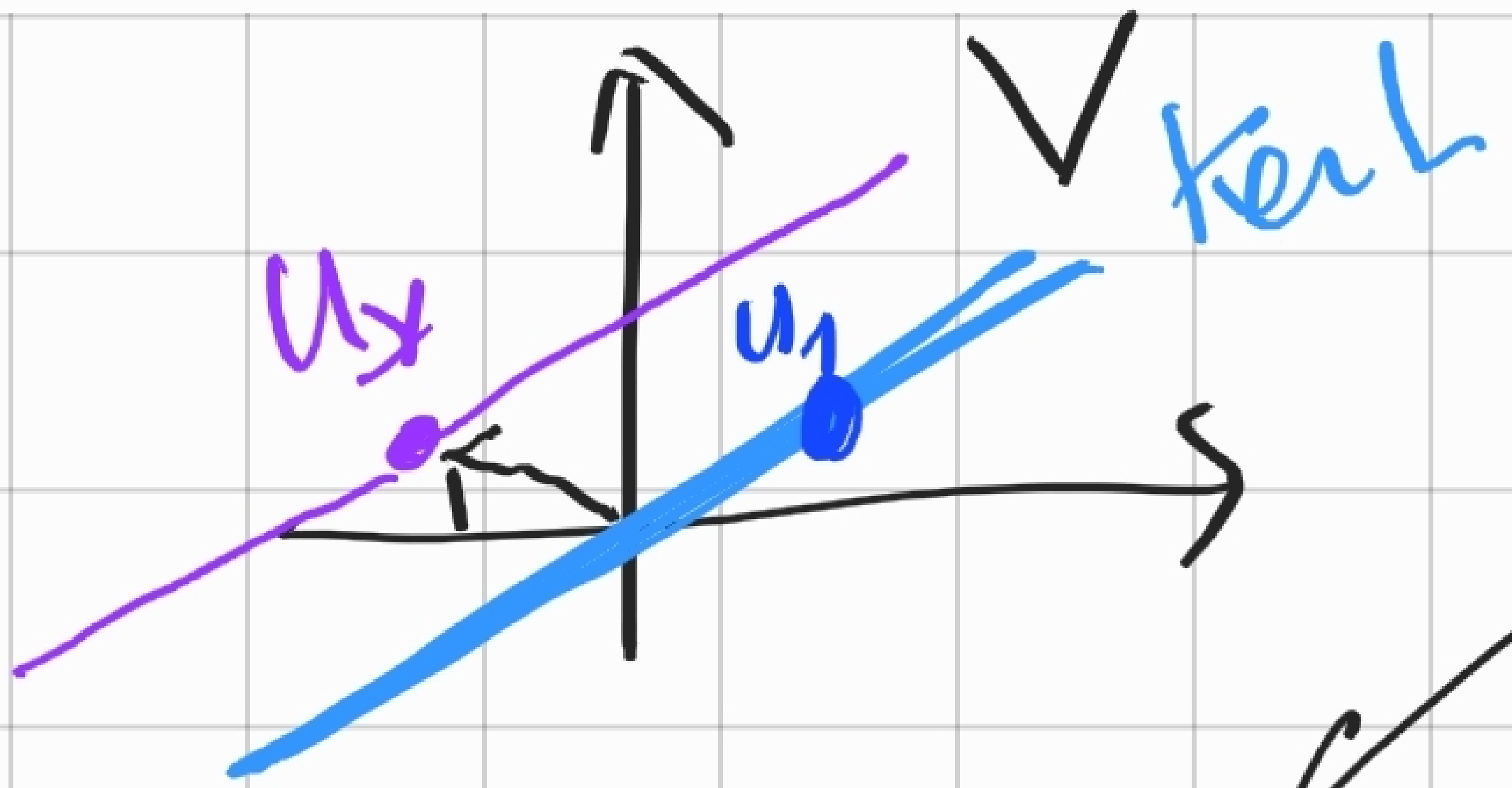
$$L[u_*] = b$$

e aggiungendo tutte le soluzioni della omogenea: $\quad L[u_0] = 0$

$$\boxed{u = u_* + u_0}$$

Se l'eq. è del I ordine $\quad u_0 = \lambda u_1$

u_1 base di $\ker L$.



$$u(x) = \underbrace{\frac{x^3}{2}}_{u_*} - \underbrace{C\cdot x}_{u_1}$$

$$L[u](x) = u'(x) - \frac{u(x)}{x}$$

$u_* = \frac{x^3}{2}$ è una soluzione della eq. $L[u](x) = x^2$.

x è una sol. di $L[u] = 0$

$$L[x] = 1 - \frac{x}{x} = 0. \quad \checkmark$$

In generale:

$$u(x) \in e^{-A(x)} \int b(x)\, e^{A(x)}\, dx.$$

$$\boxed{u = e^{-A(x)}\left(c + \int_{x_0}^{x} b(t)\, e^{A(t)}\, dt\right)}$$

$$= \underbrace{C \cdot e^{-A(x)}}_{\text{sol. della omogenea.}} + \underbrace{e^{-A(x)} \int_{x_0}^{x} b(t)\, e^{A(t)}\, dt}_{\text{sol. particolare della non omogenea.}}$$

<u>ES</u> (Rilevante) $L = D$

$Du = 0 \Rightarrow u = \text{cost}$

$Du = b \Rightarrow u \in \int b$

Autovettori di D

$$Du = \lambda u$$

$$\longrightarrow u' - \lambda u = 0$$

$$u' e^{-\lambda x} - \lambda e^{-\lambda x} u = 0$$

$$\left(u \cdot e^{-\lambda x} \right)' = 0$$

$$u e^{-\lambda x} = c$$

$$u = c \cdot \underbrace{e^{\lambda x}}$$

Eq. del I ordine a variabili separabili

ES $$u'(x) = x \cdot \underbrace{u^2(x)}_{\uparrow} = f(x, u(x))$$

$$f(x,y) = x \cdot y^2$$

Attenzione: $f(x,y) = g(x) \cdot h(y)$

se $u(x) = 0 \leftarrow \quad h(0) = 0$

$u'(x) = 0$

$$\frac{u'(x)}{u^2(x)} = x$$

INFORMALMENTE

$$\int \frac{u'(x)}{u^2(x)}\, dx = \int x \, dx$$

$$\parallel$$

$$H \in \int \frac{du}{u^2} = \int x \, dx \qquad \text{HA SENSO ?!?}$$

$$H'(u) = \frac{1}{u^2} \qquad H(u) = -\frac{1}{u}$$

$$\left(H(u(x)) \right)' = H'(u(x))\, u'(x) = \frac{u'(x)}{u^2(x)}$$

$$\left(-\frac{1}{u(x)} \right)' = \frac{u'(x)}{u^2(x)} = x$$

$$-\frac{1}{u(x)} \in \int x \, dx$$

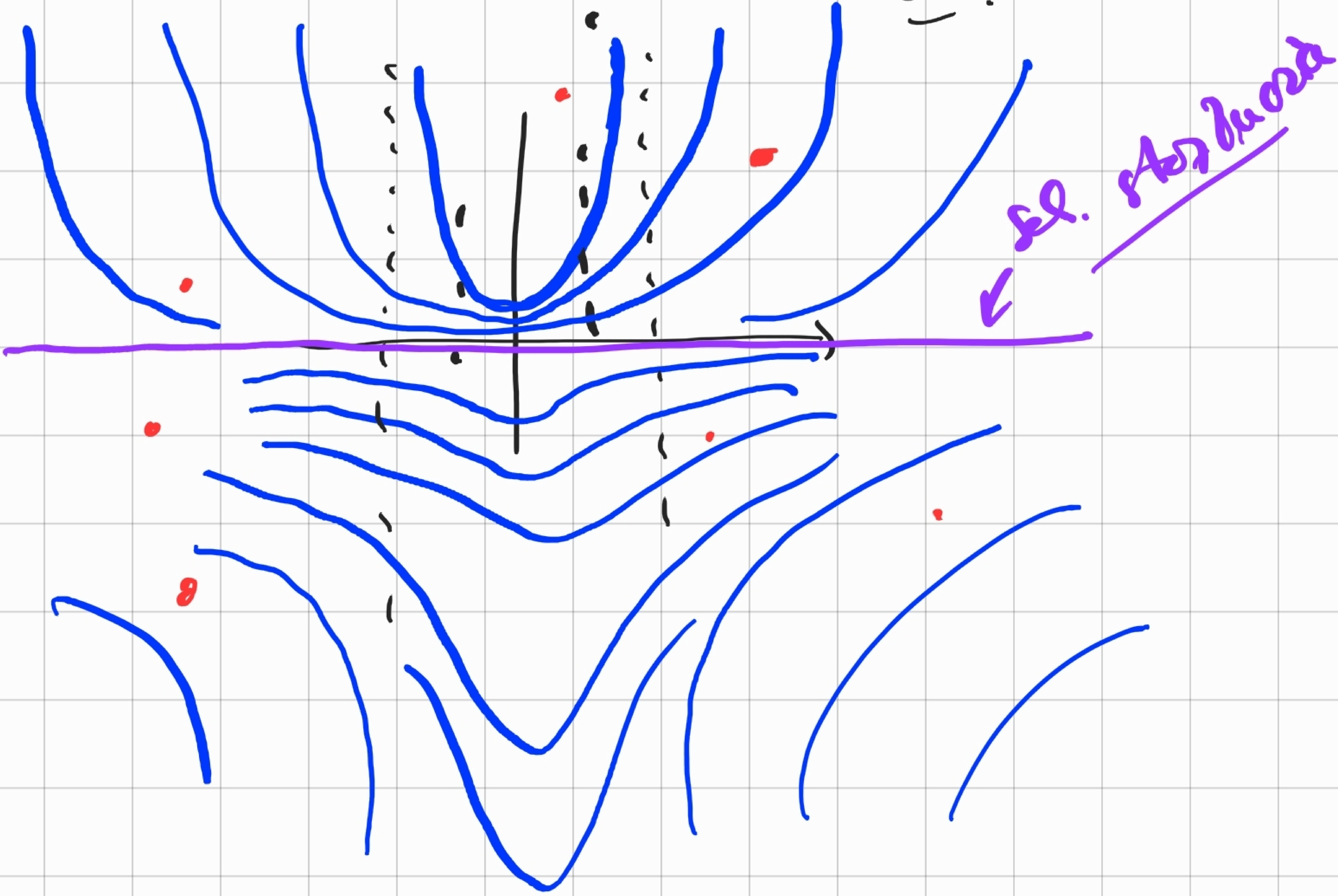
$$-\frac{1}{u(x)} = \frac{x^2}{2} + c$$

$$u(x) = \frac{1}{-\frac{x^2}{2} - c} = -\frac{2}{x^2+2c}$$

Verifica

$$u'(x) = \frac{2 \cdot 2x}{(x^2+2c)^2} \overset{?}{=} x \cdot u^2 = \frac{4x}{(x^2+2c)^2}$$

ok!



Es

$$u' = u^3$$

(eq. autonoma)

$$u' = f(x, u(x))$$

$$\uparrow$$

$$u' = h(u(x)).$$

$u = 0$ è sol.

$$\frac{u'}{u^3} = 1$$

$$\int \frac{du}{u^3} = \int 1 \, dx$$

$$\frac{u^{-2}}{-2} = x - c$$

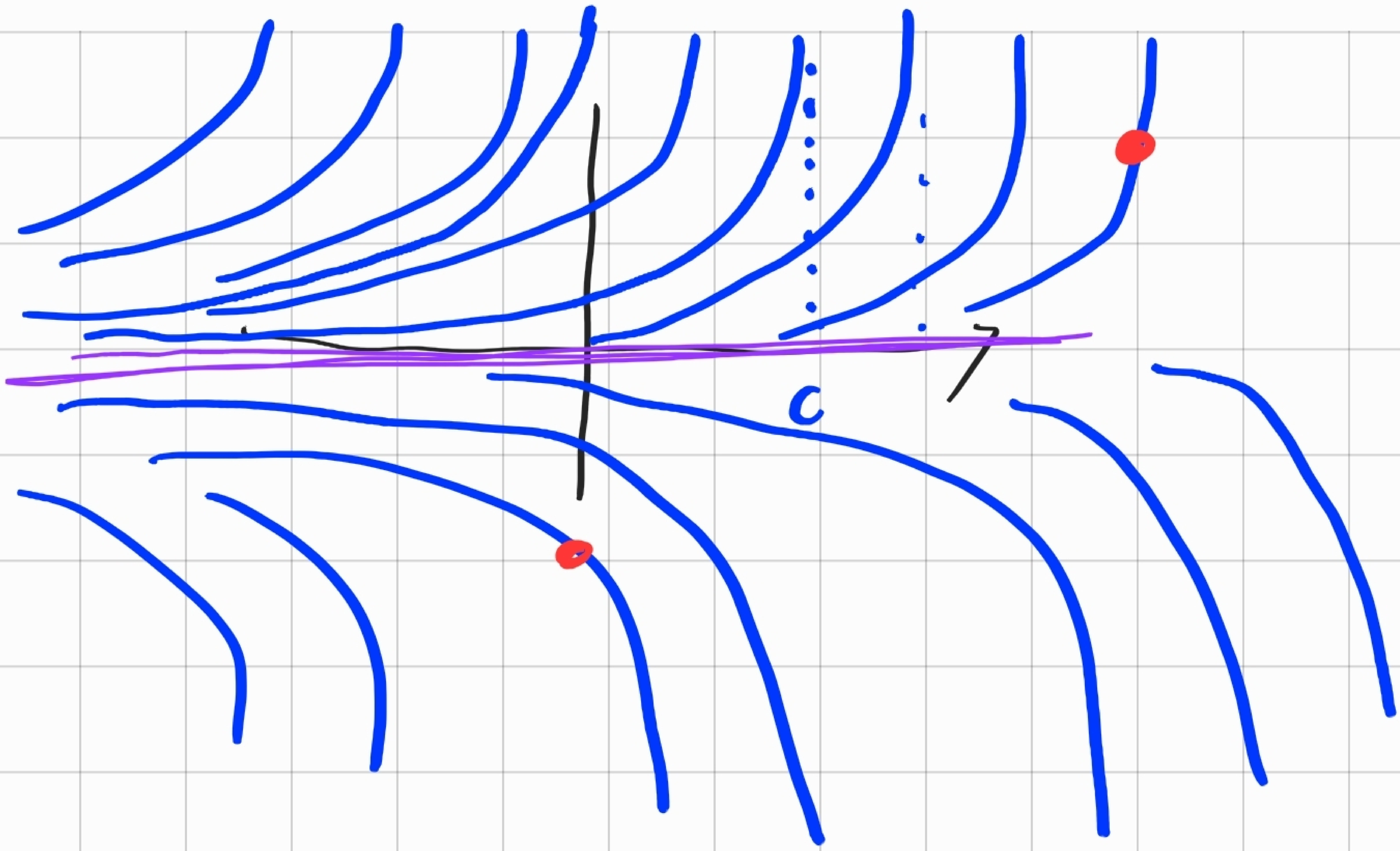
$$-\frac{1}{2u^2} = x - c$$

$$\frac{1}{2u^2} = -x + c$$

$$2u^2 = \frac{1}{-x + c}$$

$$u^2 = \frac{1}{-2x + 2c}$$

$$u(x) = \frac{1}{\pm\sqrt{2(c-x)}}$$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