ANALISI MATEMATICA B
LEZIONE 71 - 31.3.2021

Convergenza uniforme (successioni di funzioni)

Motivazione: Passaggio al limite sotto il segno di integrale.

Es

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt = \int_0^{+\infty} f(x,t)\, dt$$

$$\Gamma'(x) = \frac{d}{dx} \int_0^{+\infty} e^{-t} t^{x-1}\, dt$$

$$= \lim_{h\to 0} \frac{\Gamma(x+h)-\Gamma(x)}{h}$$

$$= \lim_{h\to 0} \int_0^{+\infty} \frac{f(x+h,t)-f(x,t)}{h}\, dt$$

$$\boxed{\overset{?}{=}} \int_0^{+\infty} \lim_{h\to 0} \frac{f(x+h,t)-f(x,t)}{h}\, dt$$

$$= \int_0^{+\infty} \frac{\partial f}{\partial x}(x,t)\, dt$$

$$\frac{d}{dx}\int_0^{+\infty} f(x,t)\,dt \overset{?}{=} \int_0^{+\infty} \frac{\partial}{\partial x} f(x,t)\,dt$$

E' giustificato questo passaggio?

Consideriamo il caso discreto $f_n(t)$ $\quad n\in\mathbb{N}$.

$$\left[\text{es: } f_n(t) = \frac{f(x+\frac{1}{n},t)-f(x,t)}{\frac{1}{n}}\right]$$

f_n è una successione di funzioni.

$$\lim_{n\to+\infty}\int_a^b f_n(t)\,dt \overset{?}{=} \int_a^b \underbrace{\lim_{n\to+\infty} f_n(t)}\,dt.$$

$$= \int_a^b f(t)\,dt$$

$$f(t)=\lim_{n\to+\infty} f_n(t).$$

$f_n: A\to\mathbb{R}$, $\quad A\subseteq\mathbb{R}$. $\qquad f_n\in\mathbb{R}^A$

$f: A\to\mathbb{R}$, $\qquad f\in\mathbb{R}^A$

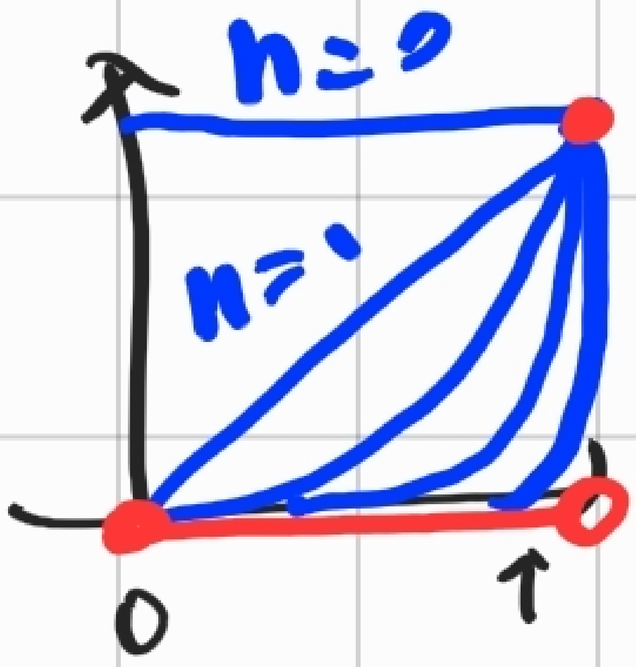
Diremo che f_n converge puntualmente ad f

se $\quad \forall t\in A: \quad f_n(t)\longrightarrow f(t)$

Es: $f_n(x) = x^n$

se $x \in [0,1]$ $\qquad \lim_{n\to+\infty} x^n = f(x) = \begin{cases} 0 & \text{se } x<1 \\ 1 & \text{se } x=1. \end{cases}$

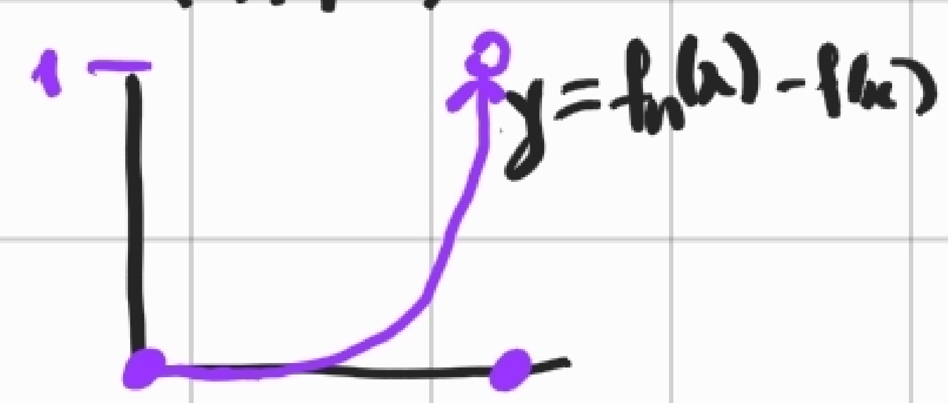
$f_n \to f$ puntualmente



$y = f_n(x)$

$y = f(x)$.

$d_\infty(f_n, f) = 1$



$$\int_0^1 f_n(x)\,dx = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} \to 0$$

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0 \qquad ✓.$$

Esempio (cattivo) $\qquad f_n : [0,1] \to \mathbb{R}$

$\qquad f : [0,1] \to \mathbb{R}$

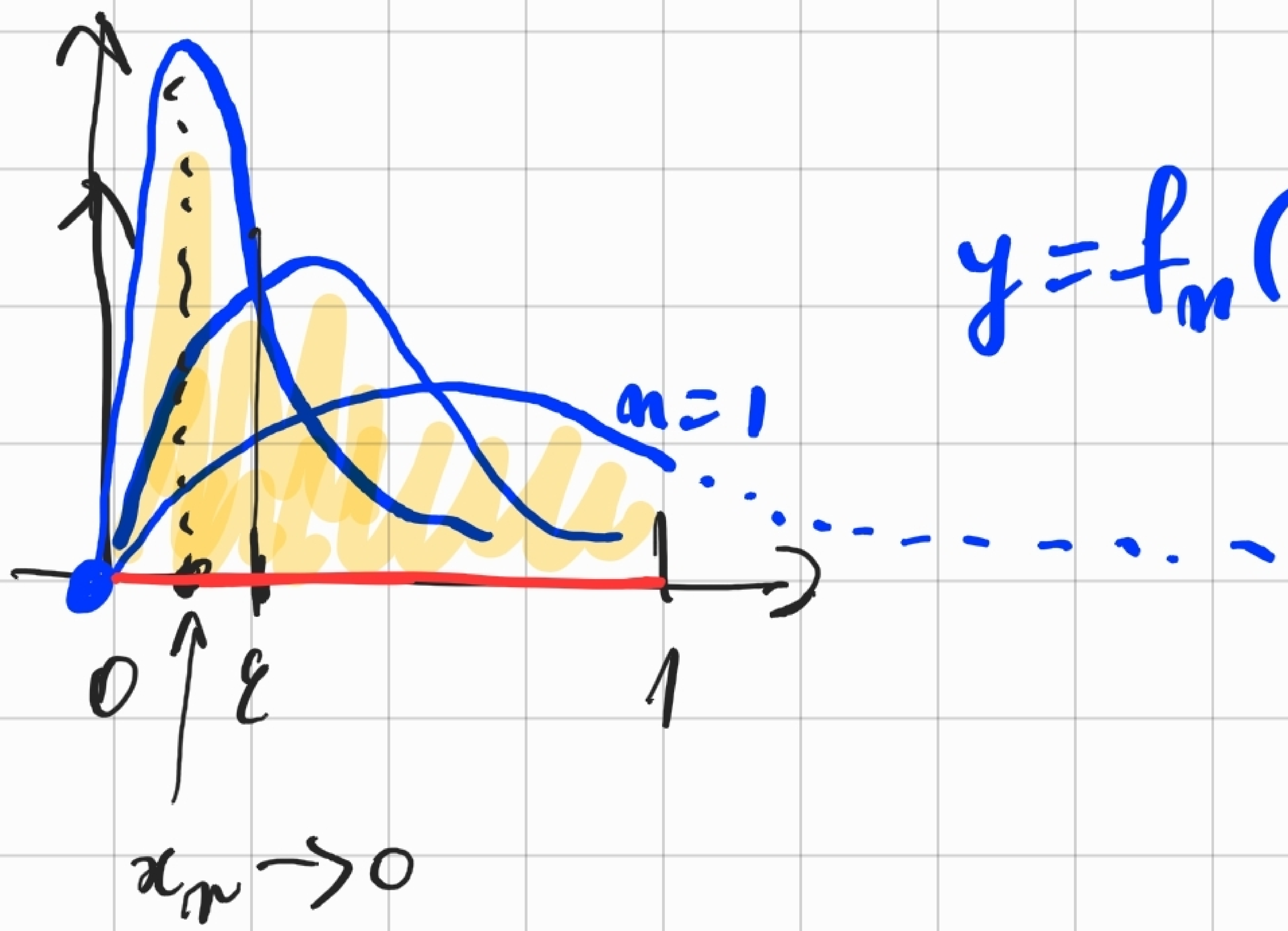
$$f_n(x) = \frac{nx}{1+n^2x^4}$$

$$f(x) = \lim_{n\to+\infty} f_n(x) = 0$$

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{nx}{1+n^2x^4}\,dx = \frac{1}{2}\left[\operatorname{arctg}(nx^2)\right]_0^1$$

$$= \frac{1}{2}\operatorname{arctg} n \longrightarrow \frac{\pi}{4} \quad \text{per } n\to+\infty$$

$$\lim_{n\to+\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to+\infty} f_n(x)\,dx$$



La convergenza puntuale non è quella "giusta".

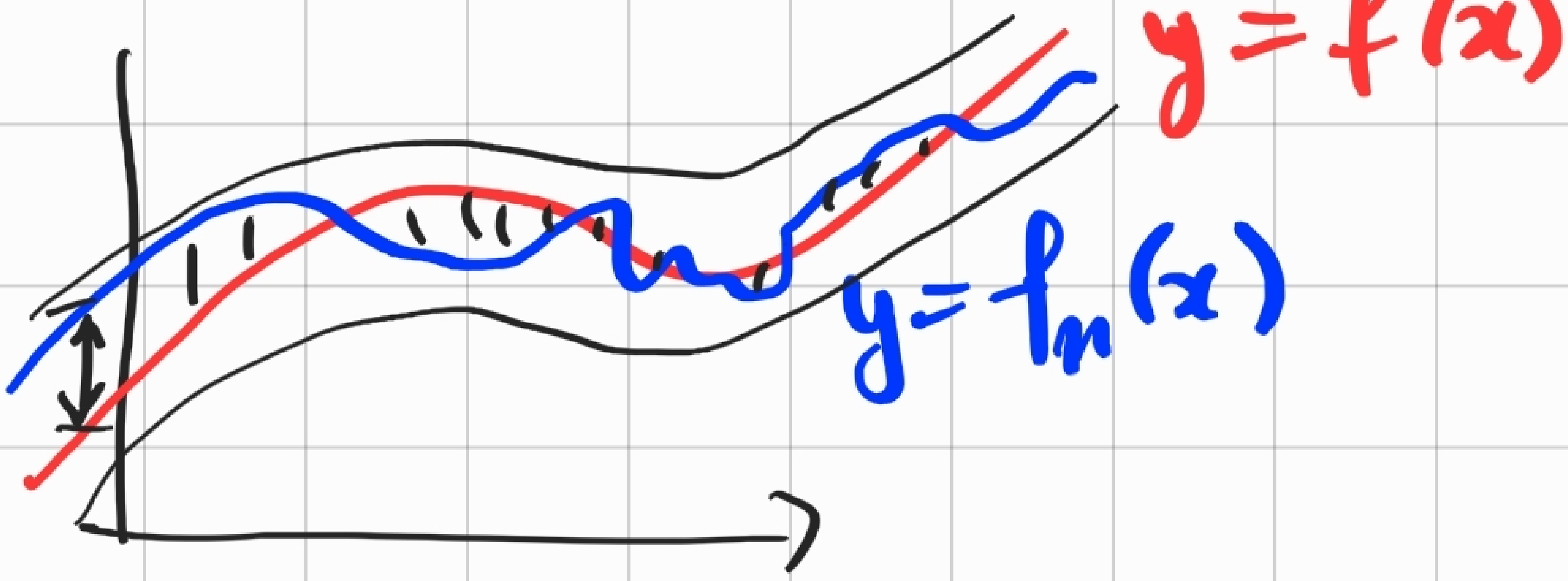
Convergenza uniforme:

$$f_n : A \to \mathbb{R}$$
$$f : A \to \mathbb{R}$$

Diremo che f_n converge uniformemente a f, scriveremo:

$$f_n \rightrightarrows f$$

Se
$$\sup_{x\in A} |f_n(x) - f(x)| \longrightarrow 0 \qquad \square$$

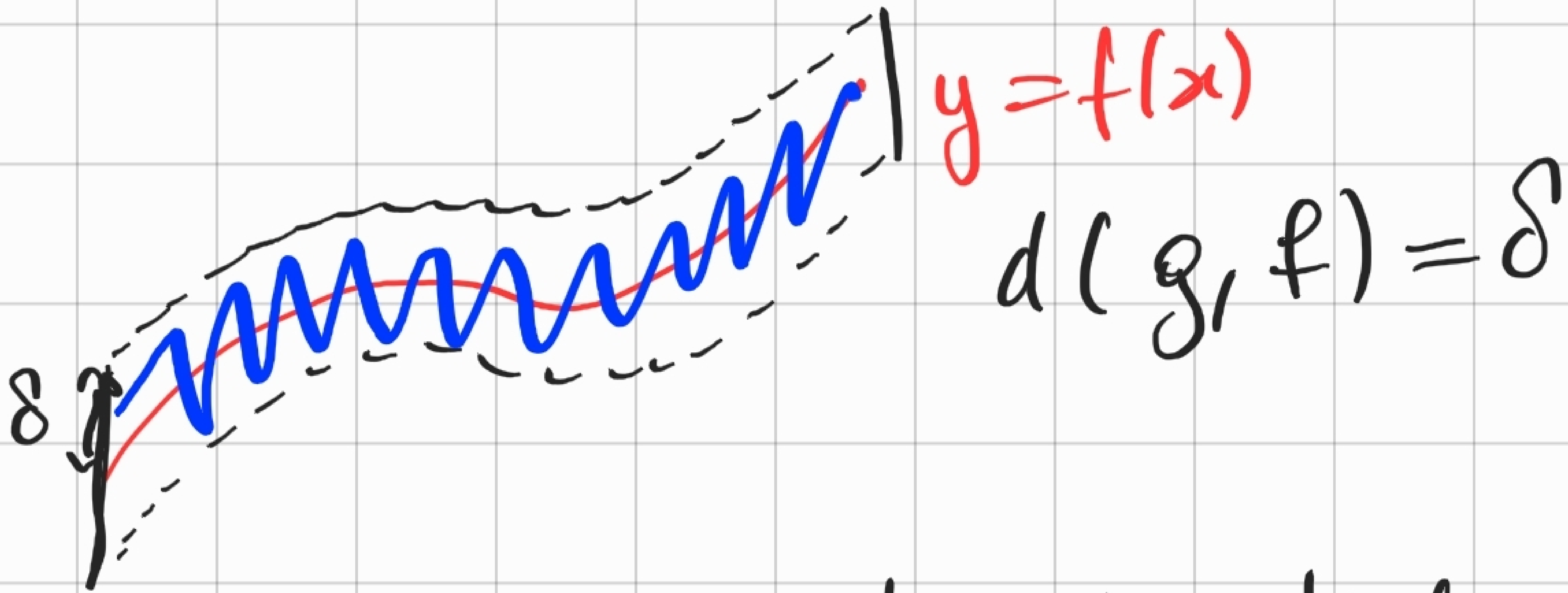


$$\|f_n-f\|_\infty = d_\infty(f_n,f) = \sup_{x\in A} |f_n(x)-f(x)|$$

$$f_n \rightrightarrows f \iff d_\infty(f_n,f)\to 0$$

Se $\delta = d_\infty(f,g)$

$$|f(x)-g(x)|\le\delta \quad \forall x\in A$$



Teorema (passaggio al limite sotto l'integrale).

Se $f_n:[a,b]\to\mathbb{R}$, $f:[a,b]\to\mathbb{R}$

se $f_n\rightrightarrows f$, f_n, f limitate R-integrabili su $[a,b]$

Allora $$\lim_{n\to+\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

dim

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b \underline{|f_n(x) - f(x)|}\,dx$$

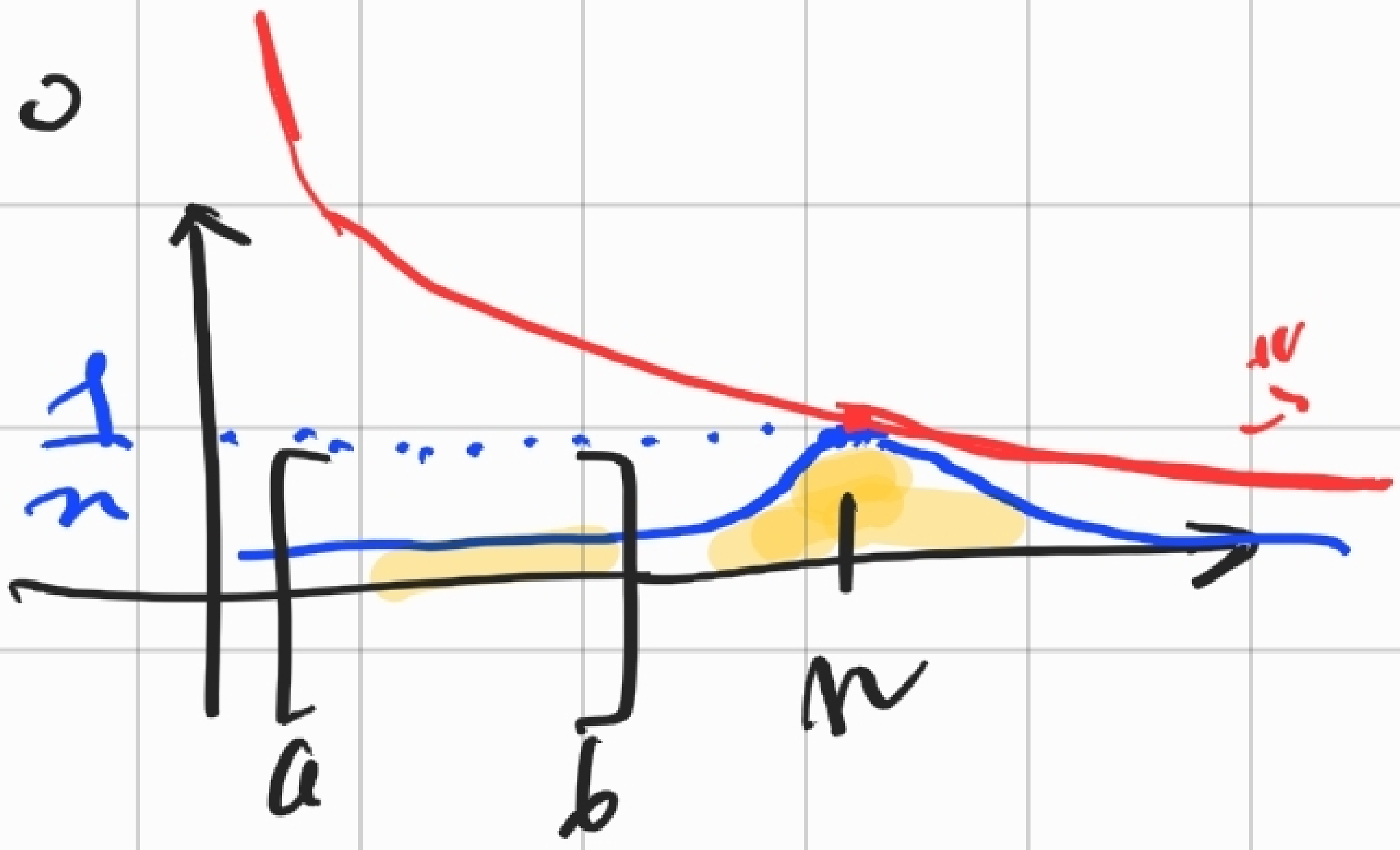
$$\le \int_a^b d_\infty(f_n, f)\,dx = d_\infty(f_n,f)\cdot|b-a|$$

$$\to 0\cdot|b-a| = 0.$$

$\square$

Es $\quad f_n(x) = \dfrac{1}{n}\,\dfrac{1}{1+\left(\frac{x-n}{n}\right)^2} \qquad f_n : \mathbb{R} \to \mathbb{R}$

$$\max_{x\in\mathbb{R}} f_n(x) = f_n(n) = \frac{1}{n} \to 0$$



$$f_n \rightrightarrows 0$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b 0\,dx = 0$$

MA

$$\int_{-\infty}^{+\infty} f_n(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{1+\left(\frac{x-n}{n}\right)^2}\,\frac{1}{n}\,dx$$

$$= \left[\operatorname{arctg}\frac{x-n}{n}\right]_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$\lim_{n\to+\infty}\int_{-\infty}^{\infty} f_n(x)\,dx = \pi \neq 0 \int_{-\infty}^{+\infty} 0\,dx$$

Teorema (convergenza dominata)

Siano $f_n, f : [a,b) \to \mathbb{R}$ $\quad a<b\leq+\infty$

f_n, f localmente R-integrabili.

Supponiamo che $\forall \beta<b$ $\quad f_n \rightrightarrows f$ su $[a,\beta]$

Supponiamo che esista $g:[a,b)\to\mathbb{R}$ loc. R-integrabile tale che

$$|f_n(x)|\leq g(x) \quad \forall x\in[a,b)$$

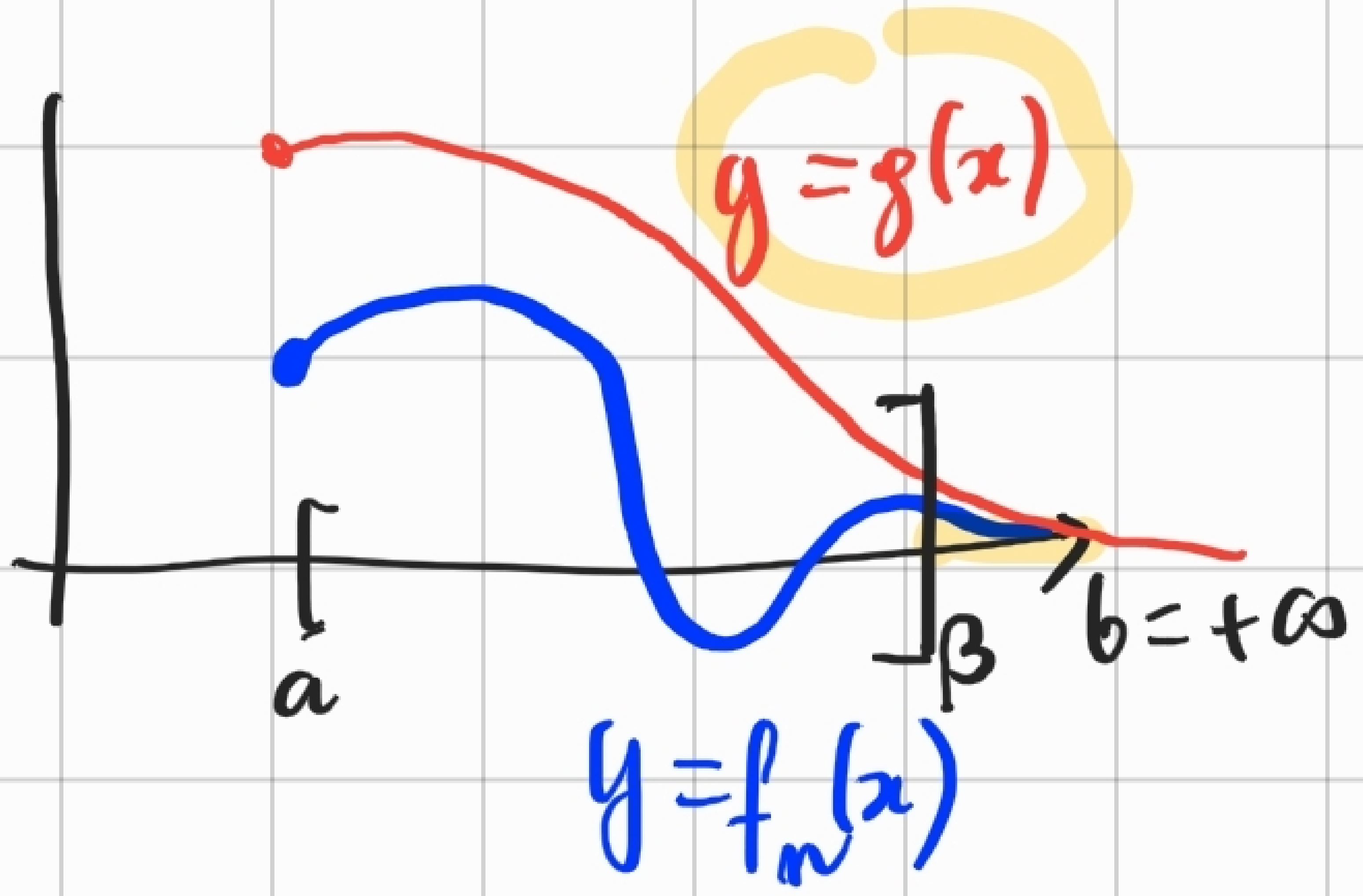
e $\quad \int_a^b g(x)\,dx$ è convergente.

Allora $\quad \lim_{n\to+\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$

Lo stesso vale per l'intervallo (a,b)

con $\quad -\infty\leq a<b\leq+\infty$

dim



$$\int_a^b g(x)\,dx = S < +\infty$$

$$\parallel$$

$$\lim_{\beta\to b} \int_a^\beta g(x)\,dx$$

$$\int_\beta^b g(x)\,dx = \int_a^b g(x)\,dx - \int_a^\beta g(x)\,dx \longrightarrow 0 \quad \text{per } \beta \to b^-$$

$\forall \varepsilon > 0 \quad \exists \beta < b \quad$ t.c. $\quad \int_\beta^b g(x)\,dx < \varepsilon$.

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq$$

$|f(x)| = \lim_{n\to+\infty} |f_n(x)| \leq g(x)$

$(|A - B| \leq |A| + |B|)$

$$\leq \left| \int_a^\beta f_n(x)\,dx - \int_a^\beta f(x)\,dx \right| + \int_\beta^b |f_n(x)|\,dx + \int_\beta^b |f(x)|\,dx$$

$$\leq \left| \int_a^\beta f_n(x)\,dx - \int_a^\beta f(x)\,dx \right| + 2\int_\beta^b g(x)\,dx$$

$$\int_\beta^b |f_n(x)|\,dx \leq \int_\beta^b g(x)\,dx$$

[Visto che $\int_a^\beta f_n(x)\,dx \to \int_a^\beta f(x)\,dx$ per $n\to+\infty$

$\forall \varepsilon > 0 \ \exists N : n > N \quad \left| \int_a^\beta f_n(x)\,dx - \int_a^\beta f(x)\,dx \right| < \varepsilon$.]

$$\leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

$\square$

Tornando alla motivazione iniziale

$$\Gamma(x) = \int_0^{+\infty} f(x,t)\,dt$$

$$f(x,t) = \underbrace{e^{-t} \cdot t^{x-1}}$$

$$\Gamma'(x) \overset{?}{=} \int_0^{+\infty} \frac{\partial f}{\partial x}(x,t)\,dt$$

$$= \int_0^{+\infty} e^{-t} \cdot \ln t \cdot t^{x-1}\,dt$$

$$\Gamma'(x) = \lim_{h\to 0} \int_0^{+\infty} \boxed{\frac{f(x+h,t)-f(x,t)}{h}}\,dt$$

x Fissato

Lagrange

$$\boxed{g_h(t)} = \frac{f(x+h,t)-f(x,t)}{h} = \frac{\partial f}{\partial x}(x+k,t)$$

$$g(t) = \frac{\partial f}{\partial x} \qquad \text{con} \quad |k|<|h|$$

Devo verificare che $\boxed{\text{① } g_h \rightrightarrows g \text{ per } h\to 0 \text{ su ogni } [\alpha,\beta]\subseteq(0,+\infty)}$

$$\boxed{\text{②} \; \exists\, G(t) \quad \int_0^{+\infty} G(t)\,dt < +\infty}$$

tale che $|g_h(t)| \le G(t)$ $\forall t>0$

$$g(t) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{-t} t^{x-1}\right) = e^{-t}\ln t\, t^{x-1}$$

$$g_h(t) = \boxed{e^{-t}\ln t\, t^{x+k-1}}$$

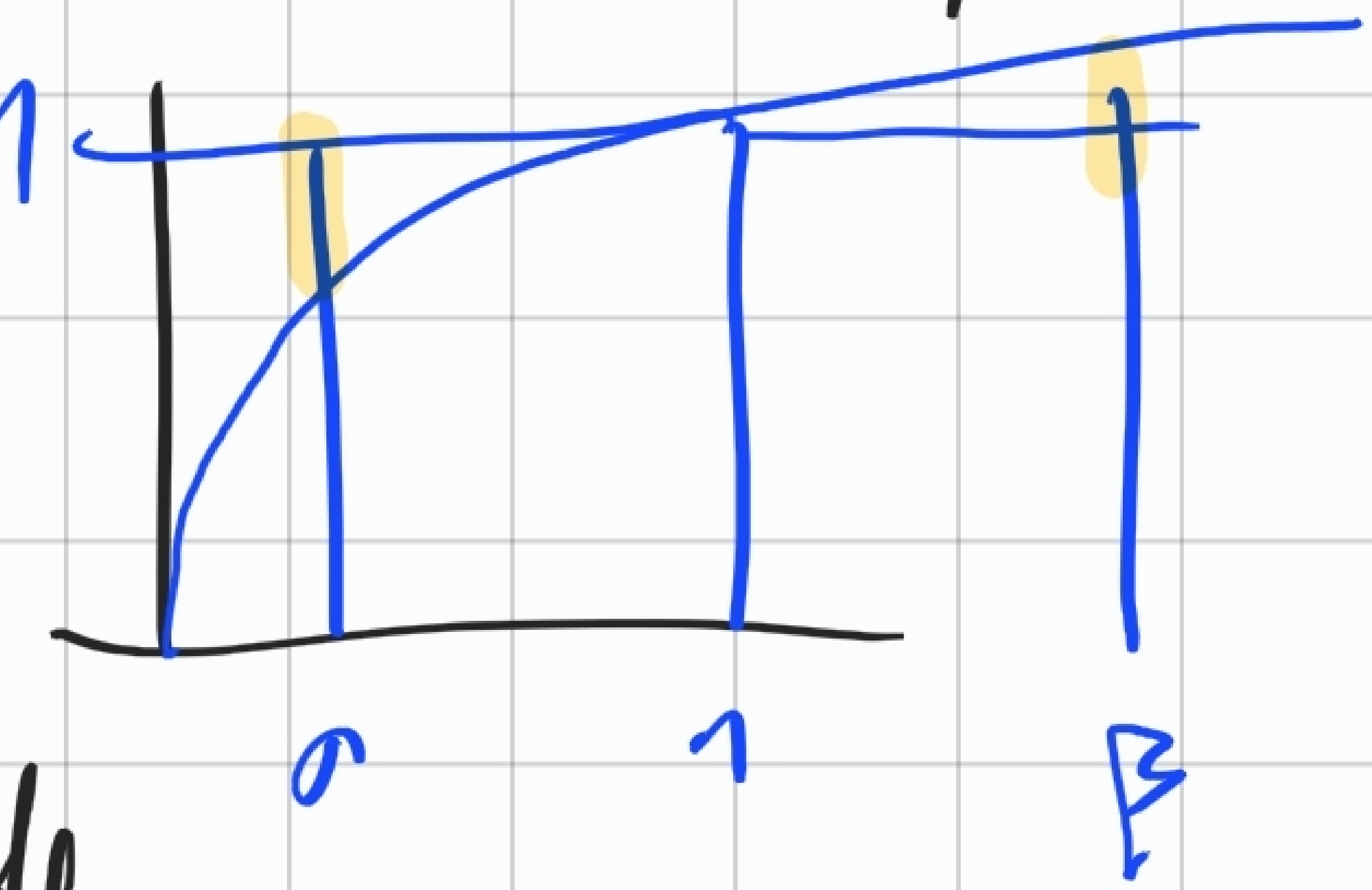
$$|g_h(t) - g(t)| = \left|e^{-t}\ln t\, t^{x-1}\left(t^k - 1\right)\right|$$

$$= e^{-t}\cdot|\ln t|\cdot t^{x-1}\cdot |t^k - 1|$$

se $t \in [\alpha, \beta]$

$$\le e^{-\alpha}\cdot\max\{|\ln\alpha|, |\ln\beta|\}\cdot\max\{\alpha^{x-1}, \beta^{x-1}\}\cdot$$

$$\cdot\max\{\beta^k - 1, \alpha^k - 1\}$$



$\longrightarrow 0$ uniformemente.

Per trovare G osservo che se h piccolo $\frac{x}{2} < x+k \le 2x$

$$\left| g_n(t) \right| \leq \ldots\ldots\ldots\ldots\ldots \quad \text{\rotatebox{0}{}}$$