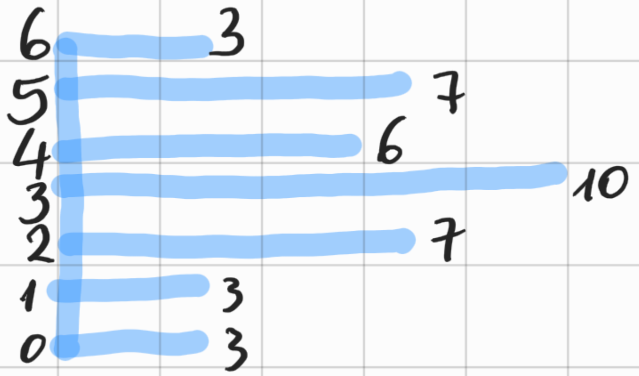
ANALISI MATEMATICA B
LEZIONE 68 – 22.3.2021

Quiz settimanale

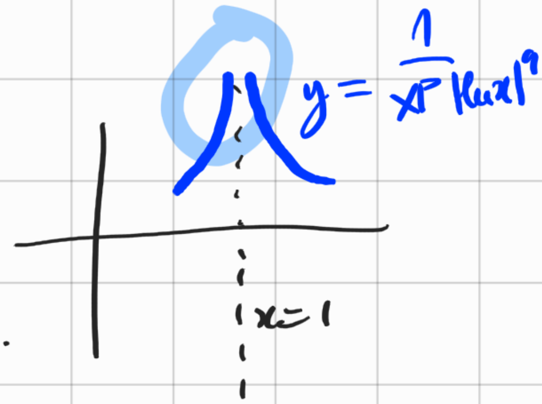


Es 6

$$\int_{\frac{1}{2}}^{2} \frac{1}{x^p |\ln x|^q}\,dx$$

per $x \to 1$

$$\ln x = \ln(1+(x-1)) \sim x-1$$
per $x \to 1$.



$$\frac{1}{x^p |\ln x|^q} \sim \frac{1}{(x-1)^q}$$

$\int_{\frac{1}{2}}^{2} \frac{1}{(x-1)^q}\,dx$ converge

$\Updownarrow$

$q < 1$.

Integrali impropri

Convergenza assoluta

Diremo che l'integrale improprio $\int_a^b f(x)\,dx$
è **assolutamente convergente** se
è convergente $\int_a^b |f(x)|\,dx$

Oss Se f è loc. R-integrabile anche $|f|$ lo è.
(già visto).
(Non è vero il viceversa: Es. $f(x) = \begin{cases} 1 & \text{se } x \notin \mathbb{Q} \\ -1 & \text{se } x \in \mathbb{Q} \end{cases}$

$|f(x)| = 1$)

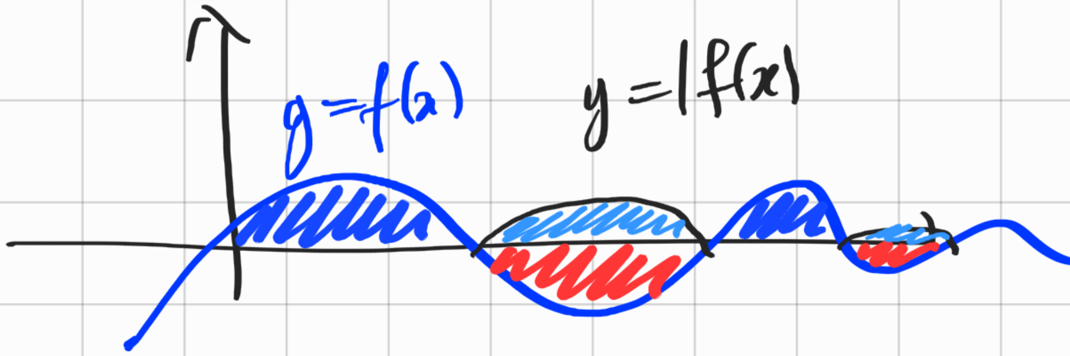
Teo (assoluta convergenza)
Sia f loc. R-integrabile su (a,b).

Se $\int_a^b |f(x)|\,dx$ converge

Allora $\int_a^b f(x)\,dx$ converge.

(*) E inoltre $\boxed{\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx}$.

dim



$$f(x) = f^+(x) - f^-(x)$$

$$f^+(x) = \begin{cases} f(x) & \text{se } f(x) > 0 \\ 0 & \text{altrimenti} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{se } f(x) < 0 \\ 0 & \text{altrimenti.} \end{cases}$$

$$|f(x)| = f^+(x) + f^-(x)$$

$$0 \le f^+(x) \le |f(x)|$$

$$\int_a^b f^+(x)\,dx \le \int_a^b |f(x)|\,dx < +\infty$$

$$0 \le f^-(x) \le |f(x)|$$

$$\int_a^b f^-(x)\,dx \le \int_a^b |f(x)|\,dx < +\infty$$

Per linearità

$$\underline{\int_a^b f(x)\,dx} = \underline{\int_a^b f^+(x)\,dx} - \underline{\int_a^b f^-(x)\,dx}$$

è convergente.

$$\left|\int_a^b f(x)\,dx\right| \leq \left|\int_a^b f^+(x)\,dx\right| + \left|\int_a^b f^-(x)\,dx\right|$$

$$\Big[\big(\,|a-b| \leq |a| + |b|\,\big)\Big]$$

$$= \int_a^b \left[f^+(x) + f^-(x)\right] dx$$

$$= \int_a^b |f(x)|\,dx \qquad \square$$

Esercizio (disuguaglianza di Jensen)

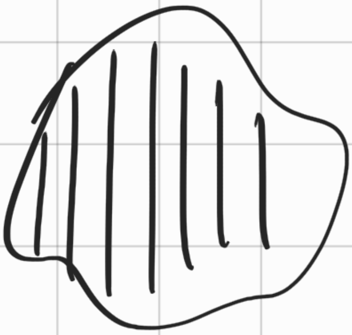
Se φ è convessa:

$$\varphi\left(\fint_a^b f(x)\,dx\right) \leq \fint_a^b \varphi(f(x))\,dx$$

$$\varphi\left(\int_a^b f(x)\cdot g(x)\,dx\right) \leq \int_a^b \varphi(f(x))\,g(x)\,dx$$

se $$\int_a^b g(x)\,dx = 1$$

$$g(x) \geq 0$$



$g(x)$ = densità di massa

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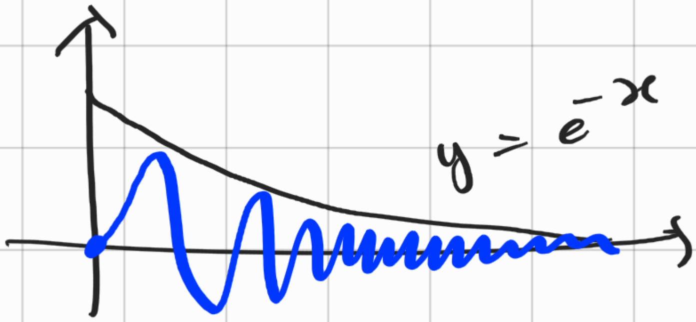
Es

$$\int_0^{+\infty} \sin(x^2)\cdot \boxed{e^{-x}}\,dx$$

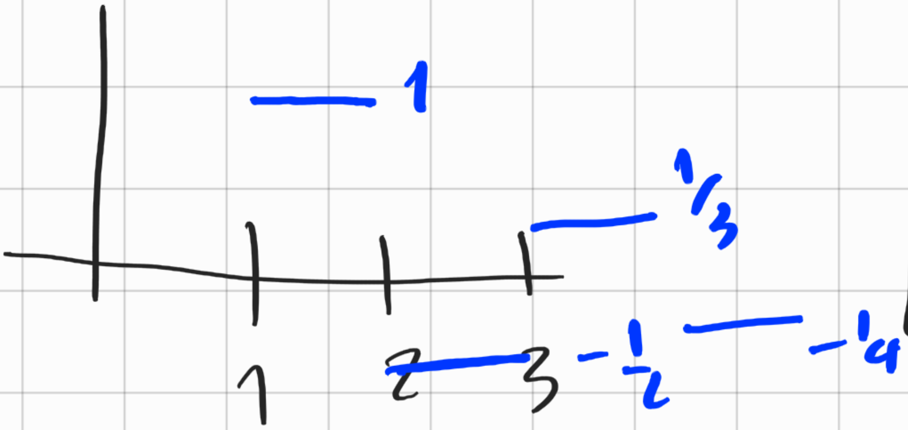


$$|\sin(x^2)\cdot e^{-x}| \leq e^{-x}.$$

$$\int_0^{+\infty} e^{-x}\,dx = 1.$$

$\int_0^{+\infty} \sin(x^2)\,e^{-x}\,dx$ è assolutamente convergente $\Rightarrow$ è convergente.

Es

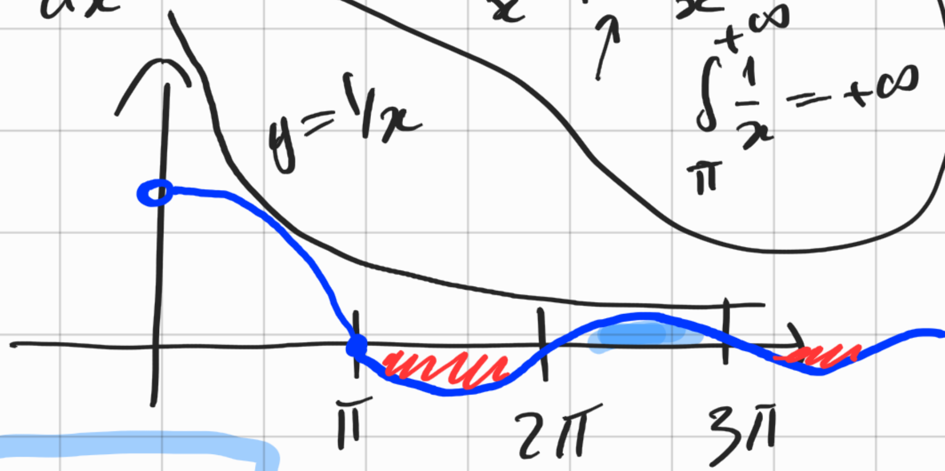


$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$$

$$f(x) = a_{\lfloor x \rfloor} \qquad a_k = \frac{(-1)^k}{k}$$

Es $\displaystyle\int_{\pi}^{+\infty} \frac{\sin x}{x}\, dx$

$$\left|\frac{\sin x}{x}\right| \leq \frac{1}{x} \qquad \int_{\pi}^{+\infty} \frac{1}{x} = +\infty$$



$$\int_{\pi}^{+\infty} \frac{\sin x}{x}\, dx = \left[-(\cos x)\cdot \frac{1}{x}\right]_{\pi}^{+\infty} - \int_{\pi}^{+\infty} (-\cos x)\cdot\left(-\frac{1}{x^2}\right) dx$$

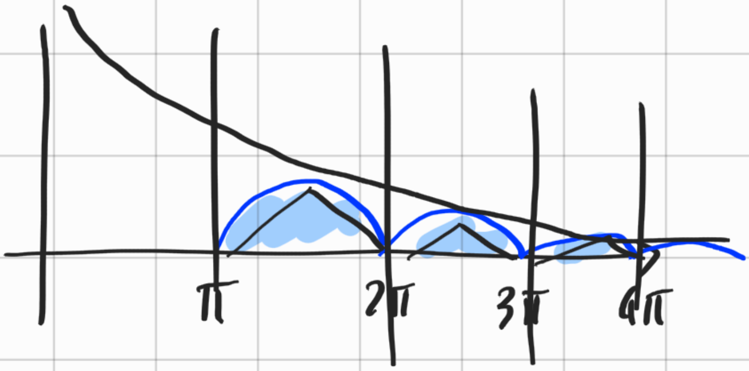
$$= 0 - \left(-\cos\pi \cdot \frac{1}{\pi}\right) - \int_{\pi}^{+\infty} \frac{\cos x}{x^2}\, dx$$

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2} \qquad \int_{\pi}^{+\infty} \frac{1}{x^2}\, dx < +\infty$$

perché $2 > 1$.

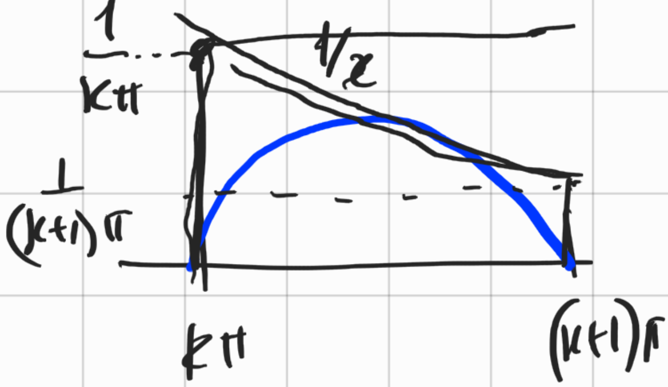
$$\Rightarrow \int_{\pi}^{+\infty} \frac{\sin x}{x}\,dx \quad \text{è convergente.}$$

$$\int_{\pi}^{+\infty} \frac{|\sin x|}{x}\,dx \quad \text{converge?} \quad \underline{\text{NO}}$$



$$\int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx$$

$$\geq \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi}\,dx = \sum_{k=1}^{n} \frac{1}{(k+1)\pi} \underbrace{\int_{0}^{\pi} \sin x\,dx}_{[-\cos]_0^{\pi} = 2}$$



$$= \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k+1}$$

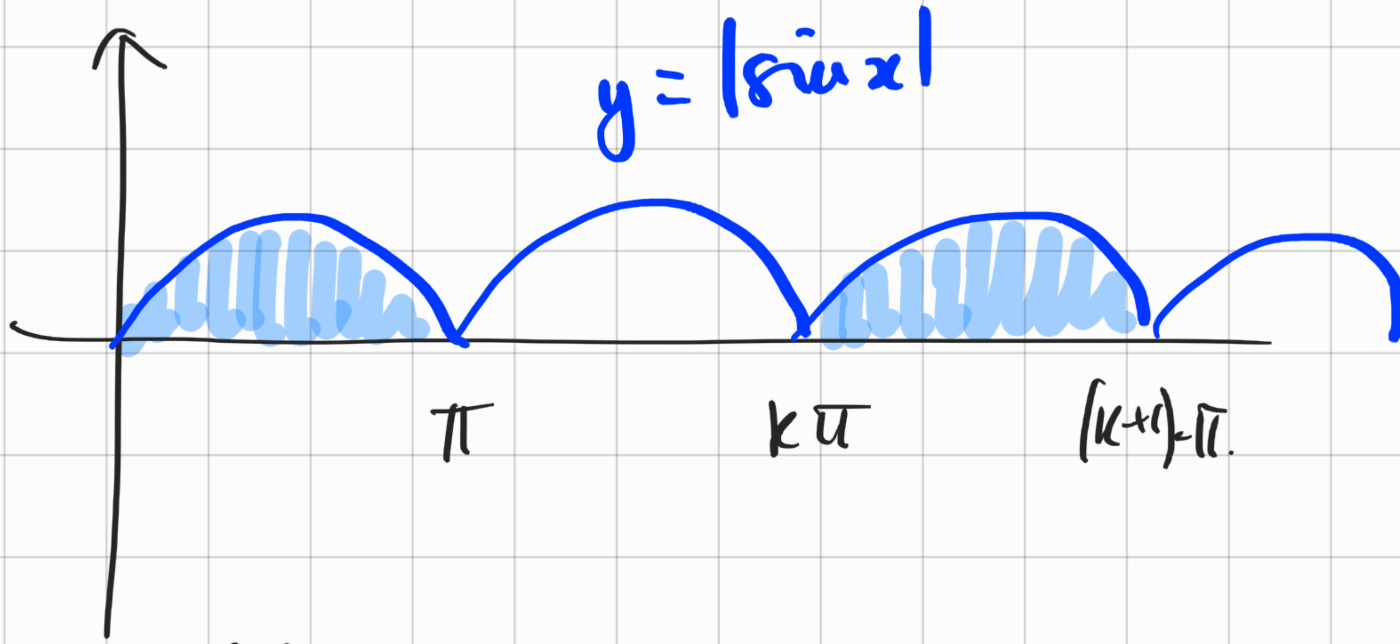
se $n \to +\infty$

$$\frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{1}{k+1} = +\infty$$

$$\int_{\pi}^{+\infty} \frac{|\sin x|}{x}\,dx = +\infty$$

$\square$

$$\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx$$



Es $\displaystyle\int_1^{+\infty} \sin(x^2)\left[\sin\left(\frac{1}{x}\right) - \frac{1}{x}\right] dx$ converge?

$|\sin(x^2)| \leq 1.$

$t = \frac{1}{x}$ per $x \to +\infty$ $t \to 0^+$

$$|\sin t - t| \sim \frac{t^3}{6} \sim \frac{1}{6x^3}$$

$$\int_1^{+\infty} \frac{1}{6x^3}\,dx = \frac{1}{6}\int_1^{+\infty} \frac{1}{x^3}\,dx < +\infty$$

$3 > 1.$

converge.

$$\left|\sin(x^2)\left[\sin\left(\frac{1}{x}\right) - \frac{1}{x}\right]\right| \leq \frac{1}{x} - \sin\frac{1}{x} \sim \frac{1}{6x^3} \text{ per } x \to +\infty$$

è integrabile per confronto puntuale

è integrabile per confronto asintotico

è integrabile (lo so)

$$\int \sin(x^2)\left[\sin\frac{1}{x} - \frac{1}{x}\right] dx$$

è assolutamente integrabile $\Rightarrow$ integrabile

$\square$

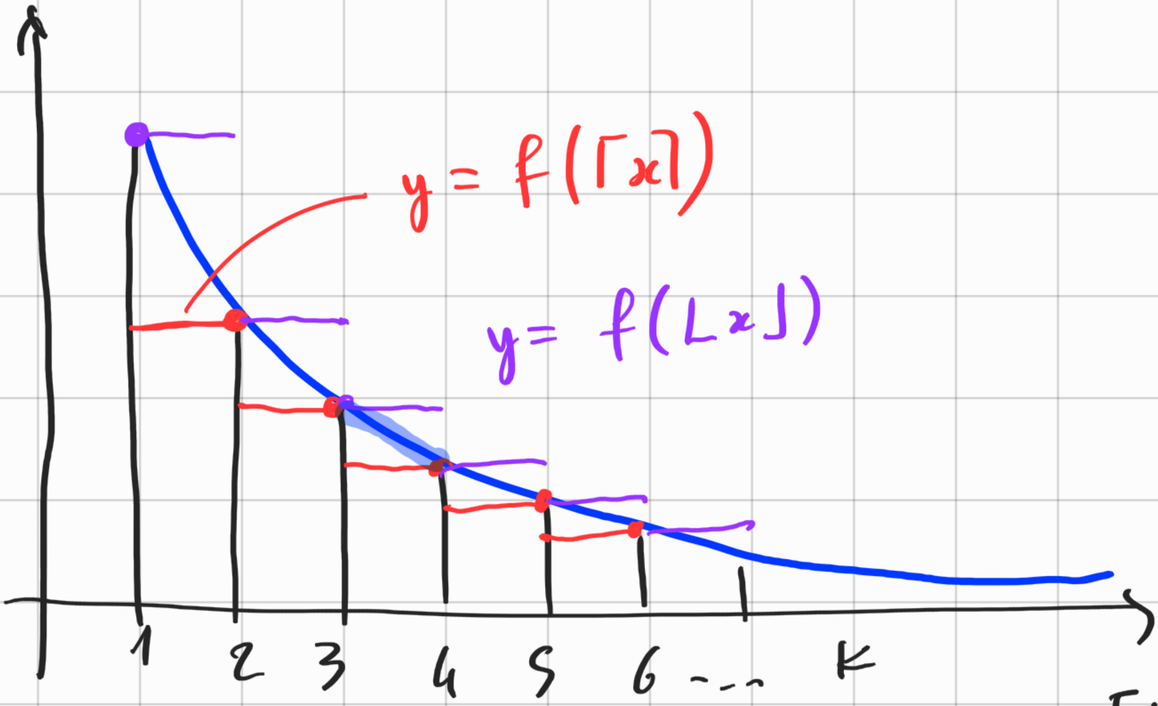
Criterio di confronto serie $\leftrightarrow$ integrali

Teorema Sia $f : [1,+\infty) \to \mathbb{R}$, $f(x) \geq 0$
f loc. R. integrabile, decrescente

$$\int_1^{+\infty} f(x)\,dx \quad e \quad \sum_{k=1}^{+\infty} f(k)$$

hanno lo stesso carattere

dim



$$f(k+1) \le f(x) \le f(k) \quad x \in [k, k+1)$$

$$f(k+1) = \int_k^{k+1} f(k+1) \le \underbrace{\int_k^{k+1} f(x)\,dx}_{} \le \int_k^{k+1} f(k)\,dx = f(k)$$

$$\int_1^{n+1} f(x)\,dx \longrightarrow \int_1^{+\infty} f(x)\,dx$$

$$\|$$

$$\sum_{k=1}^{n} f(k+1) \le \sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx \le \sum_{k=1}^{n} f(k)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\sum_{k=2}^{+\infty} f(k) \qquad\qquad \sum_{k=1}^{+\infty} f(k)$$

$\square$

ES $$\sum_{k=2}^{+\infty} \frac{1}{k\ln k}$$

$$f(x) = \frac{1}{x\ln x}$$

$$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \Big[\ln\ln x\Big]_2^{+\infty} = +\infty$$

anche la serie diverge.

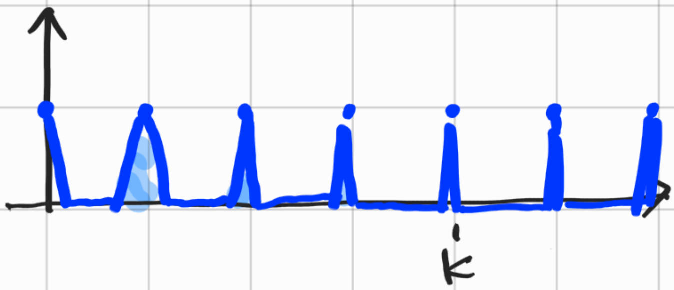
ES $$\sum_{k=1}^{+\infty} \frac{1}{k^p} \Longleftrightarrow \int_1^{+\infty} \frac{1}{x^p}$$

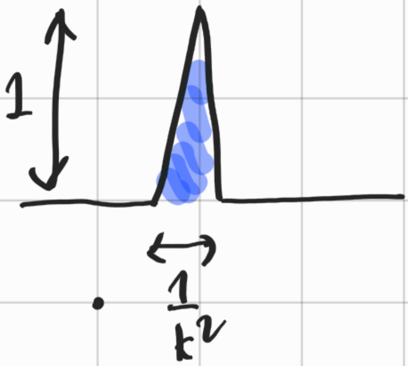
Osservazione per gli integrali non vale la "condizione necessaria" per la convergenza.

$\sum a_k$ converge $\Rightarrow$ $a_k \to 0$

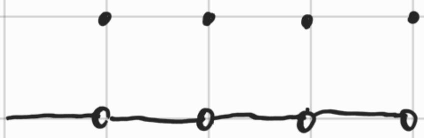
Non vale per gli integrali:

$\int_0^{+\infty} f(x)\,dx$ converge ma $\underbrace{\text{non} \lim_{x\to+\infty} f(x) = 0}$





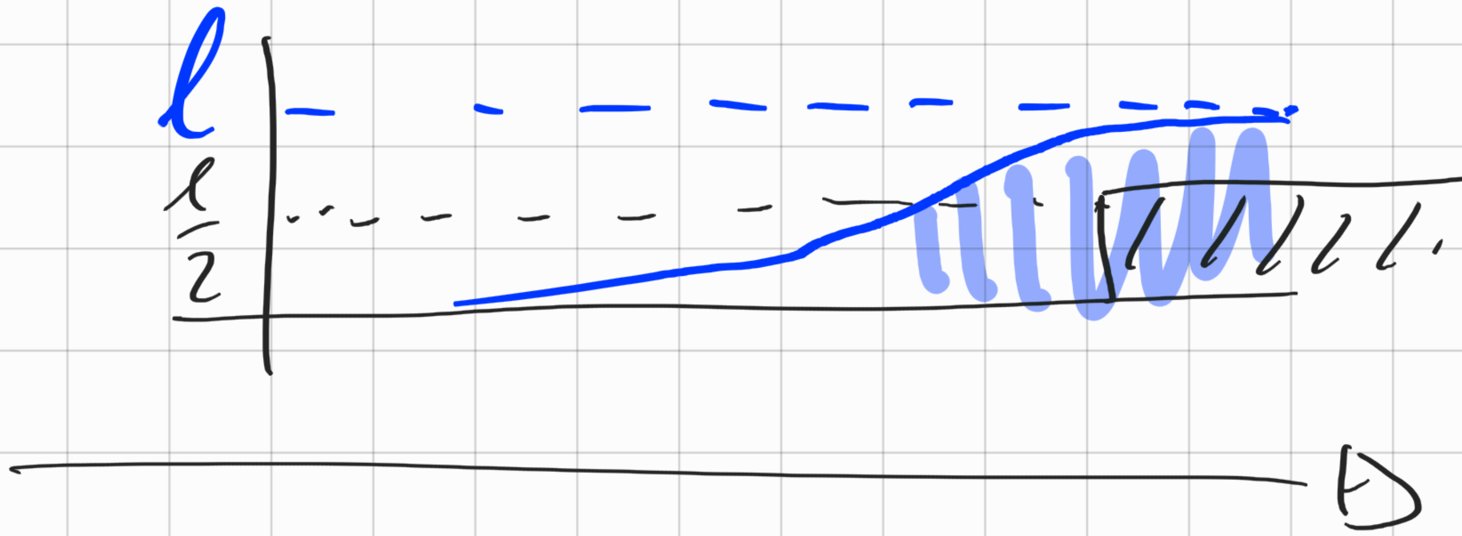
$\sum \frac{1}{k^2} < +\infty$



Però: Se $\lim_{x\to+\infty} f(x) = \ell$ (esiste)

e $\int_0^{+\infty} f(x)\,dx$ converge

allora $\ell = 0$



$$\int_0^{+\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_0^{+\infty} = 0 - (-\infty)$$

$$= +\infty$$