ANALISI MATEMATICA B
LEZIONE 63 - 10.3.2021

F è una primitiva di f se $F' = f$.

<u>Teo fondamentale</u>

Se $F(x) = \int_{x_0}^{x} f(t)\,dt$
allora F è una primitiva di f

Regole di integrazione

Cambio di variabile
(integrazione per sostituzione)

1. sostituzione diretta:

$$\int f(g(x))\, g'(x)\, dx \overset{?}{=} \Big[\int f(y)\, dy\Big]_{y=f(x)}$$

$$\begin{cases} y = g(x) \\ dy = g'(x)\, dx \end{cases}$$

2. integrale definito:

→ diretta $$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

→ inversa $$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\,dt$$

$x = g(t)$

$dx = g'(t)\,dt$

$t = g^{-1}(x)$

se g iniettiva

3. sostituzione inversa

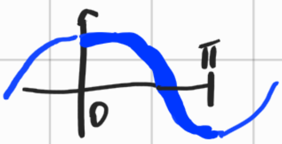
$$\int f(x)\,dx \stackrel{?}{=} \left[\int f(g(t))\,g'(t)\,dt\right]_{t=g^{-1}(x)}$$

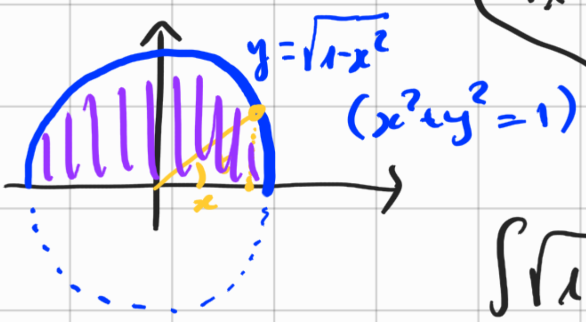
$x = g(t)$

$dx = g'(t)\,dt$

$t = g^{-1}(x)$

Esempio $\int \sqrt{1-x^2}\,dx$ $\quad (x^2 \le 1 \quad -1 \le x \le 1)$





$x = \cos t \qquad t \in [0,\pi]$

$\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = |\sin t|$

$dx = -\sin t\,dt \qquad t = \arccos x$

$$\int \sqrt{1-x^2}\,dx = \int |\sin t|(-\sin t)\,dt$$

$\sin t \geq 0$ se $t \in [0, \pi]$

$$= -\int \sin^2 t \, dt = -\int \left[\frac{1}{2} - \frac{\cos 2t}{2}\right] dt$$

$$\cos(2t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t$$

$$\sin t = \frac{1 - \cos 2t}{2}$$

$$= \ldots\ldots \quad \ldots\ldots$$

ES

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{\pi}^{0} \sin t \cdot (-\sin t)\, dt$$

$x = \cos t$

$dx = -\sin t \, dt$

$t = \arccos x$

$$= \int_0^{\pi} \sin^2 t \, dt = \int_0 \left[\frac{1}{2} - \frac{\cos 2t}{2}\right] dt$$

$$= \frac{1}{2}\left[t\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2t \, dt$$

$$= \frac{1}{2}(\pi - 0) - \frac{1}{4}\int_0^{\pi} \cos(2t)\, 2\, dt$$

$y = 2t$

$dy = 2dt$

$$= \frac{\pi}{2} - \frac{1}{4}\int_0^{2\pi} \cos y \, dy$$

$$= \frac{\pi}{2} - \frac{1}{4}\left[\sin y\right]_0^{2\pi}$$

$$= \frac{\pi}{2} - \frac{1}{4}\left[0 - 0\right] = \frac{\pi}{2} \qquad \square$$

<u>ES</u> $\displaystyle\int \frac{1}{x \ln x}\, dx$

$$\boxed{\int \frac{1}{\ln x} = ?}$$

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{\ln x}\,\frac{1}{x}\, dx = \boxed{\int \frac{1}{y}\, dy} = \ln|y|$$

$(x>0)$

$y = \ln x$

$dy = \frac{1}{x}\, dx$

$\ln|y| = \ln|\ln x|$

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{\ln x}\, d\ln x = \ln|\ln x|$$

<u>ES</u> $\displaystyle\int \frac{1}{\sqrt{x}+1}\, dx = \int \frac{1}{y+1}\, 2y\, dy =$

$y = \sqrt{x}$

$x = y^2$

$dx = 2y\, dy$

$$= \int \frac{2y}{y+1}\, dy = \int \frac{2y+2-2}{y+1}\, dy$$

$$= \int \left[2 - \frac{2}{y+1}\right] dy = 2y - 2\int \frac{1}{y+1}\, dy$$

$$= 2y - 2\ln|y+1|$$

$$= 2\sqrt{x} - 2\ln(\sqrt{x}+1)$$

$$\left[2\sqrt{x} - 2\ln(\sqrt{x}+1)\right]' = 2\,\frac{1}{2\sqrt{x}} - 2\,\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}+1}$$

$$= \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(\sqrt{x}+1)}$$

$$= \frac{\sqrt{x}+1-1}{\sqrt{x}(\sqrt{x}+1)} = \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$$

Per Parti

Derivata del prodotto: $(F\cdot g)' = F'\cdot g + F\cdot g' = fg + Fg'$

Se $F = \int f \quad F' = f$

$$f\cdot g = (F\cdot g)' - F\cdot g'$$

$$\boxed{\int f\cdot g = F\cdot g - \int F\cdot g'}$$

Se $F = \int f$

INTEGRAZIONE PER PARTI

$$\int_a^b f\cdot g = [F\cdot g]_a^b - \int_a^b F\cdot g'$$

Esempi

$$\int \underset{\substack{\uparrow\\ \text{deriva}}}{x}\cdot \underset{\substack{\uparrow\\ \text{integra}}}{\cos x}\, dx = \underbrace{(\sin x)\cdot x}_{} - \int 1\cdot \sin x\, dx$$

$$= x\cdot \sin x - (-\cos x) = x \sin x + \cos x$$

VERIFICA

$$(x\cdot \sin x + \cos x)' = \cancel{1\cdot \sin x} + x\cos x \cancel{-\sin x}$$

$$= x\cos x.$$

Es

$$\int \underset{\substack{\uparrow\\ \text{derivo}}}{(x^2+1)}\cdot \underset{\substack{\uparrow\\ \text{integro}}}{e^x}\, dx = (x^2+1)\cdot \underset{\substack{\uparrow\\ \text{integrale}}}{e^x} - \int \underset{\substack{\uparrow\\ \text{derivata}}}{2x}\cdot \underset{\substack{\uparrow\\ \text{integrale}}}{e^x} dx$$

$$= (x^2+1)e^x - \int 2x\cdot e^x\, dx$$

$$= (x^2+1)e^x - \left[2x\cdot \underset{\substack{\uparrow\\ \text{integrale}}}{e^x} - \int 2\cdot e^x\, dx\right]$$

$$= (x^2+1)e^x - 2xe^x + 2e^x$$

$$= (x^2 - 2x + 3)\, e^x .$$

VERIFICA : $(2x-2)\cdot e^x + (x^2 - 2x + 3) e^x$

$$= (\cancel{2x} - 2 + x^2 - \cancel{2x} + 3) e^x$$

$$= (x^2+1) e^x . \qquad \square$$

Es $\frac{1}{2}\int_{-1}^{1} x^3 \cos x \, dx = 0$ $\underbrace{\text{per simmetria.}}$

$y = -x$ $\quad x = -y$

$dy = -dx$

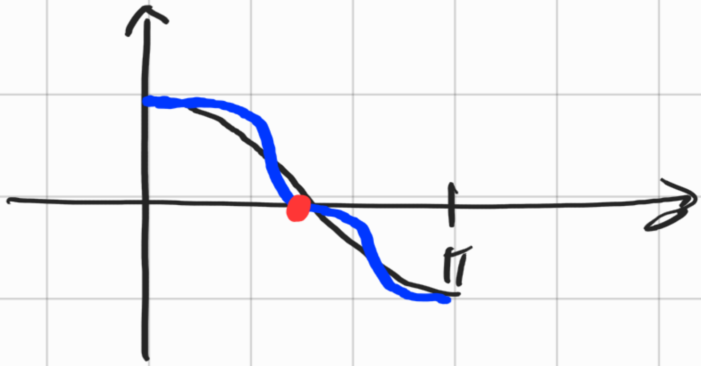
$$I = \int_{-1}^{1} x^3 \cos x \, dx = \int_{1}^{-1} (-y^3) \cos(-y) (-1)\, dy$$

$$= \int_{1}^{-1} y^3 \cos y \, dy = -\int_{-1}^{1} x^3 \cos x \, dx = -I$$

$I = 0$

Es $\int_0^{\pi} \cos^3 x \, dx$ $= \ldots = -\int_0^{\pi} \cos^3 x dx$

$y = \pi - x$



ES $\int \cos^2 x \, dx = \int \cos x \cdot \cos x \, dx$

per parti

$= \sin x \cdot \cos x - \int \sin x \, (-\sin x) \, dx$

integrale derivata

$= \sin x \cdot \cos x + \int \sin^2 x \, dx$

$= \sin x \cdot \cos x + \int [1 - \cos^2 x] \, dx$

$= \sin x \cdot \cos x + x - \int \cos^2 x \, dx$

$$2 \int \cos^2 x \, dx = \sin x \cos x + x$$

$$\int \cos^2 x \, dx = \frac{\sin x \cos x + x}{2} \qquad \square$$

ES ⚠ $\int \cos^3 x \, dx = ? \ldots$

ES [cattivo]

$$\cancel{\int \frac{1}{x \ln x}\,dx} \overset{?}{=} \int \frac{1}{x}\cdot\frac{1}{\ln x}\,dx$$

per parti

$$\overset{?}{=} \ln x \cdot \frac{1}{\ln x} - \int \cancel{\ln x} \cdot \frac{\left(-\frac{1}{x}\right)}{\ln^2 x}\,dx$$

$$\overset{?}{=} 1 + \cancel{\int \frac{1}{x \ln x}\,dx}$$

$0 = 1$??

$$\int \frac{1}{x \ln x}\,dx - \int \frac{1}{x \ln x}\,dx \overset{?}{=} 1$$

$$\int 0\,dx = 1$$

$$c(x) - c(x) \cancel{=} 0$$

Esempi importanti

- $$\int \ln x \, dx = \int 1 \cdot \ln x \, dx$$

integro   derivo

per parti

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \cdot \ln x - \int 1 \, dx$$

$$= x \ln x - x = x(\ln x - 1).$$

- $$\int \arctan x \, dx = \int 1 \cdot \arctan x$$

integro   derivo

per parti

$$= x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \cdot \arctan x - \int \frac{x}{1+x^2} \, dx$$

$$\int f(g(x)) g'(x) \, dx$$

$$= x \cdot \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$y = 1+x^2$
$dy = 2x\,dx$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) \quad \int \frac{1}{y}\, dy$$

$$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx$$

$$= x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$\left[= x \cdot \arcsin x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}}\, dx \right]$$

$$\left[= x \cdot \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}}\, d(1-x^2) \right]$$

$$= x \cdot \arcsin x + \sqrt{1-x^2} \qquad \square$$

$$D \sqrt{1-x^2} = D\,(1-x^2)^{\frac{1}{2}}$$

$$= \frac{1}{2} \frac{-2x}{\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$