ANALISI MATEMATICA B
LEZIONE 40 - 15. 1. 2021

$$\exp(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!}$$

omomorfismo

- $\exp(0) = 1$
- $\exp(z+w) = \exp(z) \cdot \exp(w)$ (*)
- $\lim_{z \to 0} \frac{\exp(z) - 1}{z} = 1$

__Teorema__ se $x \in \mathbb{R}$ $\exp(x) = e^x$.

$a^{x+y} = a^x \cdot a^y$

a^x era l'unico omomorfismo crescente

$$\mathbb{R} \to (0, +\infty)$$
$$+ \qquad \cdot$$
$$x \longmapsto a^x$$
$$1 \longmapsto a$$

exp è un omomorfismo

è crescente?

$$x_1 \le x_2 \Rightarrow \exp(x_1) \le \exp(x_2)$$

$$t = x_2 - x_1 \geq 0$$

$$\exp(x_2) = \exp(x_1 + t) \quad \checkmark$$

$$= \underbrace{\exp(x_1)}_{>0} \cdot \underbrace{\exp(t)}_{\geq 1} \geq \exp(x_1)$$

$$\left(\exp(x) \in \mathbb{R}\,? \quad \forall\, x \in \mathbb{R} \right.$$

$$\left. \exp(x) = \sum_{k=0}^{+\infty} \frac{x^k}{k!} \in \mathbb{R} \right)$$

Se $t \geq 0$ $\qquad \exp(t) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}$

$$= 1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} \geq 1$$

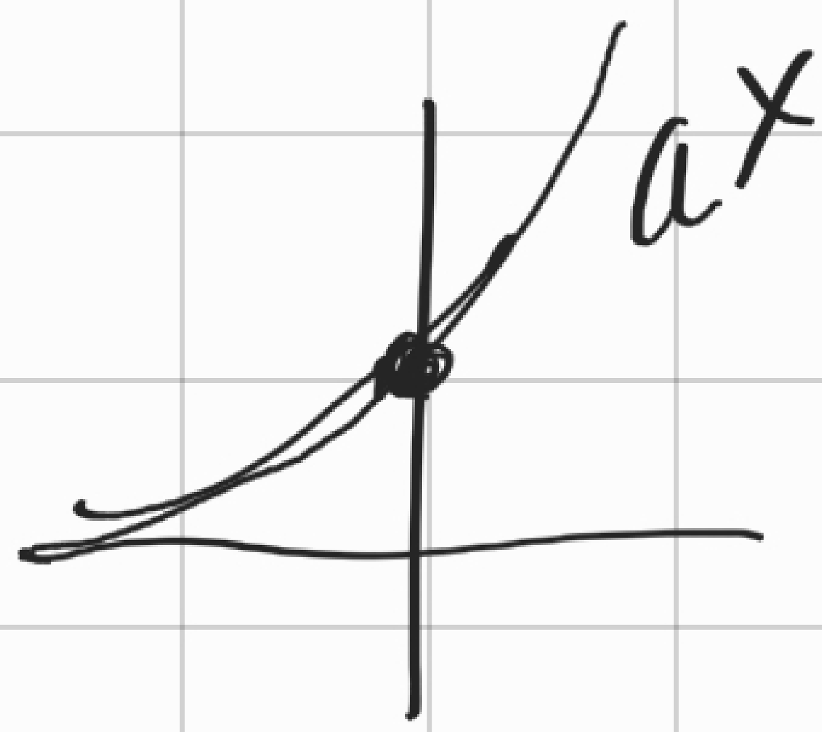
$\frac{t^k}{k!} \geq 0$ se $t \geq 0$

se $t \geq 0$ $\qquad \exp(t) \geq 1$

se $x < 0$ $\qquad \exp(x) = \frac{1}{\exp(-x)} > 0$

$$\exp(x) = a^x \qquad \text{con } a = \exp(1)$$

Dobbiamo mostrare che $\underline{\underline{a = e}}$



Noi sappiamo che:

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

$$\underline{\underline{1}} = \lim_{x \to 0} \frac{\exp(x) - 1}{x}$$

$$= \lim_{x \to 0} \frac{a^x - 1}{x}$$

$$= \lim_{x \to 0} \boxed{\frac{e^{x \ln a} - 1}{x \cdot \ln a}} \ln a = \underline{\underline{\ln a}}$$

$$\ln a = 1 \qquad a = e^1 = e. \ \square$$

$$\boxed{\sum_{k=0}^{+\infty} \frac{1}{k!} = e} = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n.$$

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} \qquad \forall x \in \mathbb{R}$$

se $z \in \mathbb{C}$ definiamo:

$$e^z = \sum_{k=0}^{+\infty} \frac{z^k}{k!} = \exp(z)$$

(Si può dimostrare: $\lim_{n\to+\infty} \left(1+\frac{z}{n}\right)^n = e^z$)

$$e = \sum_{k=0}^{+\infty} \frac{1}{k!} = \overbrace{1+1+\frac{1}{2}+\frac{1}{6}}+\dots$$

$$\sim \frac{16}{6} = 2{,}6$$

$8 : 3 = 2,\overline{6}$
20
2

Teorema (approssimazione di e)

$$0 \le \boxed{e - \sum_{k=0}^{n} \frac{1}{k!}} \le \frac{1}{n \cdot n!}$$

dim

$$R_n = \boxed{e - \sum_{k=0}^{n} \frac{1}{k!}} = \sum_{k=n+1}^{+\infty} \frac{1}{k!}$$

$$n! \cdot R_n = \sum_{k=n+1}^{+\infty} \frac{n!}{k!} = \sum_{k=n+1}^{+\infty} \frac{\cancel{n} \cdot \cancel{(n-1)} \cdots \cancel{2} \cdot \cancel{1}}{\underbrace{k(k-1) \cdots}_{(k-n) \text{ fattori } \ge n+1} \cancel{n} \cdot \cancel{(n-1)}}$$

$$\le \sum_{k=n+1}^{+\infty} \frac{1}{(n+1)^{k-n}}$$

$$\left[= \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \right]$$

$$= \frac{1}{(n+1)} \sum_{j=0}^{+\infty} \frac{1}{(n+1)^j} = \frac{1}{n+1} \frac{1}{1 - \frac{1}{n+1}}$$

$$= \frac{1}{n+1-1} = \frac{1}{n} \qquad \square$$

Per $n = 3$

$$0 \leq e - 2{,}\overline{6} \leq \frac{1}{3 \cdot 3!}$$

$$2{,}66 \leq e \leq \frac{16}{6} + \frac{1}{18} \leq 2{,}73$$

$$= \frac{49}{18} =$$

$49 : 18 = 2{,}7\overline{2}$

36

130

126

40

4

$$\begin{array}{r} 18 \\ 7 \\ \hline 126 \end{array}$$

Teorema $e \notin \mathbb{Q}$

dim per assurdo $\boxed{e = \frac{p}{q}}$

$p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q > 1$

$$\mathbb{Z} \ni q! \cdot e = q! \sum_{k=0}^{+\infty} \frac{1}{k!} = \sum_{k=0}^{+\infty} \frac{q!}{k!}$$

$$= \underbrace{\sum_{k=0}^{q} \frac{q!}{k!}}_{N \in \mathbb{N}} + \underbrace{q! \sum_{k=q+1}^{+\infty} \frac{1}{k!}}_{0 < \varepsilon \leq \frac{1}{q} < 1}$$

$$N < \underbrace{q! \cdot e} \leq N + \frac{1}{q} < N+1$$

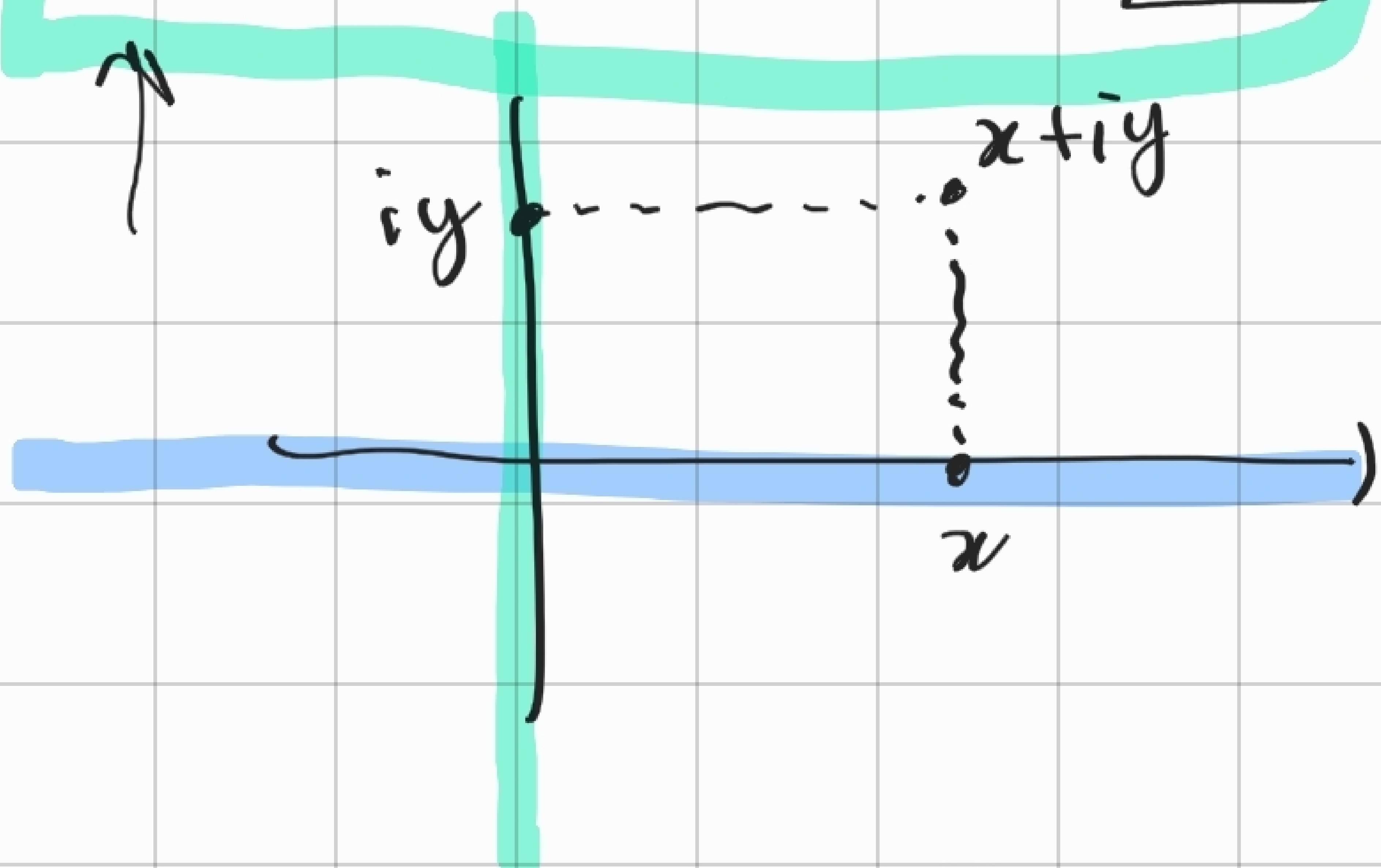
ma se $e = \frac{p}{q}$ $\quad q! \cdot e \in \mathbb{Z}$.

assurdo $\square$

Come è fatto l'esponenziale complesso

$z = x + iy$

$$\boxed{e^z = e^{x+iy} = e^x \cdot e^{iy}}$$



$z+\bar{z} = x+\cancel{iy}+x-\cancel{iy}$

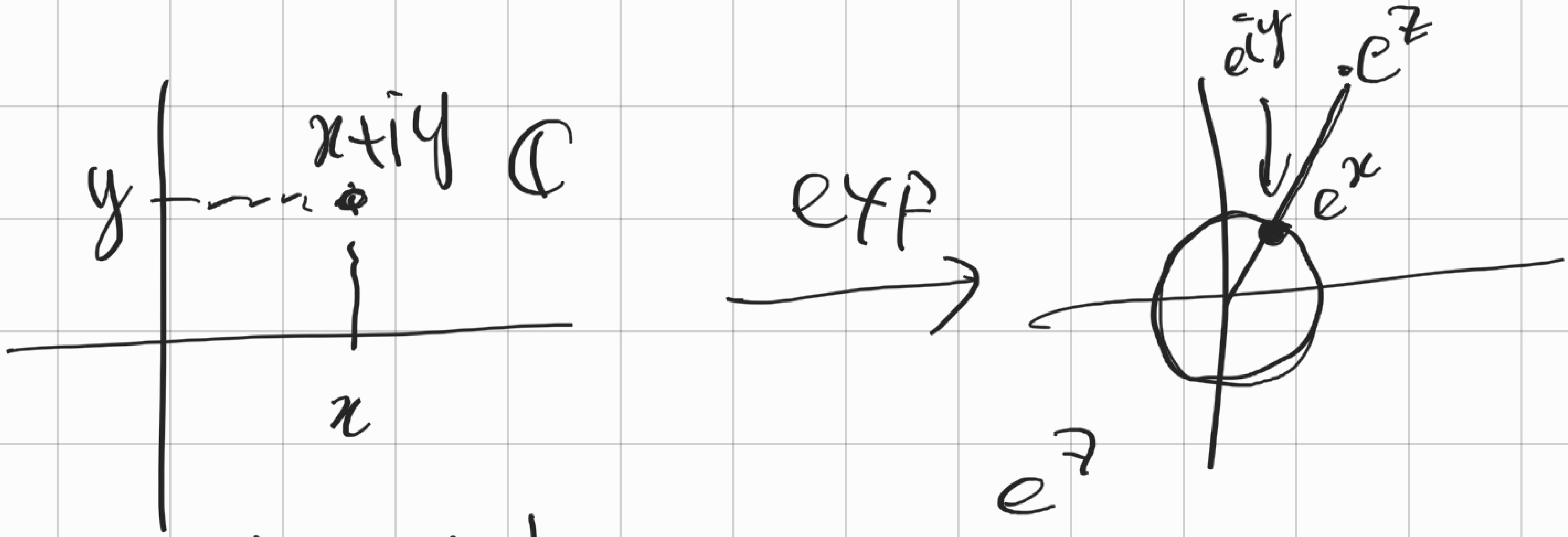
$|w| = \sqrt{w \cdot \bar{w}}$

$$|e^z| = \sqrt{e^z \cdot \overline{e^z}} = \sqrt{e^z \cdot e^{\bar{z}}} = \sqrt{e^{z+\bar{z}}}$$

$$= \sqrt{e^{2\operatorname{Re} z}} = |e^{\operatorname{Re} z}| = e^{\operatorname{Re} z} = e^x$$

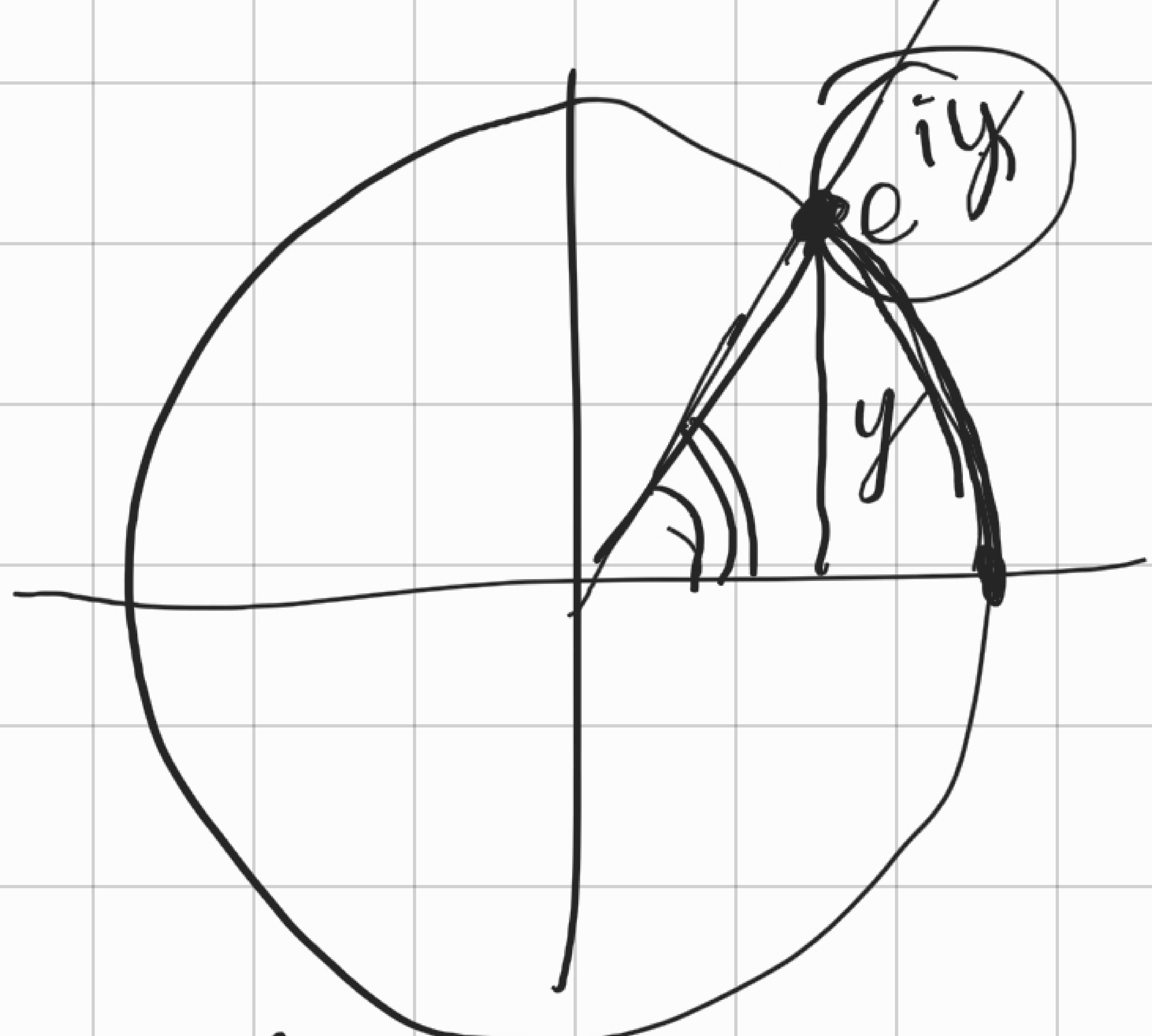
$$|e^{iy}| = 1$$

$$e^{x+iy} = \underbrace{e^x}_{\text{reale}} \cdot \underbrace{e^{iy}}_{\text{unitario}}$$



$$|e^{x+iy}| = e^x$$

$$\operatorname{Arg} e^{x+iy} = y$$



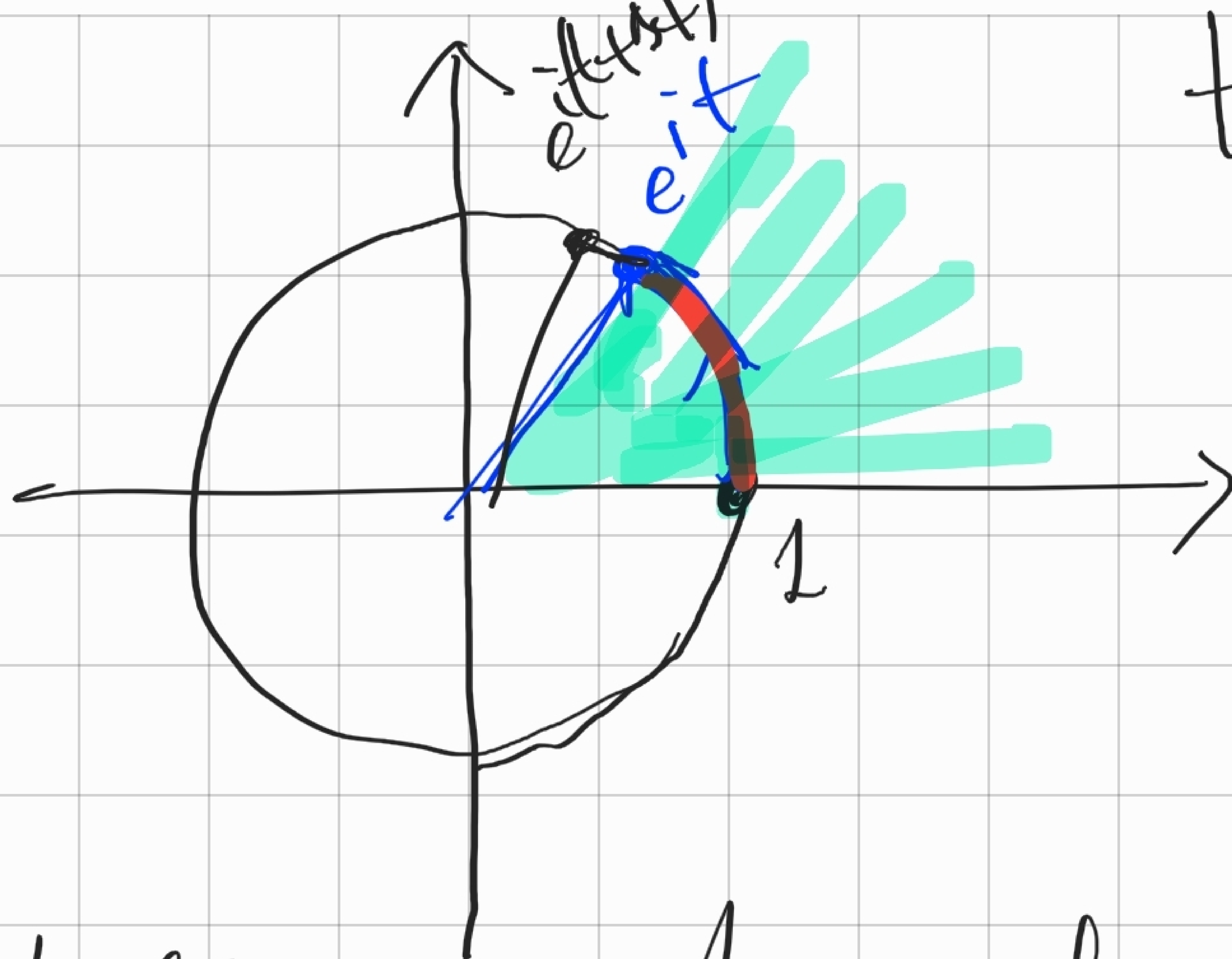
$\boxed{e^{iy}}$ sta sulla circonferenza di raggio unitario

Per definizione poniamo:

$$\begin{cases} \cos y = \mathrm{Re}\, e^{iy} \\ \sin y = \mathrm{Im}\, e^{iy} \end{cases}$$

Giustifichiamo il fatto che y è la misura in radianti dell'angolo identificato da e^{iy}.

Quello che sappiamo è che per $t \in \mathbb{R}$ e^{it} rappresenta un punto che si muove nella circonferenza unitaria di $\mathbb{C}$.



$t=0 \quad e^{i\cdot 0} = e^{0} = 1$

Vogliamo mostrare che il punto e^{it} si muove con velocità $|v| = 1$

$$v = \lim_{\Delta t \to 0} \frac{e^{i(t+\Delta t)} - e^{it}}{\Delta t}$$

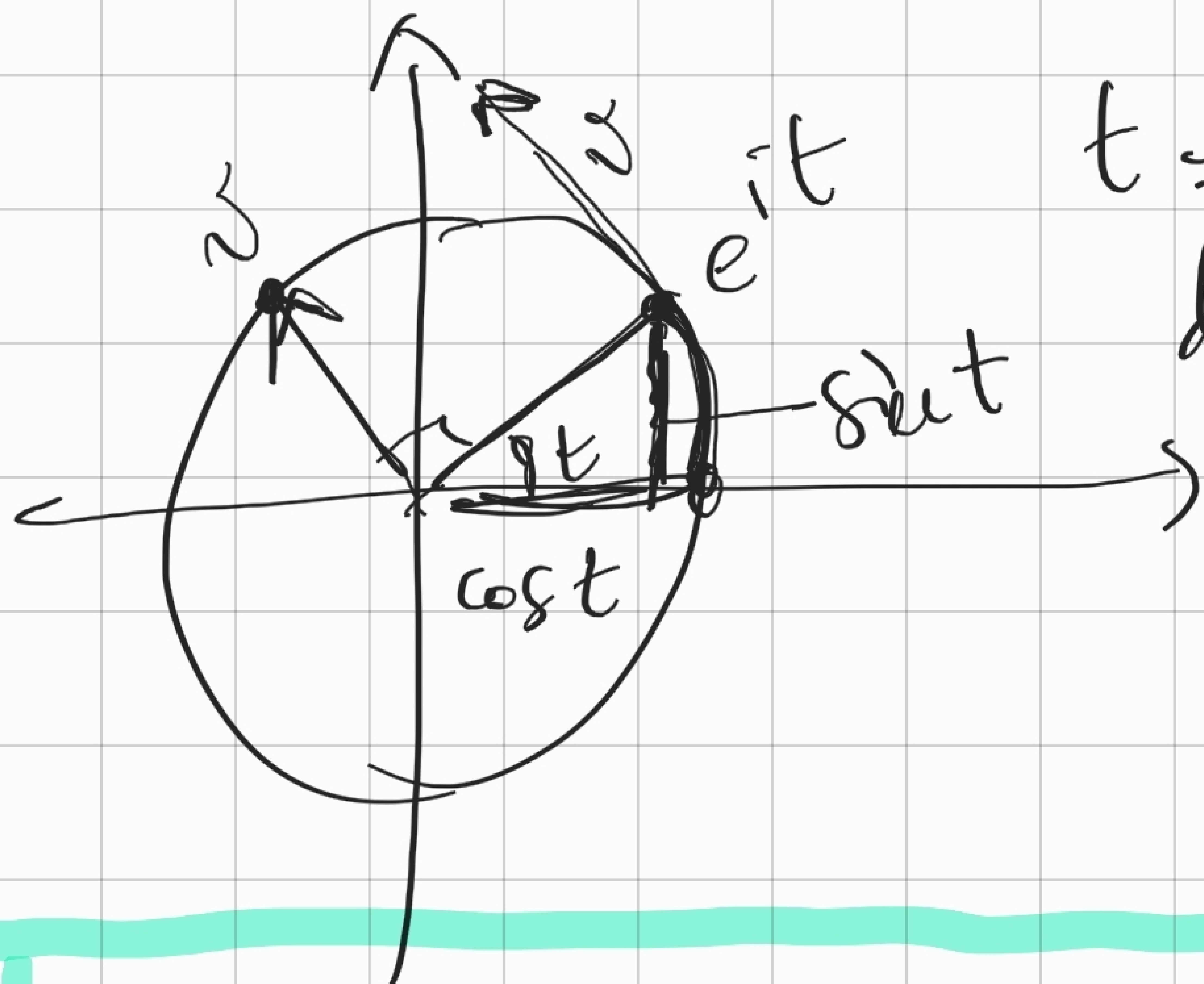
$$= \lim_{\Delta t \to 0} \frac{e^{it}\left(e^{i\Delta t} - 1\right)}{\Delta t}$$

$$= e^{it} \lim_{\Delta t \to 0} \boxed{\frac{e^{i\Delta t} - 1}{i\Delta t}} \cdot i \quad \to 1$$

$v = i \cdot e^{it}$

$$\boxed{\lim_{z \to 0} \frac{e^z - 1}{z} = 1}$$

$|v| = |e^{it}| = 1$



t = lunghezza dell'arco

$$\boxed{e^{ix} = \cos x + i \operatorname{sen} x}$$

Formula di Eulero

$$e^{x+iy} = e^x(\cos y + i \operatorname{sen} y)$$

$$\begin{cases} \cos x = \operatorname{Re} e^{ix} = \dfrac{e^{ix} + e^{-ix}}{2} \\ \sin x = \operatorname{Im} e^{ix} = \dfrac{e^{ix} - e^{-ix}}{2i} \end{cases}$$

- Proprietà di $\sin x$, $\cos x$

(1) $\begin{cases} \sin(-x) = -\sin x & \text{(dispari)} \\ \cos(-x) = \cos x & \text{(pari)} \end{cases}$

$$e^{ix} = \cos x + i \sin x$$

$$\overline{e^{ix}} = \cos x - i \sin x$$

$$\|$$

$$e^{\overline{ix}} = e^{-ix} = \cos(-x) + i \sin(-x)$$

(2) $\boxed{\sin^2 x + \cos^2 x = 1}$

$$|e^{ix}|^2 = e^{ix} \cdot \overline{e^{ix}} = e^{ix} \cdot e^{-ix}$$

$$= e^{ix-ix} = e^0 = 1$$

$$1 = e^{ix} e^{-ix} = (\cos x + i \operatorname{sen} x)(\cos x - i \operatorname{sen} x)$$

$$= \cos^2 x + \operatorname{sen}^2 x \qquad \square$$

(3) Formule di addizione

$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$$

$$\cos(\alpha+\beta) + i \operatorname{sen}(\alpha+\beta) \qquad (\cos\alpha + i \operatorname{sen}\alpha)\cdot(\cos\beta + i \operatorname{sen}\beta)$$

$$\cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta + i(\operatorname{sen}\alpha \cos\beta + \cos\alpha \operatorname{sen}\beta)$$

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta \\ \operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha\cos\beta + \cos\alpha\operatorname{sen}\beta \end{cases}$$

(4) $\sin x$ e $\cos x$ sono funzioni continue.

In quanto e^{ix} è continua

dunque $\cos x = \mathrm{Re}\, e^{ix}$ e $\sin x = \mathrm{Im}\, e^{ix}$

sono continue.

(5) Rappresentazione in serie di potenze.

$$e^{ix} = \sum_{k=0}^{+\infty} \frac{(ix)^k}{k!} = (*)$$

$$\left[\quad (ix)^k = i^k x^k \qquad \begin{aligned} i^{2k} &= (i^2)^k = (-1)^k \\ i^{2k+1} &= i \cdot (-1)^k \end{aligned} \right.$$

$$(\ast) = \sum_{n=0}^{+\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$

$$= \underbrace{\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}}_{\in \mathbb{R}} + i \underbrace{\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\in \mathbb{R}}$$

$\operatorname{Re} e^{ix}$ $\qquad\qquad$ $\operatorname{Im} e^{ix}$

$$\begin{cases} \cos x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ \sin x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{cases}$$

Più esplicitamente:

$$\begin{cases} \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\ldots \end{cases}$$

6 Limiti notevoli

$$\lim_{x \to 0} \underbrace{\frac{e^{ix} - 1}{ix}} = 1$$

$$\frac{\cos x + i \sin x - 1}{ix}$$

$$\boxed{\frac{1}{i} = -i}$$

$$= \frac{(\cos x - 1)}{ix} + \frac{\sin x}{x}$$

$$= \frac{\operatorname{sen} x}{x} - i\left(\frac{\cos x - 1}{x}\right) \longrightarrow 1 + i \cdot 0$$

$$\begin{cases} \boxed{\dfrac{\operatorname{sen} x}{x} \longrightarrow 1} & \text{per } x \to 0 \\ \dfrac{\cos x - 1}{x} \longrightarrow 0 & \text{per } x \to 0 \end{cases}$$

$\square$

[Sugli appunti: $\dfrac{1-\cos x}{x^2} \longrightarrow \dfrac{1}{2}$]