ANALISI MATEMATICA B
LEZIONE 22 - 13.11.2020

SUCCESSIONI

$\underline{a} : \mathbb{N} \to \mathbb{R}$

$a_n = \underline{a}(n)$ $\quad \underline{a} = (a_0, a_1, a_2, \ldots, a_n, \ldots)$

$l = \lim_{n \to +\infty} a_n$

carattere
- convergente $l \in \mathbb{R}$
- divergente $l = +\infty$ o $l = -\infty$
- indeterminata $\nexists\, l$

REGOLARE

Esempio $a_n = (-1)^n$
- limitata $|a_n| \leq 1$
- a segni alterni

indeterminata

Esempio $a_n = \frac{1}{n!}$

$a_0 = \frac{1}{1} = 1 \quad a_1 = 1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{6} \quad \ldots$

a_n è a termini positivi $\quad a_n > 0$

a_n è <u>monotona</u> <u>decrescente</u> $\Leftarrow$

$n \geq m \quad n! \geq m! \qquad \frac{1}{n!} \leq \frac{1}{m!}$

a_n è convergente $\quad 0 < \frac{1}{n!} \leq \frac{1}{n} \to 0$

$n! \geq n \quad$ a $l = 0 \qquad \downarrow 0$

<u>Oss</u> Se a_n è monotona allora <u>$\lim a_n$ esiste</u>

$$\begin{cases} \text{se } a_n \text{ crescente } \lim a_n = \sup a_n \\ \text{se } a_n \text{ decrescente } \lim a_n = \inf a_n \end{cases}$$

a_n <u>decrescente</u>, <u>positiva</u>

$\Rightarrow \quad \lim a_n = \inf a_n$

$a_n \geq 0 \quad \Rightarrow \lim a_n \geq 0$

$a_n \leq a_0 \qquad \lim a_n \leq a_0 < +\infty$

$\Rightarrow$ a_n converge.

Esempio $a_n = n^2 - \frac{1}{2^n}$

a_n è divergente.

$$\lim_{x \to +\infty} x^2 - \frac{1}{2^x} = +\infty$$

$$\Downarrow$$

$$\lim_{n \to +\infty} n^2 - \frac{1}{2^n} = +\infty$$

$\square$

Località del limite

$\lim_{n \to +\infty} a_n$ dipende solo dai valori di a_n per "n grande"

$U \in \mathcal{B}_{+\infty}$ $\quad U = (\alpha, +\infty]$

$n \in U \iff n > \alpha$.

Se $a_n = b_n$ per $\underbrace{n > 1000}$

allora a_n e b_n hanno lo stesso carattere e, se sono regolari, lo stesso limite.

Se $U \in B_{+\infty}$ $\mathbb{N} \setminus U$ è finito

Se a_n e b_n differiscono solo per un numero finito di termini allora hanno lo stesso limite.

$a_n = b_n$ definitivamente

$\Leftrightarrow$ non frequentemente $a_n \neq b_n$

$a_n \neq b_n$ per finiti n

$$\#\{n \in \mathbb{N} : a_n \neq b_n\} < +\infty.$$

Esempio

$$a_n = \begin{cases} 2^n & \text{se } n < 10 \\ \dfrac{1}{n^2} & \text{se } \underline{\underline{n \geq 10}} \end{cases}$$

$$\lim a_n = \lim \frac{1}{n^2}$$

In particolare non importa che a_n sia definita $\forall n \in \mathbb{N}$

Esempio $\quad a_n = \frac{1}{n} \quad \underline{a} : \mathbb{N} \setminus \{0\} \to \mathbb{R}$

MONOTONIA

a_n crescente se

$\underline{n \geq m} \Rightarrow a_n \geq a_m$

$\Updownarrow$

$a_{n+1} \geq a_n$

$a_1 \geq a_0 \qquad a_2 \geq a_1 \geq a_0$

$a_3 \geq a_2 \geq a_1 \geq a_0$

$\vdots$

$a_n \geq a_m \qquad$ se $n \geq m$.

<u>dim</u> per induzione $\square$

<u>Esercizio</u> $\quad a_n = n^2 + \frac{1}{2^n}$

è crescente?

n	a_n
0	1
1	$3/2$
2	$17/4$
3	$\vdots$

$$a_{n+1} \overset{?}{\geq} a_n \qquad \forall n \in \mathbb{N}$$

$$(n+1)^2 + \frac{1}{2^{n+1}} \geq n^2 + \underbrace{\frac{1}{2^n}}_{\leq 1}$$

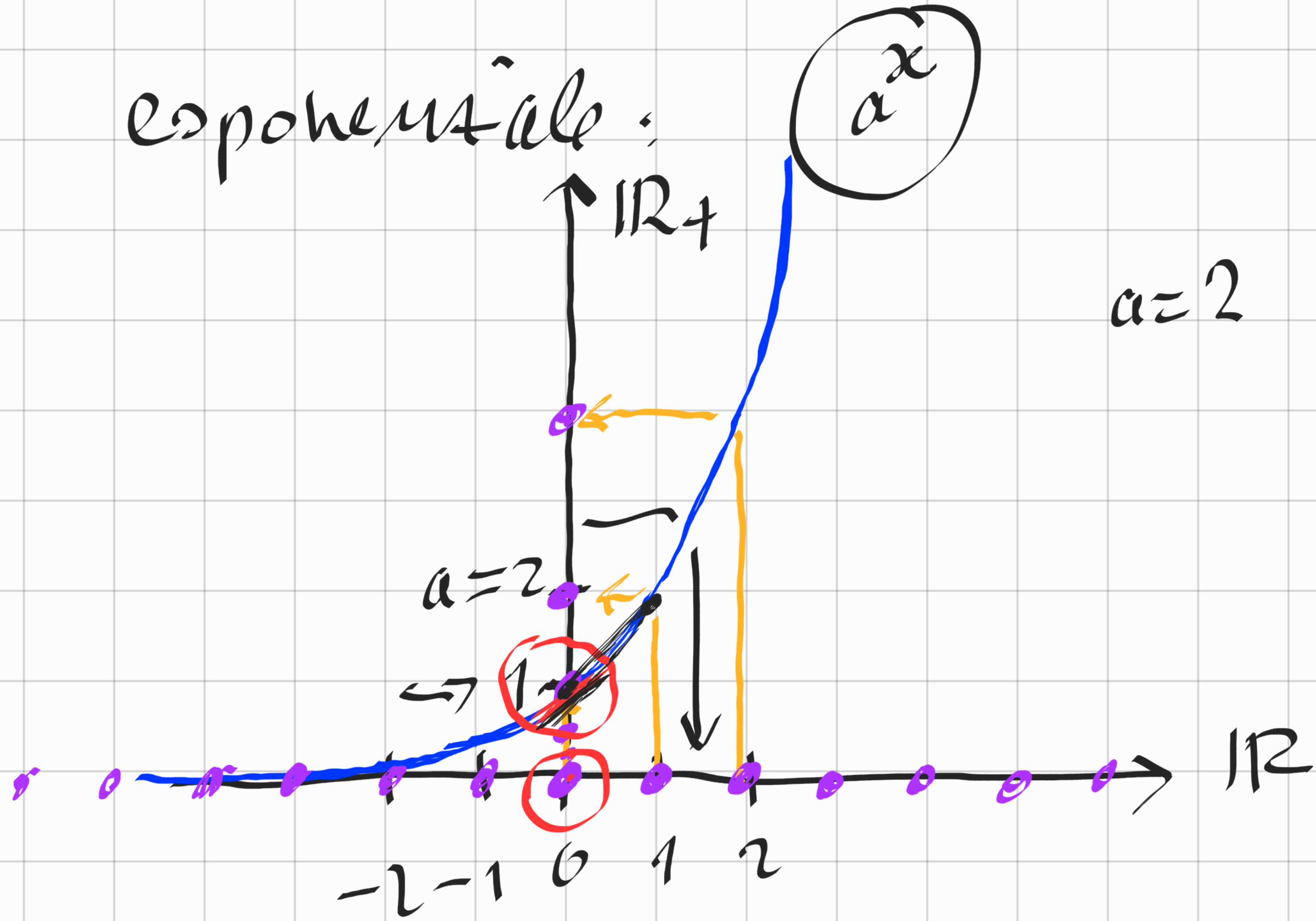
$$n^2 + \cancel{2n} + 1 + \cancel{\frac{1}{2^{n+1}}} \geq n^2 + 1$$

$$\frac{1}{2^n} \leq 1$$

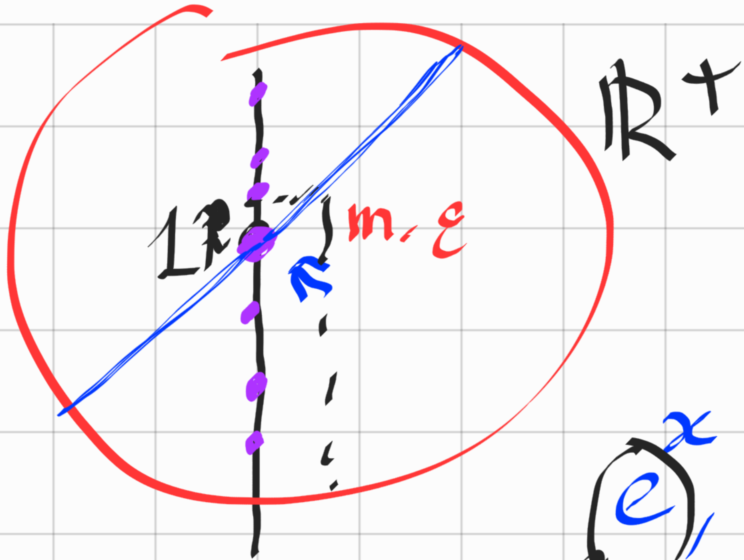
NUMERO DI NEPERO (e)

(BASE NATURALE DEI LOGARITMI)

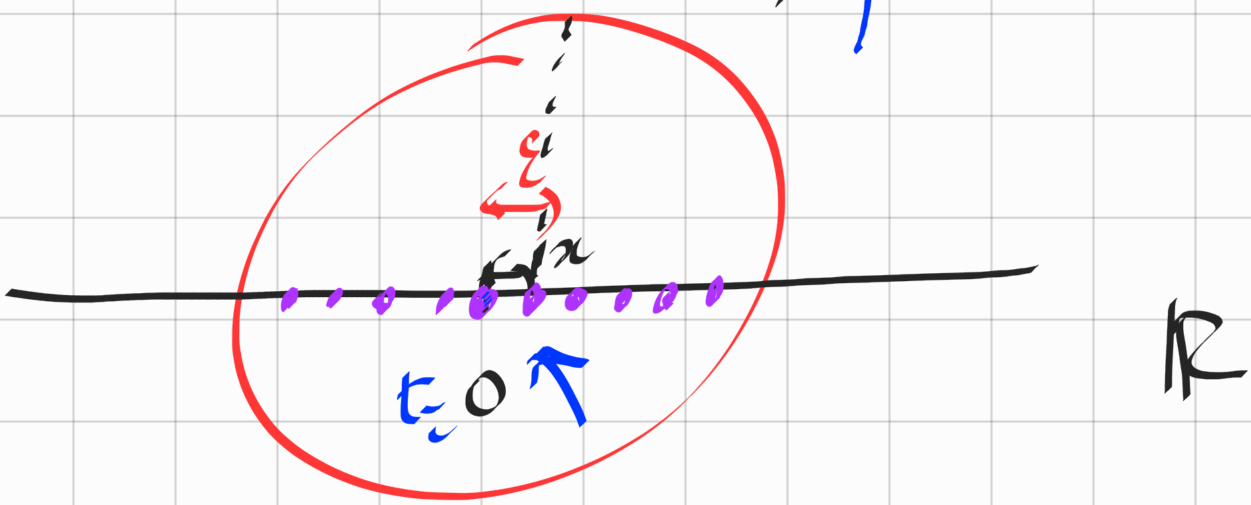
esponenziale:



$\mathbb{R}$	$\mathbb{R}_+$
$+$	$\cdot$
0	1
1	a
$x \longmapsto$	a^x ⟸ isomorfismo



e^x $\quad m = 1$



$x \to e^x$

$$\boxed{\frac{e^x - 1}{x} \to 1}$$ per $x \to 0$

$y = e^x$ $\quad x = \log_e y$

$$\frac{y-1}{\log_e y} \to 1$$ per $y \to 1$

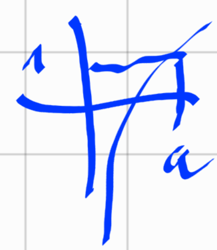
$t = y - 1$

$$\frac{t}{\log_e(1+t)} \to 1 \quad \text{per } t \to 0$$

$$\boxed{\frac{\log_e(1+t)}{t} \to 1} \quad \text{per } t \to 0$$

$t = \frac{1}{n}$ con $n \to +\infty$

$$\frac{\log_e\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1$$



$$e^{\log_e\left(1+\frac{1}{n}\right)^n} \to e^1$$

$$\boxed{e = \lim_{n \to +\infty}\left(1+\frac{1}{n}\right)^n}$$

Storicamente Bernoulli 1683.

Interessi bancari. $[r] = \frac{1}{s}$ € interesse

q capitale dopo 1 anno $q + r \cdot q$

$r = 1\% = \frac{1}{100}$ $= (1+r)\cdot q$

dopo 1 mese: $q + \frac{r}{12} q = \left(1+\frac{r}{12}\right) q$

$q, \left(1+\frac{r}{12}\right)q, \left(1+\frac{r}{12}\right)^2 q, \ldots\ldots, \left(1+\frac{r}{12}\right)^{12}$

gennaio febbraio

$$\lim_{n\to+\infty} \left(1+\frac{r}{n}\right)^n = e^r$$

vedremo

Teorema

$$a_n = \left(1+\frac{1}{n}\right)^n \text{ converge.}$$

idea a_n crescente, superiormente limitata.

Disuguaglianza di Bernoulli

$\forall n \in \mathbb{N}, \quad \forall x, \quad x \geq -1$

$$\boxed{(1+x)^n \geq 1+n\cdot x}$$

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$(x>0)$

$\geq 1+nx$

$\binom{n}{1} = n$

dim per induzione

(i) $n=0$ $\quad (1+x)^0 = 1$

$1+nx = 1$ ✓

(ii) $\underbrace{(1+x)^n \geq 1+nx}_{\text{ipotesi induttiva}} \overset{?}{\Longrightarrow} (1+x)^{n+1} \geq 1+(n+1)x$

positivo ≥ 0

$$(1+x)^{n+1} = (1+x)\cdot(1+x)^n \geq (1+x)\cdot(1+nx)$$

$$= 1+x+nx+nx^2$$

$nx^2 \geq 0$

$$\geq 1+x+nx = 1+(n+1)x \quad \square$$

$$(1+x)^2 = \underbrace{1+2x}+x^2 \geq 1+2x$$

$$(1+x)^3 = 1+3x+\underbrace{3x^2+x^3}$$

$$\geq 1+3x$$

$$(1+x)^n = 1 + nx + \underbrace{\ldots\ldots + x^n}$$

$$\geq 1 + nx$$

<u>dim</u> (teorema)

$$a_n = \left(1+\frac{1}{n}\right)^n \text{ è crescente?}$$

$$\left(a_1 = 2,\ a_2 = \frac{9}{4},\ a_3 = \frac{64}{27}, \ldots\right)$$

$$a_{n+1} \geq a_n \quad \forall n \geq 1$$

o $\underline{\underline{a_n \geq a_{n-1}}}$ $\quad \forall n \geq 2$

o $$\frac{a_n}{a_{n-1}} \geq 1$$

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}}$$

$$= \left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n-1}{n}\right)^n \cdot \left(\frac{n}{n-1}\right)$$

$$= \left(\frac{n^2-1}{n^2}\right)^n \left(\frac{n}{n-1}\right)$$

$$= \left(1-\frac{1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right)$$

Bernoulli

$$\geq \left(1 - n \cdot \frac{1}{n^2}\right) \cdot \left(\frac{n}{n-1}\right)$$

$$= \left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right) = \frac{n-1}{n}\,\frac{n}{n-1} = 1.$$

a_n crescente.

a_n superiormente limitata?

$$a_n = \left(1+\frac{1}{n}\right)^n \le b_n = \left(1+\frac{1}{n}\right)^{n+1} = a_n \cdot \left(1+\frac{1}{n}\right)$$

$a_1 \quad a_2 \quad a_3 \quad \dots \quad b_3 \quad b_2 \quad b_1$

Basta mostrare che b_n è decrescente!

Fisso

$$\boxed{a_n \le b_n \le b_1 = 4}$$

$$\frac{b_{n-1}}{b_n} \overset{?}{\ge} 1$$

$$\frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^{n+1}}$$

$$= \left(\frac{n}{n-1} \cdot \frac{n}{n+1}\right)^n \cdot \frac{n}{n+1}$$

$$= \left(\frac{n^2-1+1}{n^2-1}\right)^n \frac{n}{n+1}$$

$$= \left(1+\frac{1}{n^2-1}\right)^n \frac{n}{n+1}$$

$$\geq \left(1+n\frac{1}{n^2-1}\right) \cdot \frac{n}{n+1}$$

$$= \frac{(n^2-1+n)}{n^2-1} \cdot \frac{n}{n+1}$$

$$= \frac{n^3-n+n^2}{n^3+n^2-n-1} > 1$$

$\square$

Def

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$$

$$\frac{64}{27} \le e \le \left(1+\frac{1}{2}\right)^3 = \frac{27}{8} \simeq 3{,}375$$

s_4 $\qquad\qquad$ $\cap$

$2{,}37$ $\qquad\qquad$ b_2

$f(x) \qquad g(x)$

$f \le g$

$f(x) \le g(x) \quad \forall x$

$a_n \le b_n \quad (\forall n)$