Comp.11/09/2020.Ex.1.1

martedì 8 settembre 2020 15:42

(1)

$E(y) = y'' + y$

$\{E(y) = 0\} = \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$

1) $E(\tilde{y}) = \sin x$

$\tilde{y} = a x \sin x + b x \cos x$

$E(\tilde{y}) = 2a \cos x - 2b \sin x = \sin x \Leftrightarrow a = 0,\ b = -\frac{1}{2}$

$\tilde{y} = -\frac{1}{2} x \cos x$

2) $E(\tilde{y}) = x e^x \qquad \tilde{y} = a x e^x + b e^x$

$\tilde{y}'' = a e^x + a e^x + a x e^x + b e^x$

$E(\tilde{y}) = (2a + ax + b + ax + b) e^x = x e^x \Leftrightarrow 2a = 1 \quad 2a + 2b = 0$

$\Leftrightarrow \quad a = \frac{1}{2} \quad b = -\frac{1}{2} \qquad \tilde{y}(x) = \frac{1}{2} x e^x - \frac{1}{2} e^x$

3) $E(\tilde{y}) = \operatorname{tg} x \qquad y_1 = \sin x \quad y_2 = \cos x$

$\tilde{y}(x) = c_1 y_1 + c_2 y_2 = c_1 \sin x + c_2 \cos x$

$\tilde{y}'(x) = \underbrace{c_1' y_1 + c_2' y_2}_{= 0} + c_1 y_1' + c_2 y_2'$

$\tilde{y}'' = c_1' y_1' + c_2' y_2' + c_1 y_1'' + c_2 y_2''$

$E(\tilde{y}) = \tilde{y}'' + \tilde{y} = \underbrace{c_1 y_1'' + c_2 y_2'' + c_1 y_1 + c_2 y_2}_{= 0} + c_1' y_1' + c_2' y_2'$

$$\begin{cases} c_1' \sin x + c_2' \cos x = 0 \\ c_1' \cos x - c_2' \sin x = \operatorname{tg} x \end{cases}$$

$$\Rightarrow \begin{cases} c_1' \sin^2 x + c_2' \sin x \cos x = 0 \\ c_1' \cos^2 x - c_2' \sin x \cos x = \sin x \end{cases}$$

$$C_1' = \sin x \qquad C_1 = -\cos x$$

$$\sin^2 x + C_2' \cos x = 0$$

$$C_2' = -\frac{\sin^2 x}{\cos x} = -\frac{\sin^2 x}{1-\sin^2 x}\cos x$$

$$C_2 = -\int \frac{t^2}{1-t^2}\,dt = \int \frac{1-t^2}{1-t^2}\,dt - \int \frac{1}{1-t^2}\,dt =$$

$\sin x = t$

$$= t - \frac{1}{2}\Big(\ln(1+t) - \ln(1-t)\Big) = t - \frac{1}{2}\ln\frac{1+t}{1-t} =$$

$$= \sin x - \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow y(x) = C_1 \sin x + C_2 \cos x =$$

$$= -\cancel{\cos x \sin x} + \cancel{\sin x \cos x} - \frac{1}{2}\cos x \ln\frac{1+\sin x}{1-\sin x}$$

②

$E(y) = \sin x + x e^x + \operatorname{tg} x$

$y = -\frac{1}{2} x \cos x + \frac{1}{2} x e^x - \frac{1}{2} e^x - \frac{1}{2} \cos x \ln \frac{1+\sin x}{1-\sin x} + A \sin x + B \cos x$

$y(0) = -\frac{1}{2} + B = 0 \Rightarrow B = \frac{1}{2}$

$y'(0) = -\frac{3}{2} + A = 0 \Rightarrow A = \frac{3}{2}$

y soluzione di

$E(y) = \begin{cases} \sin x + x e^x + \operatorname{tg} x \\ y(0) = y'(0) = 0 \end{cases}$

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$y(x) = -\frac{1}{2} x \cos x + \frac{1}{2} x e^x - \frac{1}{2} e^x$

$- \frac{1}{2} \cos x \ln \frac{1+\sin x}{1-\sin x} + \frac{3}{2} \sin x + \frac{1}{2} \cos x$

definita in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Calcolare

$$\lim_{x\to 0} \frac{\cos x^2}{(\operatorname{arctg}(\sin x))^2} - \frac{e^{x^2}(1+\cos x)}{\sin x \arcsin(e^x-e^{-x})} =$$

$$= \lim_{x\to 0} \frac{\cos x^2 \sin x \arcsin(e^x-e^{-x}) - e^{x^2}(1+\cos x)(\operatorname{arctg}(\sin x))^2}{(\operatorname{arctg}(\sin x))^2 \sin x \arcsin(e^x-e^{-x})} \overset{\otimes}{=}$$

$$\arcsin(t) = t + \frac{t^3}{6} + o(t^3)$$

$$\Rightarrow \arcsin(e^x - e^{-x}) = 2x + \frac{x^3}{3} + \frac{8}{6}x^3 + o(x^3) = 2x + \frac{5}{3}x^3 + o(x^4)$$

$$\operatorname{arctg} t = t - \frac{t^3}{3} + o(t^4)$$

$$\Rightarrow \operatorname{arctg}(\sin x) = x - \frac{x^3}{6} - \frac{x^3}{3} + o(x^4) = x - \frac{x^3}{2} + o(x^4)$$

$$\overset{\otimes}{=} \lim_{x\to 0} \Big[\Big(1 - \frac{x^4}{2} + o(x^7)\Big)\Big(x - \frac{x^3}{6} + o(x^4)\Big)\Big(2x + \frac{5}{3}x^3 + o(x^4)\Big) -$$

$$- \Big(1 + x^2 + \frac{x^4}{2} + o(x^4)\Big)\Big(2 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\Big)\Big(x^2 - x^4 + o(x^4)\Big)\Big] /$$

$$/ \Big[\Big(x^2 - x^4 + o(x^4)\Big)\Big(x - \frac{x^3}{6} + o(x^3)\Big)\Big(2x + \frac{5}{3}x^3 + o(x^3)\Big)\Big] =$$

$$= \lim_{x\to 0} \frac{2x^2 + x^4\left(-\frac{1}{3} + \frac{5}{3}\right) - 2x^2 + \frac{1}{2}x^4 + o(x^4)}{2x^4 + o(x^4)} = \frac{11}{12}$$

Comp.11/09/2020.Ex.3

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Sia $x_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \dfrac{1}{\sqrt{n}\sqrt{n+1}\,(\sqrt{n+1}+\sqrt{n})} \sim \dfrac{1}{n^{3/2}}$

$$\frac{\left(\frac{1}{n}\right)^\beta}{\left(\sin\frac{1}{n}\right)^\beta} \xrightarrow[n\to\infty]{} 1 \quad (\beta \geq 0)$$

I) Per $\gamma = 2$ la serie è a termini positivi.

Sia $a_n = \dfrac{x_n^\alpha}{\left(\sin\frac{1}{n}\right)^\beta} \sim \dfrac{1}{n^{\alpha 3/2 - \beta}} \Rightarrow \sum a_n$ converge $\Leftrightarrow \alpha\frac{3}{2} - \beta > 1$

II) Per $\gamma = 1$ la serie è a segni alterni.

$x_n^\alpha \to 0$ decrescente. Per quanto detto sopra

la serie converge assolutamente per $\alpha\frac{3}{2} > 1$: $\alpha > 2/3$

per Leibniz converge semplicemente per $0 < \alpha \leq 2/3$

III Sia ora $\gamma = 2, \beta = 0, \alpha = 1$

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \quad : \text{serie telescopica}$$

$$\Rightarrow \sum_{h=1}^{n} a_h = \frac{1}{1} - \frac{1}{\sqrt{n+1}}$$

$$\Rightarrow \sum_{h=1}^{\infty} a_h = 1$$