lezione 76

ANALISI MATEMATICA (B)
LEZIONE 76
6.4.2020

Es 3 test settimanale

$$\begin{cases} u^{IV}(x) - 2u''(x) + u(x) = e^x \\ u(0)=0, \quad u'(0)=\frac{7}{8}, \quad u''(0)=-2, \quad u'''(0)=\frac{27}{8} \end{cases}$$

$$\rightarrow u^{IV}(0) - 2(-2) + 0 = e^0 = 1$$

$$u^{IV}(0) = -3.$$

forma normale

$$\rightarrow u^{IV}(x) = 2u''(x) - u(x) + e^x$$

$$\rightarrow u^{V}(x) = 2u'''(x) - u'(x) + e^x$$

$$u^{V}(0) = 2\cdot\frac{27}{8} - \frac{7}{8} + 1$$

$$= \frac{54-7+8}{8} = \frac{55}{8}$$

$\vdots$

$u^{VI}(0)$

$u^{VII}(0)$

$\vdots$

Sviluppo di Taylor (senza risolvere l'equazione)

$$u(x) = \frac{7}{8}x - \frac{2}{2}x^2 + \frac{\frac{27}{8}}{3!}x^3 + \frac{-3}{4!}x^4 + \frac{\frac{55}{8}}{5!}x^5 + o(x^5)$$

$$\|$$

$$u(x) = \frac{x^2-x}{8}e^x + xe^x$$

$\leftarrow$ questa era la soluzione

Metodo di similarità

Es 4.
$$\begin{cases} u'' + u = \sin(x+1) \\ u(0) = 0, \ u'(0) = 0 \end{cases}$$

$$L[u] = P(D)[u] = b(x)$$

se $b(x) = q(x) e^{\mu x}$, q polinomio
cerco una soluzione particolare
del tipo $u_*(x) = x^m q_*(x) e^{\mu x}$

m molteplicità di μ come radice
di P ($m=0$ se $P(\mu) \neq 0$).

$$P(D)[u] = b_1(x) + b_2(x)$$

per trovare una sol. particolare
basta sommare soluzioni particolari

(1) $P(D)[u_*] = b_1(x)$

(2) $P(D)[u_{**}] = b_2(x)$

principio di sovrapposizione

$\Downarrow$

$$P(D)[u_* + u_{**}] = P(D)[u_*] + P(D)[u_{**}] = b_1(x) + b_2(x).$$

$$P(\lambda) = \lambda^2 + 1 \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$u'' + u = \underline{\sin(x+1)} = \frac{e^{i(x+1)} - e^{-i(x+1)}}{2i}$$

$$= c_1 \underbrace{e^{ix}}_{b_1(x)} + c_2 \underbrace{e^{-ix}}_{b_2(x)} \qquad c_1, c_2 \in \mathbb{C}$$

$$e^{i(x+1)} = e^{ix} \cdot e^{i}$$

$$c_1 = \frac{e^{i}}{2i}$$

$$c_2 = \dots$$

Cerco u_* della forma:

soluzioni complesse

$$u_*(x) = \boxed{a \cdot x \cdot e^{ix} + b \cdot x \cdot e^{-ix}}$$

$i, -i$ sono radici di $P(\lambda)$.

$$= x \cdot \left(a e^{ix} + b e^{-ix} \right)$$

$$= \boxed{x \left(A \cdot \sin(x+1) + B \cos(x+1) \right)}$$

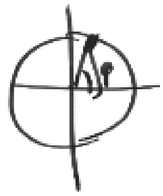
soluzioni reali

$$= \boxed{x \cdot C \cdot \sin(x + \varphi)} = x \cdot C \cdot \left[\cos\varphi \sin x + \sin\varphi \cos x \right]$$

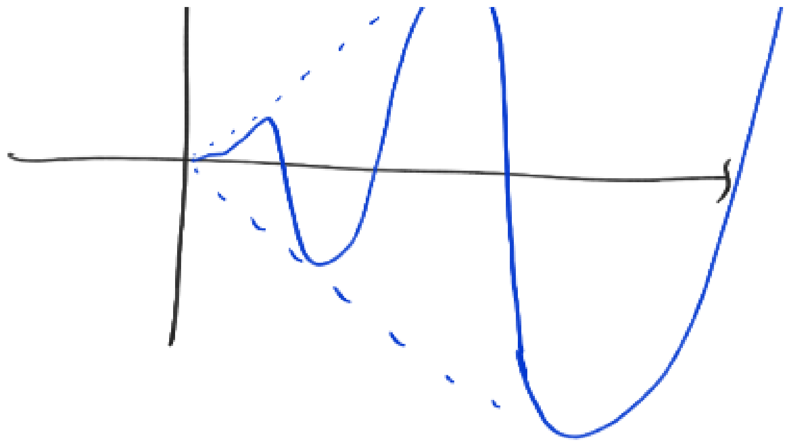
$$\boxed{\cos^2\varphi + \sin^2\varphi = 1}$$

$$A \cdot \sin x + B \cos x = \sqrt{A^2+B^2} \left[\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\varphi} \sin x + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\varphi} \cos x \right]$$

$$= \sqrt{A^2+B^2} \cdot \sin(x + \varphi)$$



con $\begin{cases} \cos\varphi = \dfrac{A}{\sqrt{A^2+B^2}} \\ \sin\varphi = \dfrac{B}{\sqrt{A^2+B^2}}. \end{cases}$



Metodo della variazione delle costanti

$$P(D)[u] = b(x)$$

coefficienti costanti
$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$

$$u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_1 u' + a_0 u = b(x)$$

$a_0, \dots, a_{n-1} \in \mathbb{R}$ (coefficienti costanti).

Le soluzioni della omogenea sono della forma:

$$u(x) = c_1 u_1(x) + \dots + c_n u_n(x).$$

$c_1 \dots c_n \in \mathbb{R}$ costanti

Cerco una soluzione della forma:

$$\boxed{u_*(x) = c_1(x) \cdot u_1(x) + \dots + c_n(x) u_n(x).}$$

$$u_*'(x) = c_1(x) u_1'(x) + \dots + c_n(x) u_n'(x) + c_1'(x) u_1 + \dots + c_n'(x) u_n$$

$$= c_1 u_1' + \dots + c_n u_n' +$$

$$u_*''(x) = c_1 u_1'' + \dots + c_n u_n'' +$$

$$\begin{cases} c_1' u_1 + \dots + c_n' u_n = 0 \\ c_1' u_1' + \dots + c_n' u_n' = 0 \\ \vdots \end{cases}$$

$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$

$$= 0$$

$$u_y^{(n)}(x) = \boxed{c_1 u_1^{(n)} + \dots + c_n u_n^{(n)}} + \boxed{c_1' u_1^{(n-1)} + \dots + c_n' u_n^{n-1} = b(x)}$$

$$P(D)[u] = \underline{0 + 0 + \dots 0} + b(x).$$

$$c_1(x)\left[u_1^{(n)} + a_{n-1} u_1^{(n-1)} + \dots + a_1 u_1' + a_0 u_1\right]$$

$$= c_1(x) \cdot [0] = 0$$

Il sistema BLU ha questa forma:

$$\begin{pmatrix} u_1(x), u_2(x), \dots & u_n(x) \\ u_1'(x), u_2', \dots & u_n'(x) \\ \vdots & \\ u_1^{(n-1)}, u_2^{(n-1)}, \dots & u_n^{(n-1)}(x) \end{pmatrix} \cdot \begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix}$$

$\uparrow$

$A(x)$

Teo Se $u_1 \dots u_n$ sono soluzioni indipendenti di una eq. diff. $P(D)[u] = 0$.

Allora $\det A(x) \neq 0 \ \forall x$.

vedremo la dimostrazione più avanti.

Es se
$$u_1(x) = e^{\lambda_1 x}$$
$$u_2(x) = e^{\lambda_2 x}$$
$$\vdots$$
$$u_n(x) = e^{\lambda_n x}$$

$$A(x) = \begin{pmatrix} e^{\lambda_1 x} & \dots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \dots & \lambda_n e^{\lambda_n x} \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \dots & \lambda_n^{n-1} e^{\lambda_n x} \end{pmatrix}$$

$$A(0) = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & & \\ \lambda_1^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

Vandermonde (?)

$$\det A(0) = \cdots \neq 0$$

Es

$$\boxed{u'' + u = \frac{1}{\cos x}} \qquad P(\lambda) = \lambda^2 + 1$$

$$\lambda_{1,2} = \pm i$$

$u_1(x) = \sin x$

$u_2(x) = \cos x$ sono due sol. indipendenti dell'omogenea: $u'' + u = 0$

$$u_*(x) = c_1(x) \cdot \sin x + c_2(x) \cos x$$

a_0 $u_* = c_1 \sin x + c_2 \cos x$

a_1 $u_*' = -c_2 \sin x + c_1 \cos x + \left\{ c_1' \sin x + c_2' \cos x = 0 \right.$

1 $u_*'' = \underbrace{-c_1 \sin x - c_2 \cos x}_{0} + \left\{ -c_2' \sin x + c_1' \cos x = \frac{1}{\cos x} \right.$

$$P(D)[u_*] = u_*'' + u_* = 0 + \frac{1}{\cos x} \qquad \text{ok!}$$

$$\begin{cases} c_1' \sin x + c_2' \cos x = 0 \\ -c_2' \sin x + c_1' \cos x = \dfrac{1}{\cos x} \end{cases}$$

$\cos x \neq 0$

$c_1'(x) = -\frac{\sin x}{\cdots}$ ...

$$\begin{cases} c_2'(x) = -\dfrac{\sin x}{\cos x} c_1' \\ \dfrac{\sin x}{\cos x} \cdot \sin x \, c_1' + \cos x \, c_1' = \dfrac{1}{\cos x} \end{cases}$$

$$\begin{cases} \%_0 \\ (\underbrace{\sin^2 x + \cos^2 x}_{1}) c_1' = 1 \end{cases} \qquad \begin{cases} c_2' = -\mathrm{tg}\, x \\ c_1' = 1 \end{cases}$$

$$\begin{cases} c_1(x) = x \\ c_2(x) = -\int \mathrm{tg}\, x = \ln(\cos x) \end{cases}$$

$$u_*(x) = x \cdot \sin x + \ln(\cos x) \cdot \cos x$$

sol. particolare di $u_*'' + u_* = \dfrac{1}{\cos x}$

$$u(x) = x \cdot \sin x + \ln(\cos x| \cdot \cos x + a \cdot \sin x + b \cdot \cos x$$

$a, b \in \mathbb{R}$

$$u(x) = (x + a) \sin x + (\ln(\cos x| + b) \cos x.$$

$a, b \in \mathbb{R}$.

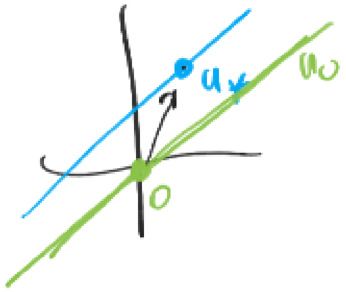
soluzione generale della eq. non omogenea.

$$\boxed{u'' + u = \frac{1}{\cos x}}$$

□

In generale le sol. della non-omogenea:

$$u(x) = u_*(x) + u_0(x)$$



1 sol. particolare della non omogenea

soluzione generale della omogenea.

Es

$u^{IV}(x) - 2u''(x) + u(x) = e^x$

$P(\lambda) = \lambda^4 - 2\lambda^2 + 1$

$= (\lambda^2 - 1)^2 = (\lambda+1)^2(\lambda-1)^2$

$\lambda_{1,2} = \pm 1$

$m_{1,2} = 2$

Sol. generale della omogenea:

$u_0(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$.

Metodo della var. delle costanti:

$u_* = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$

$u_*' = c_1 e^x + c_2(e^x + x e^x) - c_3 e^{-x} + c_4(e^{-x} - x e^{-x})$

$(c_1' e^x + c_2' x e^x + c_3' e^{-x} + c_4' x e^{-x} \equiv 0)$

$= (c_1 + c_2)e^x + c_2 x e^x + (c_4 - c_3)e^{-x} - c_4 x e^{-x}$

$u_*'' = (c_1 + c_2 + c_2)e^x + c_2 x e^x + (c_3 - c_4 - c_4)e^{-x} + c_4 x e^{-x}$

$(c_1' + c_2')e^x + c_2' x e^x + (c_4' - c_3')e^{-x} - c_4' x e^{-x} \equiv 0$

$u_*''' = (c_1 + 3c_2)e^x + c_2 x e^x + (c_4 - c_3)e^{-x} - c_4 x e^{-x}$

$((c_1' + 2c_2')e^x + c_2' x e^x + (c_3' - 2c_4')e^{-x} + c_4' x e^{-x} \equiv 0$

$$u_4^{IV} = (c_1 + 4c_2)e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$$

$$(c_1' + 3c_2')e^x + c_2' x e^x + (c_4' - c_3')e^{-x} - c_4' x e^{-x} \equiv e^x.$$

$$\begin{cases} c_1' e^x + c_2' x e^x + c_3' e^{-x} + c_4' x e^{-x} \equiv 0 \\ (c_1' + c_2')e^x + c_2' x e^x + (c_4' - c_3')e^{-x} - c_4' x e^{-x} \equiv 0 \\ (c_1' + 2c_2')e^x + c_2' x e^x + (c_3' - 2c_4')e^{-x} + c_4' x e^{-x} \equiv 0 \\ (c_1' + 3c_2')e^x + c_2' x e^x + (c_4' - c_3')e^{-x} - c_4' x e^{-x} \equiv e^x \end{cases}$$

$\vdots$