LEZIONE n. 68

ANALISI MATEMATICA (B)

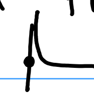
18.3.2020

(cds FISICA)

INIZIO PREVISTO ore 11:10

(Es. lezione scorsa

$\int_0^1 \frac{1}{x^2}\,dx \qquad \int_0^1 f(x)\,dx \qquad f(x)=\begin{cases} 1/x^2 & x\neq 0 \\ 7 & x=0 \end{cases}$)



INTEGRALI IMPROPRI

f non limitata

I intervallo non limitato es. $I=[0,+\infty)$

$f:[a,b)\to\mathbb{R} \qquad \int_a^b f(x)\,dx = \lim_{\beta\to b^-} \int_a^\beta f(x)\,dx \quad (*)$

f localmente Riemann-integrabile — basta f continua o f monotona

cioè se $f:A\to\mathbb{R}$ è R-integrabile su ogni intervallo chiuso e limitato.

se il limite $(*)$ esiste e finito diremo che
- la funzione è integrabile in senso improprio
- l'integrale converge

se il limite esiste ma è $+\infty$ o $-\infty$
- l'integrale diverge

se il limite non esiste
- l'integrale è indeterminato

carattere dell'integrale improprio

Se $f:(a,b]\to\mathbb{R}$ — $[\alpha,\beta]\subset(a,b]$ — localmente R-integrabile

$\int_a^b f(x)\,dx = \lim_{\alpha\to a^+} \int_\alpha^b f(x)\,dx$

f localmente R-integrabile.

Se $f:(a,b)\to\mathbb{R}$

si sceglie un punto $c\in(a,b)$

e si pone $(a,c]\cup[c,b)$

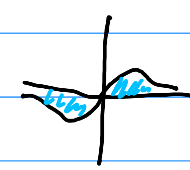
$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

integrali impropri

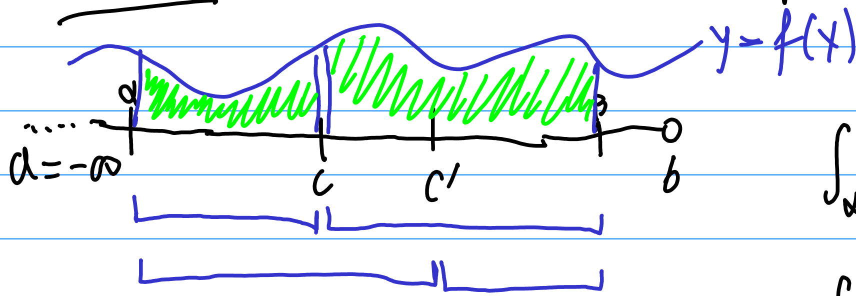
se entrambi gli integrali esistono e non sono "$(+\infty)+(-\infty)$" o "$(-\infty)+(+\infty)$"

es: $\mathbb{R}=(-\infty,+\infty)$

$\int_{-\infty}^{+\infty} \frac{x^3}{1+x^4}\,dx$



Osservazione la definizione non dipende dal punto c scelto



$$L_1 + L_2 = L$$

$$\int_\alpha^c + \int_c^\beta = \int_\alpha^\beta$$

$$\int_\alpha^{c'} + \int_{c'}^\beta = \int_\alpha^\beta$$

$$L_1' + L_2' = L$$

Es
$$\int_{-\infty}^{+\infty} \frac{x^3}{1+x^4}\,dx \underset{\text{non esiste}}{=} \underbrace{\int_{-\infty}^{0} \frac{x^3}{1+x^4}\,dx}_{= -\infty} + \underbrace{\int_{0}^{+\infty} \frac{x^3}{1+x^4}\,dx}_{= +\infty}$$

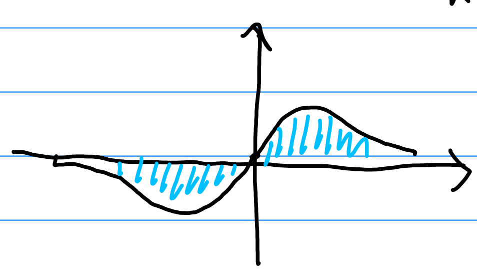
$$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\ln(1+x^4)$$

$$\int_0^{+\infty} \frac{x^3}{1+x^4}\,dx = \lim_{\beta\to+\infty} \int_0^\beta \frac{x^3}{1+x^4}\,dx = \lim_{\beta\to+\infty} \left[\frac{1}{4}\ln(1+x^4)\right]_0^\beta$$

$$= \lim_{\beta\to+\infty} \left(\frac{1}{4}\ln(1+\beta^4) - \cancel{\frac{1}{4}\ln 1}\right) = +\infty$$

$$\int_{-\infty}^0 \frac{x^3}{1+x^4}\,dx = \lim_{\alpha\to-\infty} \int_\alpha^0 \frac{x^3}{1+x^4}\,dx = \lim_{\alpha\to-\infty} \left[\frac{1}{4}\ln(1+x^4)\right]_\alpha^0$$

$$= \lim_{\alpha\to-\infty} \cancel{\frac{1}{4}\ln 1} - \frac{1}{4}\ln(1+x^4) = -\infty$$



(FACOLTATIVO)

La definizione che *non* usiamo è questa:

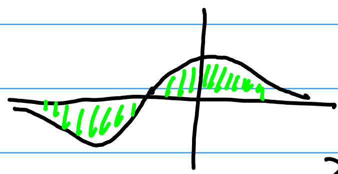
$$P.V. \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\alpha \to +\infty} \int_{-\alpha}^{\alpha} f(x)\,dx$$

$$P.V. \int_{-\infty}^{+\infty} \frac{x^3}{1+x^4}\,dx = 0 \qquad \int_{-\alpha}^{\alpha} \frac{x^3}{1+x^4} = \left[\frac{1}{4}\ln(1+x^4)\right]_{-\alpha}^{\alpha}$$

$$= \frac{1}{4}\ln(1+\alpha^4) - \frac{1}{4}\ln(1+|-\alpha|^4)$$

$$= 0.$$

$$P.V. \int_{-\infty}^{+\infty} \frac{(x+1)^3}{1+(x+1)^4}\,dx \overset{?}{=} 0$$



$$\lim_{\alpha \to +\infty} \int_{-\alpha}^{\alpha} \frac{(x+1)^3}{1+(x+1)^4}\,dx = \lim_{\alpha \to +\infty} \left[\frac{1}{4}\ln\left(1+(x+1)^4\right)\right]_{-\alpha}^{\alpha}$$

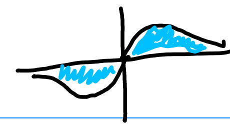
$$= \lim_{\alpha \to +\infty} \frac{1}{4}\left(\ln\left(1+(\alpha+1)^4\right) - \ln\left(1+(-\alpha+1)^4\right)\right)$$

$$= \lim_{\alpha \to +\infty} \frac{1}{4}\ln\frac{1+(\alpha+1)^4}{1+(-\alpha+1)^4} = 0$$

$$\boxed{\lim_{\alpha \to +\infty} \int_{-\alpha}^{\alpha^2} \frac{x^3}{1+x^4}\,dx \overset{?}{=} 0 \quad (+\infty)}$$

$$\boxed{PV \int_{-\infty}^{+\infty} \frac{x^3}{1+x^4}\,dx \overset{?}{=} PV \int_{-\infty}^{+\infty} \frac{(e^y+y)^3}{1+(e^y+y)^4}(e^y+1)\,dy \qquad x = e^y + y}$$

Esempio $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{1+x^4}\,dx$



$$= \int_{-\infty}^{0} \frac{x}{1+x^4}\,dx + \int_{0}^{+\infty} \frac{x}{1+x^4}\,dx$$

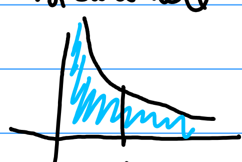
$$= \lim_{\alpha\to-\infty} \int_{\alpha}^{0} \frac{x}{1+x^4}\,dx + \lim_{\beta\to+\infty} \int_{0}^{\beta} \frac{x}{1+x^4}\,dx$$

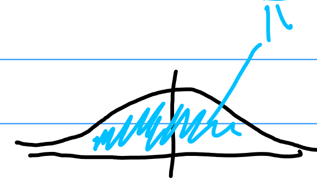
$$= \lim_{\alpha\to-\infty} \left[\frac{1}{2}\operatorname{arctg} x^2\right]_{\alpha}^{0} + \lim_{\beta\to+\infty} \left[\frac{1}{2}\operatorname{arctg} x^2\right]_{0}^{\beta}$$

$$= \left(0 - \frac{1}{2}\frac{\pi}{2}\right) + \left(\frac{1}{2}\frac{\pi}{2} - 0\right) = 0.$$

Es $\displaystyle\int_{0}^{+\infty} \frac{1}{x^2}\,dx = \underbrace{\int_{0}^{1} \frac{1}{x^2}\,dx}_{=+\infty} + \underbrace{\int_{1}^{+\infty} \frac{1}{x^2}\,dx}_{=1} = +\infty$

Bi-laterale





Es $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{\alpha\to-\infty} \int_{\alpha}^{0} \frac{1}{1+x^2}\,dx + \lim_{\beta\to+\infty} \int_{0}^{\beta} \frac{1}{1+x^2}\,dx$

$$\left[= \lim_{\alpha\to-\infty} \left[\operatorname{arctg} x\right]_{\alpha}^{0} + \lim_{\beta\to+\infty} \left[\operatorname{arctg} x\right]_{0}^{\beta}\right]$$

NOTAZIONE $\to = \left[\operatorname{arctg} x\right]_{-\infty}^{0} + \left[\operatorname{arctg} x\right]_{0}^{+\infty} =: \left[\operatorname{arctg} x\right]_{-\infty}^{+\infty}$

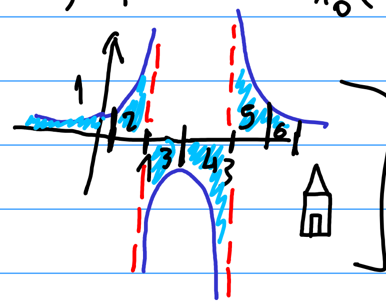
$[F(x)]_a^b = F(b) - F(a)$ se F non è definita in a poniamo $[F]_a^b = \lim_{\alpha\to a^+}\left(F(b) - F(\alpha)\right)$

$$\left([F(x)]_a^b \overset{\text{PER DEFINIZIONE}}{=} \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x) \right)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \qquad \square$$

$$f : (a,b) \setminus \{x_0, x_1, x_2, \dots, x_n\} \to \mathbb{R} \qquad x_0 < x_1 < \dots < x_n$$

Es $\quad f(x) = \dfrac{1}{(x-1)(x-3)}$

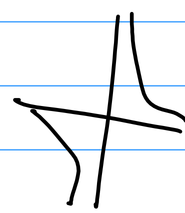


$$\int_a^b f(x)\,dx = \underbrace{\int_a^{x_0} f(x)\,dx}_{\text{integrali impropri}} + \int_{x_0}^{x_1} f(x)\,dx + \dots + \int_{x_n}^{b} f(x)\,dx$$

se ogni integrale esiste e la somma è definita.

Es

$$\int_{-\infty}^{+\infty} \frac{1}{x}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{x}\,dx}_{= -\infty} + \underbrace{\int_{0}^{+\infty} \frac{1}{x}\,dx}_{= +\infty}$$



non esiste (indeterminato)

$$\int_{-\infty}^{+\infty} \frac{1}{x}\,dx = [\ln|x|]_{-\infty}^{0} + [\ln|x|]_{0}^{+\infty}$$

$$\int_0^{+\infty} \frac{1}{x}\,dx = \int_0^1 \frac{1}{x}\,dx + \int_1^{+\infty} \frac{1}{x}\,dx$$

ES

$$\int_{-\infty}^{+\infty} \frac{1}{x^2}\,dx \neq \left[-\frac{1}{x}\right]_{-\infty}^{+\infty} = 0 - 0$$

Sospetto

$$\int_{-\infty}^{+\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-\infty}^{0} + \left[-\frac{1}{x}\right]_{0}^{+\infty} = (+\infty - 0) + (0 + \infty) = +\infty$$