# Esercizio 1

$$a_n = \frac{(2n)!}{n^{(2n)}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{(2n+2)}} \cdot \frac{n^{(2n)}}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)!}\, n^{(2n)}}{(n+1)^{2n}(n+1)^2 \cancel{(2n)!}}$$

$$= \frac{4n^2+5n+2}{\left(\frac{n+1}{n}\right)^{2n}(n^2+2n+1)} = \frac{4n^2+5n+2}{n^2+2n+1}\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)^2$$

$$\longrightarrow 4\left(\frac{1}{e}\right)^2 = \left(\frac{2}{e}\right)^2 < 1$$

Per il criterio del rapporto $a_n \to 0$

## Esercizio 2

$$f(x) = \frac{x+1}{|\log x|}$$ è definita per $x>0$, $x \neq 1$.

$$= \begin{cases} \dfrac{x+1}{\log x} & \text{se } x>1 \\[2ex] -\dfrac{x+1}{\log x} & \text{se } 0<x<1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{\log x - (x+1)/x}{\log^2 x} & \text{se } x > 1 \\[2ex] \dfrac{(x+1)/x - \log x}{\log^2 x} & \text{se } x < 1 \end{cases}$$

(a) se $x \in (0,1)$ $\quad \frac{x+1}{x} = 1 + \frac{1}{x} > 1$, $\log x < 0$

$\Rightarrow f'(x) > 0 \Rightarrow f$ è strettamente crescente su $(0,1)$

(b) se $x>1$

Posto $g(x) = \log x - (x+1)/x = \log x - 1 - \frac{1}{x}$

si ha $f'(x) = \dfrac{g(x)}{\log^2 x}$ segno di $f'(x)$ = segno di $g(x)$.

$$g'(x) = \frac{1}{x} + \frac{1}{x^2} > 0 \qquad \text{per } x>0$$

$$g(1) = -1, \qquad \lim_{x\to+\infty} g(x) = +\infty$$

$\Rightarrow$ $g$ è strettamente crescente su $[1,+\infty)$
e per il teorema dei valori intermedi
esiste un unico $\bar{x} \in (1,+\infty)$ t.c. $g(\bar{x})=0$.

~~Abbiamo~~ Inoltre

$$g(2) = (\log 2) - \frac{3}{2} < (\log e) - \frac{3}{2} = 1 - \frac{3}{2} < 0 = g(\bar{x})$$

Dunque $\bar{x}>2$.

| | $0$ | | $1$ | | $2$ | | $\bar{x}$ | |
|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | $\nexists$ | $+$ | $\nexists$ | $-$ | $-$ | $-$ | $0$ | $+$ |
| $f(x)$ | | $\nearrow$ | | $\searrow$ | | | min | $\nearrow$ |

$\bar{x}$ è un punto di minimo relativo per $f(x)$.

(c)

$$g'(y_0) = g'(f(2)) = \frac{1}{f'(2)} = \frac{1}{f'(2)} = \frac{\log^2 2}{\frac{3}{2} - \log 2}$$

## Esercizio 3

$$0 \le \frac{1}{x+1} \le \frac{1}{x+\cos x} \le \frac{1}{x-1} \le \frac{1}{n-1}$$

$x > -1$ se $x \ge n$

Quindi se ~~[illegible]~~ $n \ge 1$ si ha

$$0 \le \int_n^{n+1} \frac{1}{x+\cos x}\, dx \le \int_n^{n+1} \frac{1}{n-1}\, dx = \frac{1}{n-1} \xrightarrow{\text{per } n\to+\infty} 0$$

quindi $\lim\limits_{n\to+\infty} \int_n^{n+1} \frac{1}{x+\cos x}\, dx = 0.$

## Soluzione alternativa

Per il teorema della media integrale:

Esiste $x_n \in [n, n+1]$ tale che

$$\int_n^{n+1} \frac{1}{x+\cos x}\, dx = \frac{1}{x_n + \cos x_n} \longrightarrow 0 \quad \text{per } n \to +\infty$$

in quanto $x_n \to +\infty$ se $n \to +\infty$.

# Esercizio 4

$$\sin n^{-\alpha} = n^{-\alpha} + o(n^{-\alpha}) \qquad \text{per } n \to +\infty$$

$$\log(1+\sin n^{-\alpha}) = \sin n^{-\alpha} + o(\sin n^{-\alpha})$$

$$= n^{-\alpha} + o(n^{-\alpha})$$

$$\cos(\log(1+\operatorname{sen} n^{-\alpha})) = 1 - \frac{1}{2}(\log(1+\operatorname{sen} n^{-\alpha}))^2 + o\left((\log(1+\operatorname{sen} n^{-\alpha}))^2\right)$$

$$= 1 - \frac{1}{2} n^{-2\alpha} + o(n^{-2\alpha})$$

$$\tan(n^{-\beta}) = n^{-\beta} + o(n^{-\beta})$$

$$a_n = (1-\cos(\log(1+\sin(n^{-\alpha}))))\tan(n^{-\beta})$$

$$= \left(-\frac{1}{2} n^{-2\alpha} + o(n^{-2\alpha})\right)\left(n^{-\beta} + o(n^{-\beta})\right)$$

$$= -\frac{1}{2} n^{-(2\alpha+\beta)} + o\left(n^{-(2\alpha+\beta)}\right)$$

$\sum a_n$ converge $\iff \sum(-a_n)$ converge

$\iff$ $1 . \sum n^{-(2\alpha+\beta)}$ converge $\iff$ $\boxed{2\alpha+\beta > 1}$