Arrangements with up to 7 Hyperplanes

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Main Definitions

- A **complex hyperplane arrangement** is a finite collection $\mathcal{A} = \{H_1, \ldots, H_m\}$ of affine hyperplanes in $\mathbb{C}^d$.
- The **complement manifold** $M(\mathcal{A})$ is $\mathbb{C}^d \setminus \bigcup_{j=1}^{m} H_j$.
- **Problem:** study the topology of $M(\mathcal{A})$.

The Central Case

- A complex hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ is **central** if all the H_j's contain the **origin**.
- **Result:** to understand $M(\mathcal{A})$ we can study the central case.

Milnor Fiber and Fibration

For a complex central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ let $\alpha_i \in (\mathbb{C}^d)^*$ be linear formswith $H_i = \ker \alpha_i$. The polynomial $Q_{\mathcal{A}} = \prod_{i=1}^m \alpha_i$ is **homogeneous** of degree m and can be considered as a map $Q_{\mathcal{A}} : M(\mathcal{A}) \longrightarrow \mathbb{C}^*$ that is the projection of a **fiber bundle** called the **Milnor fibration** of the arrangement.

Isotopic Hyperplane Arrangements (Part 1)

Theorem ([Ran89])

Let $\mathcal{A}_t$ be a ***smooth one-parameter*** *family of central complex hyperplane arrangements in $\mathbb{C}^d$. If the underlying matroid $M_{\mathcal{A}_t}$ does* ***not*** *depend on t, so does the* ***diffeomorphism*** *type of $M(\mathcal{A}_t)$.*

Isotopic Hyperplane Arrangements (Part 2)

Theorem ([Ran97])

Let $\mathcal{A}_t$ be a ***smooth one-parameter*** *family of central complex hyperplane arrangements in $\mathbb{C}^d$. If the underlying matroid $M_{\mathcal{A}_t}$ does* ***not*** *depend on t, so does the* ***isomorphism*** *type of $Q_{\mathcal{A}_t}$.*

Main Results

Theorem ([GS16])

Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be ***rank*** *d central hyperplane arrangements in $\mathbb{C}^d$ with* ***same*** *underlying matroid. If $1 \leqslant d \leqslant m \leqslant 7$, then $\mathcal{A}$ and $\mathcal{B}$ are* ***isotopic*** *arrangements.*

Some Corollaries (Part 1)

Corollary ([GS16])

*Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be **rank** d central hyperplane arrangements in $\mathbb{C}^d$ with **same** underlying matroid. If $1 \leqslant d \leqslant m \leqslant 7$, then the complement manifolds $M(\mathcal{A})$ and $M(\mathcal{B})$ are **diffeomorphic**.*

Some Corollaries (Part 2)

Corollary ([GS16])

*Let $\mathcal{A} = \{H_1, \dots, H_m\}$ and $\mathcal{B} = \{K_1, \dots, K_m\}$ be **rank** d central hyperplane arrangements in $\mathbb{C}^d$ with **same** underlying matroid. If $1 \leqslant d \leqslant m \leqslant 7$, then the Milnor fibrations $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ are **isomorphic** fiber bundles.*

Further Questions

- Find a **non-case-by-case** proof of these results.
- Find more **refined** techniques to study **non-connected** matroid realization spaces.
- Study the **Rybnikov** matroid realization space.

A Small Bibliography

Matteo Gallet and Elia Saini, *The diffeomorphism type of small hyperplane arrangements is combinatorially determined*, [arXiv:1601.05705] (2016).

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