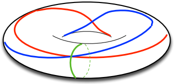
Combinatorial algebraic topology

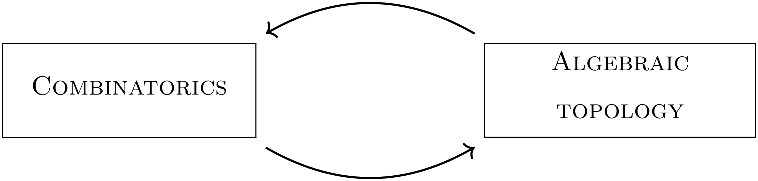


# of toric arrangements.

Emanuele Delucchi
(SNSF / Université de Fribourg)

Università di Pisa
February 3., 2016

Combinatorial algebraic topology



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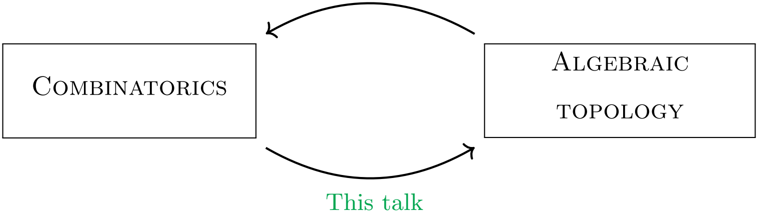


Outline:

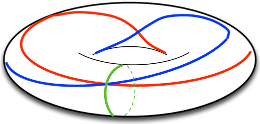
1. Problem, context
2. Our tools
3. Our solution

TORIC ARRANGEMENTS

A toric arrangement in the complex torus $T := (\mathbb{C}^*)^d$ is a set

$$\mathscr{A} := \{K_1, \ldots, K_n\}$$

of 'hypertori' $K_i = \chi_i^{-1}(b_i)$ with $\chi_i \in \mathrm{Hom}_{\neq 0}(T, \mathbb{C}^*)$ and $b_i \in \mathbb{C}^* / = 1 / \in S^1$



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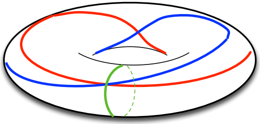
of 'hypertori' $K_i = \{z \in T \mid z^{a_i} = b_i\}$ with $a_i \in \mathbb{Z}^d \backslash 0$ and $b_i \in \mathbb{C}^*$

For simplicity assume that the matrix $[a_1, \dots, a_n]$ has rank d.



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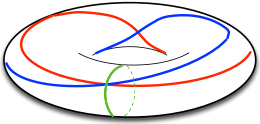

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PROBLEM: Determine the ring $H^*(M(\mathscr{A}), \mathbb{Z})$.

General problem

Let X be a complex manifold, $\mathscr{A} := \{L_i\}_i$ a family of submanifolds of X. Determine the topology of

$$M(\mathscr{A}) := X \setminus \bigcup_i L_i.$$

Examples: normal crossing divisors (Deligne), arrangements of hypersurfaces (Dupont), configuration spaces (e.g., Totaro), affine subspace arrangements (e.g., Goresky-MacPherson, De Concini-Procesi), toric arrangements, arrangements of hyperplanes, etc.

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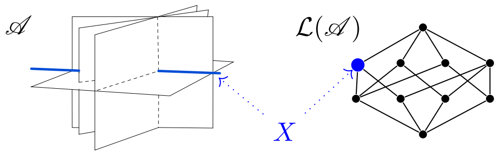
What can combinatorial models tell us?

“Exhibit A”: Arrangements of (real) pseudospheres $\leftrightarrow$ Oriented matroids.

Hyperplanes: Brieskorn

$\mathscr{A} := \{H_1, \ldots, H_d\}$: set of (affine) hyperplanes in $\mathbb{C}^d$,

$\mathcal{L}(\mathscr{A}) := \{\cap\mathscr{B} \mid \mathscr{B} \subseteq \mathscr{A}\}$: (po)set of intersections (reverse inclusion).



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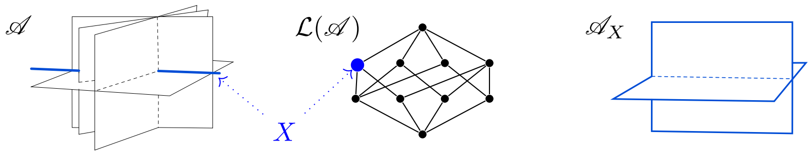

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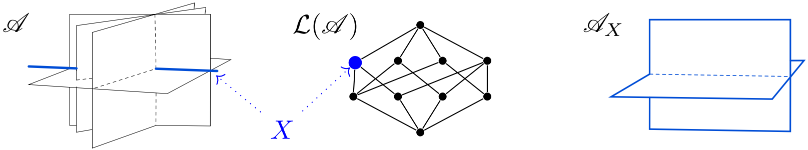

Theorem (Brieskorn 1972). The inclusions $M(\mathscr{A}) \hookrightarrow M(\mathscr{A}_X)$ induce, for every k, an isomorphism of free abelian groups

$$b : \bigoplus_{\substack{X \in \mathcal{L}(\mathscr{A}) \\ \operatorname{codim} X = k}} H^k(M(\mathscr{A}_X), \mathbb{Z}) \xrightarrow{\cong} H^k(M(\mathscr{A}), \mathbb{Z})$$

Hyperplanes: The Orlik-Solomon algebra

[Arnol'd '69, Orlik-Solomon '80]

$$H^*(M(\mathscr{A}), \mathbb{Z}) \simeq E/\mathcal{J}(\mathscr{A}), \text{ where}$$

E: exterior $\mathbb{Z}$-algebra with degree-1 generators $e_1, \ldots, e_n$ (one for each H_i);

$\mathcal{J}(\mathscr{A})$: the ideal $\langle \sum_{l=1}^{k} (-1)^l e_{j_1} \cdots \widehat{e_{j_l}} \cdots e_{j_k} \mid \operatorname{codim}(\cap_{i=1\ldots k} H_{j_i}) = k-1 \rangle$

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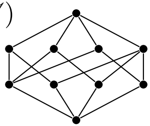
This is fully determined by $\mathcal{L}(\mathscr{A})$.

For instance:

codim X

$$P(M(\mathscr{A}),t) = \sum_{X\in\mathcal{L}(\mathscr{A})} \underbrace{\mu_{\mathcal{L}(\mathscr{A})}(\hat{0},X)}_{\substack{\text{Möbius}\\ \text{function}\\ \text{of } \mathcal{L}(\mathscr{A})}} (-t)^{\operatorname{rk} X}$$

$\mathcal{L}(\mathscr{A})$

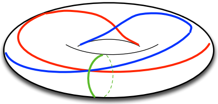


$$\operatorname{Poin}(M(\mathscr{A}),t) = 1+4t+5t^2+2t^3$$

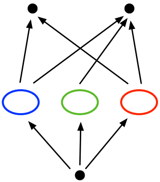
TORIC ARRANGEMENTS

Here the role of the intersection poset is played by $\mathcal{C}(\mathscr{A})$, the poset of *layers* (i.e. connected components of intersections of the K_i).

$\mathscr{A}$:



$\mathcal{C}(\mathscr{A})$:



Toric arrangements

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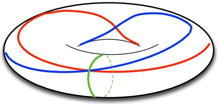
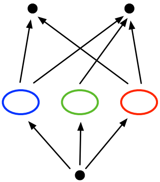
$\mathscr{A}$: 

$\mathcal{C}(\mathscr{A})$: 

Theorem [Looijenga '95, De Concini-Procesi '05]

$$\mathrm{Poin}(M(\mathscr{A}),\mathbb{Z}) = \sum_{Y\in\mathcal{C}(\mathscr{A})} \underbrace{\mu_{\mathcal{C}(\mathscr{A})}(\hat{0},Y)}_{\substack{\text{Möbius}\\ \text{function}\\ \text{of } \mathcal{C}(\mathscr{A})}}(-t)^{\mathrm{rk}\,Y}(1+t)^{d-\mathrm{rk}\,Y}.$$

Toric arrangements

[De Concini – Procesi '05] compute the cup product in $H^*(M(\mathscr{A}), \mathbb{C})$ when the matrix $[a_1, \ldots, a_n]$ is totally unimodular.

[Moci – Settepanella, '11] Combinatorial models for "thick" arrangements.

[Bibby '14] $\mathbb{Q}$-cohomology algebra of unimodular abelian arrangements

[Dupont '14, '15] Algebraic model for $\mathbb{C}$-cohomology algebra of complements of hypersurface arrangements in manifolds with hyperplane-like crossings; formality (coming up!),

We strive for a (combinatorial) presentation of the integer cohomology ring.

Posets and categories

P - a partially ordered set

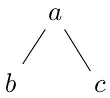
$\Delta(P)$ - the *order complex* of P

(abstract simplicial complex

of totally ordered subsets)

$||P|| := |\Delta(P)|$

its geometric realization



P

$$\left\{ \begin{matrix} a & b & c \\ ab & ac & (\emptyset) \end{matrix} \right\}$$

ΔP

$||P||$

Posets and categories

P - a partially ordered set

$\mathcal{C}$ - a *s.c.w.o.l.* / "acyclic category"

(all invertibles are endomorphisms,
all endomorphisms are identities)

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$\Delta\mathcal{C}$ - the nerve

(simplicial set of composable chains)

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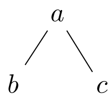

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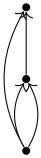

$\mathcal{C}$

Posets and categories

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its geometric realization

- Posets are special cases of s.c.w.o.l.s;
- Every functor $F : \mathcal{C} \to \mathcal{D}$ induces a continuous map $||F|| : ||\mathcal{C}|| \to ||\mathcal{D}||$.
- *Quillen-type theorems* relate properties of $||F||$ and F.

Face categories

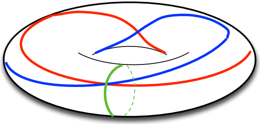
Let X be a polyhedral complex. The *face category* of X is $\mathcal{F}(X)$, with

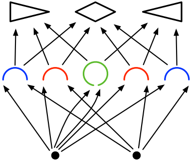
- $\mathrm{Ob}(\mathcal{F}(X)) = \{X_\alpha, \text{ polyhedra of } X\}$.
- $\mathrm{Mor}_{\mathcal{F}(X)}(X_\alpha, X_\beta) = \{ \text{ face maps } X_\alpha \to X_\beta\}$

Theorem. There is a homeomorphism $||\mathcal{F}(X)|| \cong X$. [Kozlov / Tamaki]

A toric arrangement $\mathscr{A} = \{\chi_i^{-1}(b_i)\}$ with $b_i \in S^1$ is called *complexified*.

It induces a polyhedral cellularization of $(S^1)^d$: call $\mathcal{F}(\mathscr{A})$ its face category.

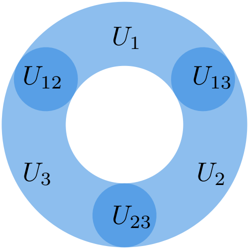




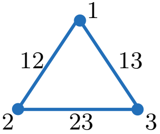
The Nerve Lemma

Let X be a paracompact space with a (locally) finite open cover $\mathcal{U} = \{U_i\}_I$.

For $J \subseteq I$ write $U_J := \bigcap_{i \in J} U_i$.



$$\mathscr{N}(\mathcal{U}) = \left\{ \begin{array}{ccc} 12 & 13 & 23 \\ 1 & 2 & 3 \end{array} \right\}$$



Nerve of $\mathcal{U}$: the abstract simplicial complex $\mathscr{N}(\mathcal{U}) = \{\emptyset \neq J \subseteq I \mid U_J \neq \emptyset\}$

Theorem (Weil '51, Borsuk '48). If U_J is contractible for all $J \in \mathscr{N}(\mathcal{U})$,

$$X \simeq |\mathscr{N}(\mathcal{U})|$$

The Generalized Nerve Lemma

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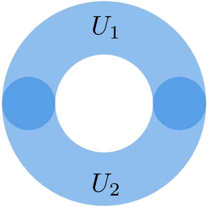
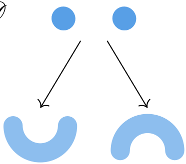


$$\mathscr{N}(\mathcal{U}) = \left\{ \begin{matrix} 1 & 2 \\ 12 & \end{matrix} \right\}$$

$\mathscr{D}$



Consider the *diagram* $\mathscr{D} : \mathscr{N}(\mathcal{U}) \to \mathrm{Top}$, $\mathscr{D}(J) := U_J$ and inclusion maps.

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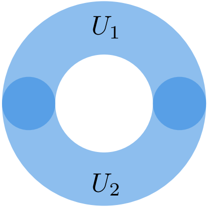

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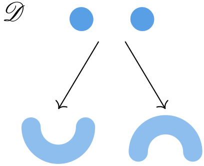

Consider the *diagram* $\mathscr{D} : \mathscr{N}(\mathcal{U}) \to \mathrm{Top}$, $\mathscr{D}(J) := U_J$ and inclusion maps.

$$X = \operatorname{colim}\mathscr{D}$$

$$\uparrow$$

$$\biguplus_J \mathscr{D}(J) \Big/ \begin{matrix}\text{identifying} \\ \text{along maps}\end{matrix}$$

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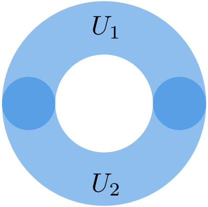

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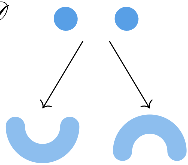

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$$\begin{tikzcd} X = \operatorname{colim}\mathscr{D} & \operatorname{hocolim}\mathscr{D} \arrow[l] \\ \biguplus_J \mathscr{D}(J) \Big/ \text{identifying along maps} \arrow[u] & \biguplus_{J_0\subseteq\dots\subseteq J_n} \Delta^{(n)} \times \mathscr{D}(J_n) \Big/ \text{glue in mapping cylinders} \arrow[u] \end{tikzcd}$$

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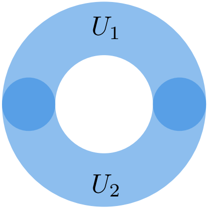

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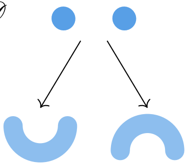

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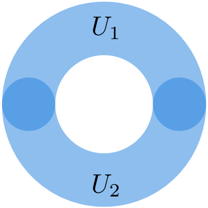

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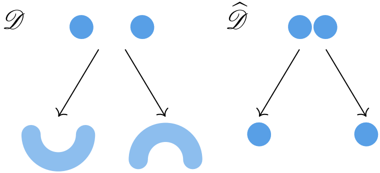

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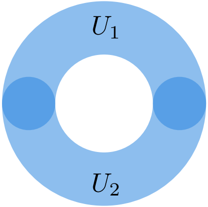

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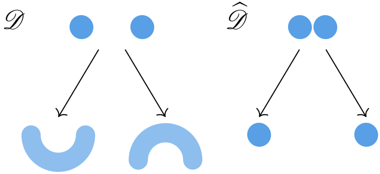

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The Generalized Nerve Lemma

Application: the Salvetti complex

Let $\mathscr{A}$ be a *complexified* arrangement of hyperplanes in $\mathbb{C}^d$ (i.e. the defining equations for the hyperplanes are real).

[Salvetti '87] There is a poset $\mathrm{Sal}(\mathscr{A})$ such that

$$||\,\mathrm{Sal}(\mathscr{A})|| \simeq M(\mathscr{A}).$$

The Generalized Nerve Lemma

Application: the Salvetti complex

Let $\mathscr{A}$ be a *complexified* arrangement of hyperplanes in $\mathbb{C}^d$ (i.e. the defining equations for the hyperplanes are real).

[Salvetti '87] There is a poset $\mathrm{Sal}(\mathscr{A})$ such that

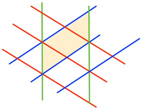
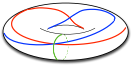
$$|| \, \mathrm{Sal}(\mathscr{A}) || \simeq M(\mathscr{A}).$$

[Callegaro-D. '15] Let $X \in \mathcal{L}(\mathscr{A})$ with $\operatorname{codim} X = k$.
There is a map of posets $\mathrm{Sal}(\mathscr{A}) \to \mathrm{Sal}(\mathscr{A}_X)$ that induces the Brieskorn inclusion $b_X : H^k(M(\mathscr{A}_X), \mathbb{Z}) \hookrightarrow H^k(M(\mathscr{A}), \mathbb{Z})$.

Salvetti Category

[d'Antonio-D., '11]

Any complexified toric arrangement $\mathscr{A}$ lifts to a complexified arrangement of affine hyperplanes $\mathscr{A}^{\upharpoonright}$ under the universal cover

$\mathbb{C}^d \to T,\qquad \mathscr{A}^{\upharpoonright}:$  $\xrightarrow{/\mathbb{Z}^d}$ $\mathscr{A}:$ 

The group $\mathbb{Z}^d$ acts on $\mathrm{Sal}(\mathscr{A}^{\upharpoonright})$ and we can define the *Salvetti category* of $\mathscr{A}$:

$$\mathrm{Sal}(\mathscr{A}) := \mathrm{Sal}(\mathscr{A}^{\upharpoonright})/\mathbb{Z}^d$$

(quotient taken in the category of scwols).

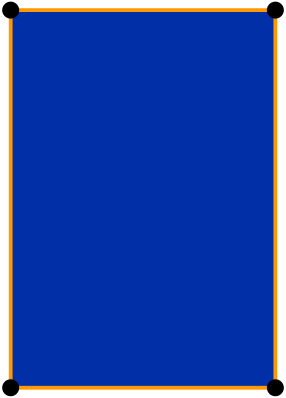
Here the realization commutes with the quotient [Babson-Kozlov '07], thus

$$\|\,\mathrm{Sal}(\mathscr{A})\| \simeq M(\mathscr{A}).$$

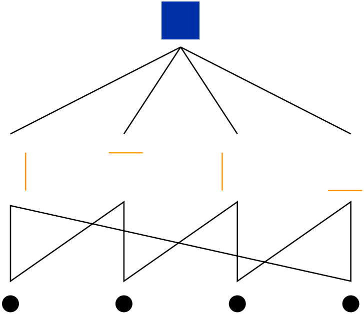
Discrete Morse Theory

[Forman, Chari, Kozlov,...; since '98]

Here is a regular CW complex



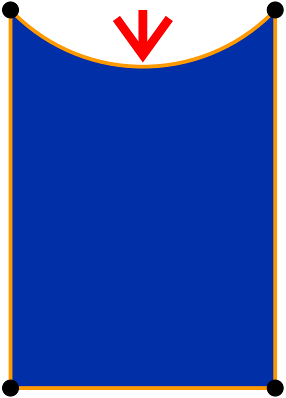
with its poset of cells:

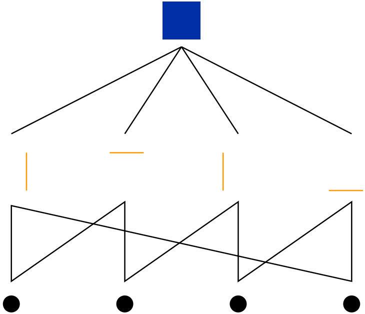


Discrete Morse Theory

[Forman, Chari, Kozlov,...; since '98]

Elementary collapses...





... are homotopy equivalences.

Discrete Morse Theory

[Forman, Chari, Kozlov,...; since '98]

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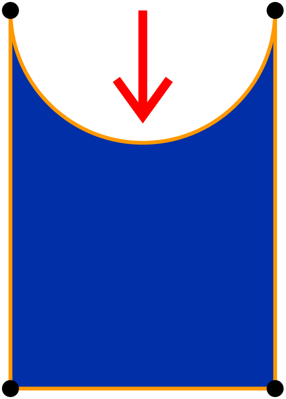

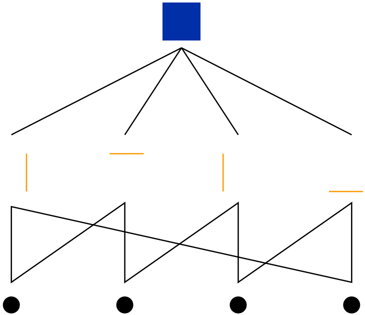

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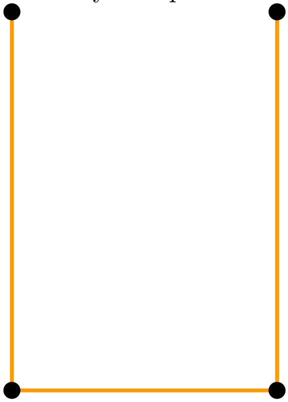

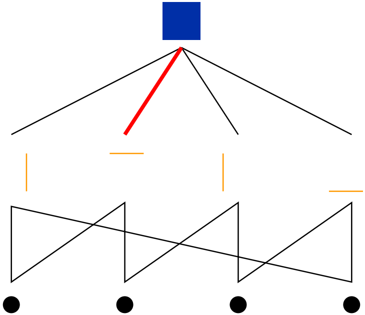

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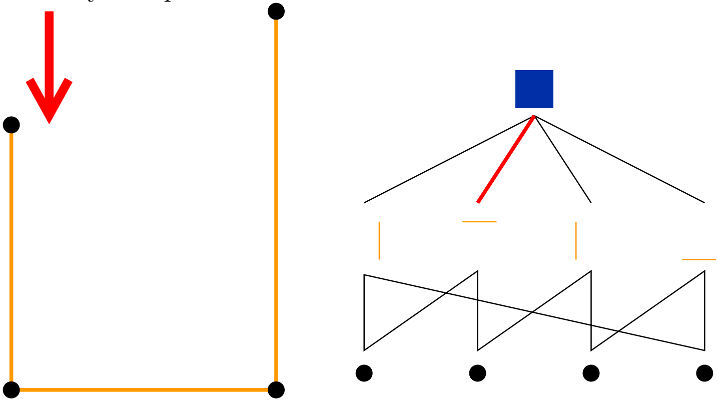

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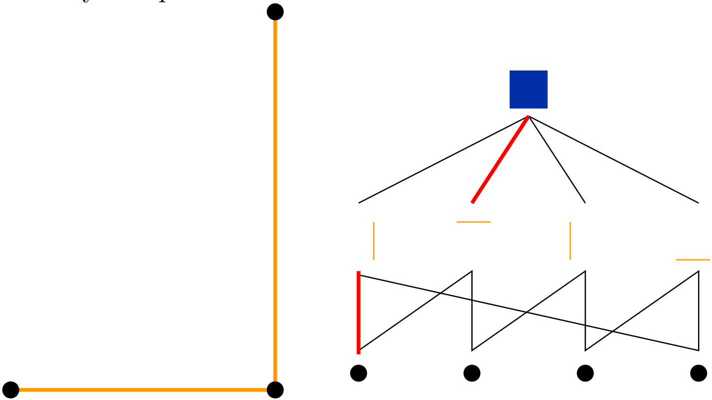

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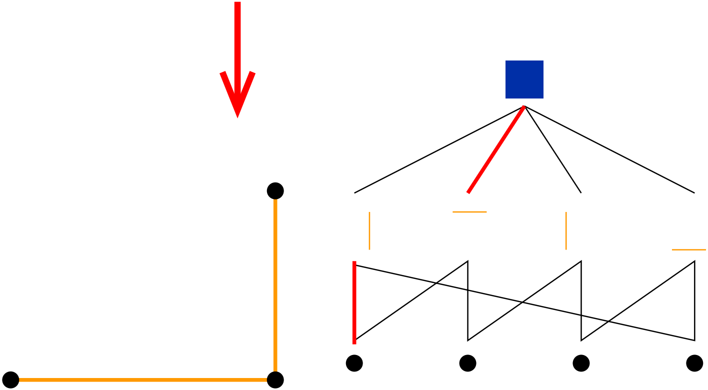

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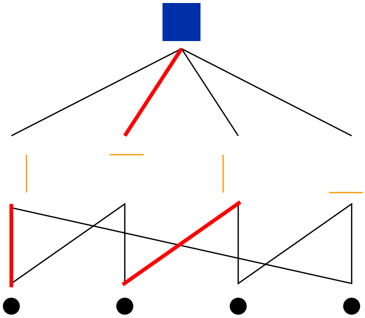

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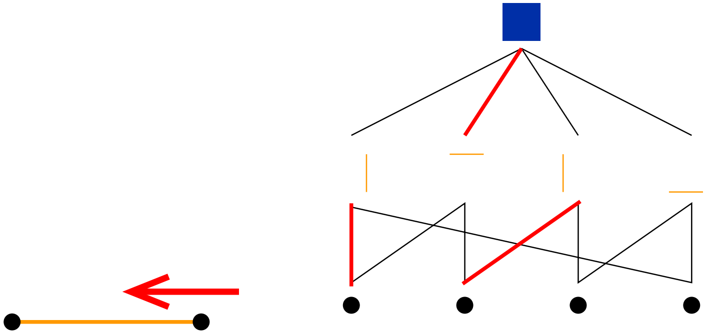

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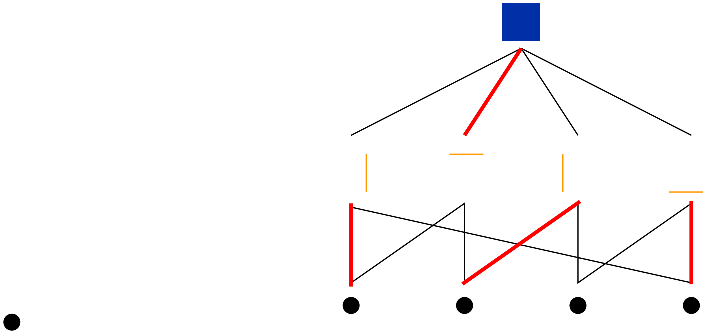

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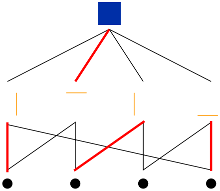
Discrete Morse Theory

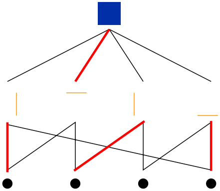
The sequence of collapses is encoded in a matching of the poset of cells.



Question: Does every matchings encode such a sequence?

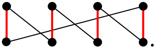
Discrete Morse Theory

The sequence of collapses is encoded in a matching of the poset of cells.



Question: Does every matchings encode such a sequence?

Answer: No. Only (and exactly) those without "cycles" like



Acyclic matchings $\leftrightarrow$ discrete Morse functions.

- The theory **extends to face categories** [d'Antonio-D. '15]

Discrete Morse Theory

Application: minimality of $\mathrm{Sal}(\mathscr{A})$

Let $\mathscr{A}$ be a complexified toric arrangement.

Theorem. [d'Antonio-D., '15] The space $M(\mathscr{A})$ is *minimal*.

In particular, its cohomology groups $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion-free.

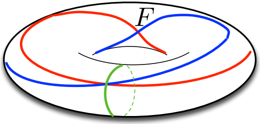
Here "minimal" means: has the homotopy type of a CW-complex with one cell for each generator in homology.

Proof. Construction of an acyclic matching of the Salvetti category with $\mathrm{Poin}(M(\mathscr{A}), 1)$ critical cells.

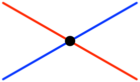
(Uses: minimality of Salvetti complexes of abstract oriented matroids [D.'08])

The Salvetti category - again

For $F \in \mathrm{Ob}(\mathcal{F}(\mathscr{A}))$ consider the hyperplane arrangement $\mathscr{A}[F]$:



$\mathscr{A}[F]$



THE SALVETTI CATEGORY - AGAIN

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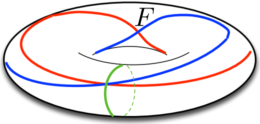
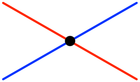


$\mathscr{A}[F]$ 

[Callegaro – D. '15] $||\,\mathrm{Sal}(\mathscr{A})|| \simeq \mathrm{hocolim}\,\mathscr{D}$, where

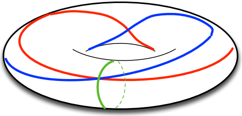
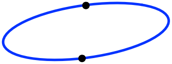
$$\begin{aligned} \mathscr{D}: \quad \mathcal{F}(\mathscr{A}) &\to \mathrm{Top} \\ F &\mapsto ||\,\mathrm{Sal}(\mathscr{A}[F])|| \end{aligned}$$

Call ${}_{\mathscr{D}}E_*^{p,q}$ the associated cohomology spectral sequence [Segal '68].

(equivalent to the Leray Spectral sequence of the canonical proj to $||\mathcal{F}(\mathscr{A})||$)

The Salvetti category - ...and again

For $Y \in \mathcal{C}(\mathscr{A})$ define $\mathscr{A}^Y = \mathscr{A} \cap Y$, the arrangement induced on Y.

$\mathscr{A}$:  $\mathscr{A}^Y = \mathscr{A} \cap Y$: 

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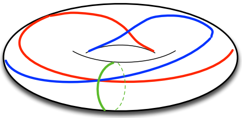
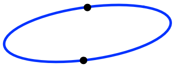


$\mathscr{A}^Y = \mathscr{A} \cap Y$:



For every $Y \in \mathcal{C}(\mathscr{A})$ there is a subcategory $\Sigma_Y \hookrightarrow \mathrm{Sal}(\mathscr{A})$ with

$$||\Sigma_Y|| \simeq ||\mathcal{F}(\mathscr{A}^Y) \times \mathrm{Sal}(\mathscr{A}[Y])|| \simeq Y \times M(\mathscr{A}[Y])$$

and we call ${}_Y E_*^{p,q}$ the Leray spectral sequence induced by the canonical projection

$$\pi_Y : \Sigma_Y \to \mathcal{F}(\mathscr{A}^Y).$$

Spectral sequences

For every $Y \in \mathcal{C}(\mathscr{A})$, the following commutative square

$$\begin{tikzcd} M(\mathscr{A}) \simeq \|\mathrm{Sal}(\mathscr{A})\| \arrow[d, "\pi"] & \|\Sigma_Y\| \arrow[l, "\supseteq"'] \arrow[d, "\pi_Y"] \\ \|\mathcal{F}(\mathscr{A})\| & \|\mathcal{F}(\mathscr{A}^Y)\| \arrow[l, "\supseteq"'] \end{tikzcd}$$

induces a morphism of spectral sequences ${}_{\mathscr{D}}E_*^{p,q} \to {}_Y E_*^{p,q}$.

Next, we examine the morphism of spectral sequences associated to the corresponding map from $\biguplus_{Y \in \mathcal{C}(\mathscr{A})} \|\Sigma_Y\|$ to $\|\mathrm{Sal}(\mathscr{A})\|$.

Spectral sequences

[Callegaro – D., '15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{tikzcd}
H^*(M(\mathscr{A})) \arrow[r] \arrow[d] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \arrow[d] \\
{}_{\mathscr{D}}E_2^{p,q} = \displaystyle\bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y=q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \arrow[r] & \displaystyle\bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y]))
\end{tikzcd}$$

Spectral sequences

[Callegaro – D., '15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{array}{ccc}
H^*(M(\mathscr{A})) & \longrightarrow & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \\
\Big\downarrow & & \Big\downarrow \\
{}_{\mathscr{D}}E_2^{p,q} = & & \displaystyle\bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \\
\displaystyle\bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y=q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) & \longrightarrow & \displaystyle\bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y]))
\end{array}$$

$$\text{On } Y_0\text{-summand:}\quad \omega\otimes\lambda \longmapsto \begin{pmatrix} i^*(\omega)\otimes b(\lambda) & \text{if } Y_0\leq Y \\ 0 & \text{else.}\end{pmatrix}_Y$$

Our solution

Spectral sequences

[Callegaro – D., '15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{tikzcd}
H^*(M(\mathscr{A})) \arrow[r] \arrow[d] & \bigoplus_{Y \in \mathcal{C}(\mathscr{A})} H^*(Y) \otimes H^*(M(\mathscr{A}[Y])) \arrow[d] \\
\begin{array}{c} {}_{\mathscr{D}}E_2^{p,q} = \\ \bigoplus\limits_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \operatorname{rk} Y = q}} H^p(Y) \otimes H^q(M(\mathscr{A}[Y])) \end{array} \arrow[r] & \begin{array}{c} \bigoplus\limits_{Y \in \mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \\ \bigoplus\limits_{Y \in \mathcal{C}(\mathscr{A})} H^p(Y) \otimes H^q(M(\mathscr{A}[Y])) \end{array}
\end{tikzcd}$$

On Y_0-summand: $\quad \omega \otimes \lambda \longmapsto \begin{pmatrix} i^*(\omega) \otimes b(\lambda) & \text{if } Y_0 \leq Y \\ 0 & \text{else.} \end{pmatrix}_Y$

$i : Y \hookrightarrow Y_0$

"Brieskorn" inclusion

Spectral sequences

[Callegaro – D., '15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$
\begin{tikzcd}
H^*(M(\mathscr{A})) \arrow[r, "\text{Hom. of rings}"] \arrow[d, "\text{bij.}"] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \arrow[d, "\text{bij.}"] \\
{}_{\mathscr{D}}E_2^{p,q} = \bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y=q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \arrow[r, "\text{Hom. of rings}"] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_{Y}E_2^{p,q} = \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y]))
\end{tikzcd}
$$

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"Brieskorn" inclusion

Spectral sequences

[Callegaro – D., '15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{tikzcd} H^*(M(\mathscr{A})) \arrow[r, "\text{Hom. of rings}", "\text{Injective}"'] \arrow[d, "\text{bij.}"] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \arrow[d, "\text{bij.}"] \\ {}_{\mathscr{D}}E_2^{p,q} = \bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A}) \\ \operatorname{rk} Y = q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \arrow[r, "\text{Hom. of rings}"] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_{Y}E_2^{p,q} = \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \end{tikzcd}$$

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"Brieskorn" inclusion

A presentation for $H^*(M(\mathscr{A}),\mathbb{Z})$

The inclusions $\phi_\bullet : \Sigma_\bullet \hookrightarrow \mathrm{Sal}(\mathscr{A})$ give rise to a commutative triangle

$$\begin{tikzcd}
\displaystyle\bigoplus_{\substack{Y'\in\mathcal{C},Y'\supseteq Y\\ \mathrm{rk}\,Y'=q}} H^*(Y')\otimes H^q(M(\mathscr{A}[Y'])) \arrow[d, "\sum f_{Y\supseteq Y'}"] & & H^*(\|\,\mathrm{Sal}(\mathscr{A})\|) \arrow[ll, "\oplus\phi^*_{Y'}"'] \arrow[dll, "\phi^*_Y"] \\
H^*(Y)\otimes H^q(M(\mathscr{A}[Y])) & &
\end{tikzcd}$$

with $f_{Y\supseteq Y'} := \iota^*\otimes b_{Y'}$ obtained from $\iota : Y \hookrightarrow Y'$ and the Brieskorn map b.

Proof. Carrier lemma and ‘combinatorial Brieskorn’.

A presentation for $H^*(M(\mathscr{A}),\mathbb{Z})$

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with $f_{Y\supseteq Y'} := \iota^*\otimes b_{Y'}$ obtained from $\iota : Y \hookrightarrow Y'$ and the Brieskorn map b.

Proof. Carrier lemma and 'combinatorial Brieskorn'.

This defines a 'compatibility condition' on $\oplus_Y H^*(Y)\otimes H^*(M(\mathscr{A}[Y]))$;

the (subalgebra of) compatible elements is isomorphic to $H^*(M(\mathscr{A}),\mathbb{Z})$.

A presentation for $H^*(M(\mathscr{A}), \mathbb{Z})$

More succinctly, define an 'abstract' algebra as the direct sum

$$\bigoplus_{Y \in \mathcal{C}(\mathscr{A})} H^*(Y, \mathbb{Z}) \otimes H^{\operatorname{codim} Y}(M(\mathscr{A}[Y]), \mathbb{Z})$$

with multiplication of α, α' in the Y, resp. Y' component, as

$$(\alpha * \alpha')_{Y''} := \begin{cases} f_{Y \supseteq Y''}(\alpha) \smile f_{Y' \supseteq Y''}(\alpha') & \text{if } Y \cap Y' \supseteq Y'' \text{ and} \\ & \operatorname{rk} Y'' = \operatorname{rk} Y + \operatorname{rk} Y', \\ 0 & \text{else.} \end{cases}$$

Question: is this completely determined by $\mathcal{C}(\mathscr{A})$?

Partial answer: yes, if "$\mathscr{A}$ has a unimodular basis".

Some references

Combinatorial algebraic topology:

- D. Kozlov, *Combinatorial Algebraic Topology*, Springer 2010.

Toric arrangements:

- d'Antonio, D., *A Salvetti complex for toric arrangements and its fundamental group*, IMRN 2011
- d'Antonio, D., *Minimality of toric arrangements*, Journal of the E.M.S., 2015
- Callegaro, D., *The integer cohomology algebra of toric arrangements*. ArXiv e-prints 2015.