Monodromy of projective hypersurfaces

Maria Gioia Cifani
j.w.w. A.Cuzzucoli and R.Moschetti

Università degli Studi di Pavia

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We will work over $\mathbb{C}$.
Let $f : X \to Y$ a finite morphism between irreducible varieties of the same dimension; to $f$ we can associate the Galois group

$$G = \text{Gal}(K/K(Y))$$

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In particular our maps will be the following:
Let $X$ be an irreducible, reduced, projective hypersurface of dimension $n$ and degree $d$.
Take $p \notin X$ a point and let $\pi_p$ be the restriction to $X$ of the linear projection from $p$:
\[ \pi_p : X \subset \mathbb{P}^{n+1} \to \mathbb{P}^n \]
that is a finite map of degree $d$.
Over the open $U = \mathbb{P}^n \setminus B$, where $B$ is the branch of $\pi_p$, the map is unramified.
The monodromy group of linear projections

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Over the open $U = \mathbb{P}^n \setminus B$, where $B$ is the branch of $\pi_p$, the map is unramified.
In the preimage of a general point \( y \in U \) there are \( d \) distinct points \( \Gamma := \{ q_1, \ldots, q_d \} \) corresponding to the intersection of the line \( \langle p, y \rangle \) with \( X \).

We can associate permutations of the general fiber \( \Gamma \) to loops in \( U \) centered in \( y \): e.g. if we fix \( \tilde{\gamma}(0) \in \Gamma \) where \( \tilde{\gamma} \) is the lift of a loop \( \gamma \) centered in \( y \), we can define a permutation inside \( \Gamma \) sending

\[
\tilde{\gamma}(0) \mapsto \tilde{\gamma}(1) \in \Gamma
\]
Hence we can define a map

\[ \mu : \pi_1(U, y) \rightarrow S_d \]

**Definition**
The monodromy group of \( \pi_p \) is \( M(\pi_p) := \mu(\pi_1(U)) \leq S_d \).

**Proposition**
The monodromy group \( M(\pi_p) \) is isomorphic to the Galois group \( G \).
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Properties

- $M(\pi_p)$ is a transitive subgroup of $S_d$
- A map $\pi_p$ is said *decomposable* if it admits a non trivial factorization, i.e.
  $$X \xrightarrow{\alpha} X' \xrightarrow{\beta} \mathbb{P}^n$$
  with $\deg(\alpha), \deg(\beta) > 1$.
  If $M(\pi_p) = S_d$ then $\pi_p$ is indecomposable.
  Being indecomposable is equivalent to say that $M(\pi_p)$ is primitive, i.e. it does not preserve non trivial blocks.
- Conversely, if $\pi_p$ is indecomposable and $M(\pi_p)$ contains a trasposition, then it is the whole $S_d$. 
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Dobbiaco 6 / 16
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- Conversely, if \( \pi_p \) is indecomposable and \( M(\pi_p) \) contains a trasposition, then it is the whole \( S_d \).
Definition

We will say that a point $p \notin X$ is uniform if $M(\pi_p) = S_d$; non uniform otherwise.

We are interested in computing the dimension of the locus inside $\mathbb{P}^{n+1}$ of the non uniform points. Indeed:

Proposition

If $p$ is general then it is uniform.
Non uniform projections

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**Proposition**

*If $p$ is general then it is uniform.*
Main Theorem

**Theorem (CCM)**

Let $X$ be an irreducible, reduced, smooth, non developable, projective hypersurface of dimension $n$ and degree $d$; let $p \in \mathbb{P}^{n+1} \setminus X$ be a point and $\pi_p : X \to \mathbb{P}^n$ be the projection from $p$.

Then $X$ admits at most a finite number of non uniform points.

$X$ is non developable if its Gauss map has maximal rank, i.e. the dual of $X$ is an hypersurface.

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The smoothness hypothesis can be relaxed.
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Remark

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Known results

- Pirola and Schlesinger showed the result for plane curves. Moreover, they allow $p \in X \subset \mathbb{P}^2$.
  ('Monodromy of projective curves'. J. Algebraic Geom, 2005)

- Cuzzuoli, Moschetti and Serizawa proved the result for smooth surfaces in $\mathbb{P}^3$.
  ('Non-uniform projections of surfaces in $\mathbb{P}^3$', Le matematiche, 2017)
Recall that if $M(\pi_p)$ is primitive and contains a trasposition then $M(\pi_p) = S_d$.

We first study the existence of traspositions.

**Proposition**

In the above setting, take a point $y \in \mathbb{P}^n$ such that $\pi_p^{-1}(y)$ is made by $d - 1$ distinct points (i.e. $y$ is a simple branch point). Then there is a trasposition in $M(\pi_p)$.
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Simple branch points correspond to simple tangent lines to the hypersurface $X$; then we study families of lines tangent to $X$ by means of the classical theory of focal loci due to Segre.

**Lemma**

*There are at most a finite number of points $p$ such that there are no transposition inside $M(\pi_p)$.*

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*The same holds also for singular hypersurfaces.*
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**Lemma**

*There are at most a finite number of points $p$ such that there are no transposition inside $M(\pi_p)$.***

**Remark**

*The same holds also for singular hypersurfaces.*
Let $W$ be the closure of the locus of non uniform points in $\mathbb{P}^{n+1}$. From an induction argument we know that $\dim(W) \leq 1$. Assume $\dim(W) = 1$.

The general point of $W$ contains a trasposition in its monodromy group, hence to be non uniform it must be non primitive. To conclude we have to show that the general point of $W$ has primitive monodromy group.
take a general point $p \in \mathbb{P}^{n+1}$ and project from it. Let $B$ be the branch locus of the projection $\pi_p$ and $C := \pi_p(W)$ be the image of the curve $W$. 

The picture is the following:
We have a map

\[ \pi_1(C \setminus B, t) \rightarrow \pi_1(\mathbb{P}^n \setminus B, t) \]

\[ \mu \]

\[ M(\pi_p) = S_d \]

If \( C \) and \( B \) intersect transversally everywhere than we have

**Nori's Lemma**

\[ \pi_1(C \setminus B) \rightarrow \pi_1(\mathbb{P}^n \setminus B) \]
Proof of the main theorem

If we are in the previous situation, we are able to study the monodromy of $\pi_p$ just looking at

$$\mu : \pi_1(C \setminus B) \to S_d$$

Consider a general fiber over $C$ of our projection from $p$; the line will meet $W$ in a point $y$.

Note that the $d$ points $x_1 \ldots, x_d$ in which the line meets $X$ are the same if we project from $p$ or from $y$. 
Cukiermann showed that for $X \subset \mathbb{P}^2$ general curve, $W = \emptyset$. Our first aim is to generalize this result for general hypersurfaces in every dimension.

Estimate the dimension of the locus of non uniform centers of projections for higher codimension varieties.
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