On the De Concini-Procesi models for reflection groups

Giovanni Gaiffi

University of Pisa

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Let V be a finite dimensional vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$).
Let us consider a finite subspace arrangement $\mathcal{G}$ in V^* and, for every $A \in \mathcal{G}$, let us denote by $A^\perp$ its annihilator in V.
Let $M(\mathcal{G}) := V - \bigcup_{A\in\mathcal{G}} A^\perp$ and consider the embedding

$$\phi_{\mathcal{G}} : M(\mathcal{G}) \longrightarrow V \times \prod_{A\in\mathcal{G}} \mathbb{P}\left(V/A^\perp\right).$$

Definition (De Concini-Procesi 1995)

The model $Y_{\mathcal{G}}$ associated to $\mathcal{G}$ is the closure of $\phi_{\mathcal{G}}\,(M(\mathcal{G}))$ in $V \times \prod_{A\in\mathcal{G}} \mathbb{P}\left(V/A^\perp\right)$.

If $\mathcal{G}$ is a building set, these *wonderful models* turn out to be smooth varieties and the complement of $M(\mathcal{G})$ in $Y_{\mathcal{G}}$ is a divisor with normal crossings, described in terms of $\mathcal{G}$-nested sets. These models can be obtained by a series of blow-ups and can be related to other constructions of models of stratified varieties (Fulton-MacPherson 1994, MacPherson-Procesi 1998, Ulyanov 2002, Hu 2003, Li 2009 etc...).

There is also a compact construction

$$\phi_{\mathcal{G}} : \mathbb{P}(M(\mathcal{G})) \longrightarrow \mathbb{P}(V) \times \prod_{A \in \mathcal{G}} \mathbb{P}\left(V/A^{\perp}\right)$$

that gives compact models $\overline{Y}_{\mathcal{G}}$.

Finally, when $\mathbb{K} = \mathbb{R}$, there is a spherical construction (G. 2003):

$$\phi : \mathcal{M}(\mathcal{G}) \cap S(V) \longrightarrow S(V) \times \prod_{A \in \mathcal{G}} S(A)$$

We denote by $CY_{\mathcal{G}}$ the closure of the image of ϕ. If $\mathcal{G}$ is building this is a smooth manifold with corners with a 'nice' boundary described by nested sets.

Building sets

If $\mathcal{A}$ is a set of subspaces of V^* we denote by $\mathcal{C}_{\mathcal{A}}$ its closure under the sum.

Definition

A collection $\mathcal{G}$ of subspaces of V^* is called **building** if every element $C \in \mathcal{C}_{\mathcal{G}}$ is the direct sum $G_1 \oplus \cdots \oplus G_k$ of the set of maximal elements $G_1, \cdots, G_k$ of $\mathcal{G}$ contained in C.

For instance let $V = \mathbb{K}^2$. An arrangement made by three distinct lines is described by $\mathcal{G} = \{A_1, A_2, A_3\}$ where $A_i \subset V^*$. This is NOT a building set: $A_1 + A_2 + A_3$ is equal to V^*, the A_i are maximal, but their sum is not direct.

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In general there are several building sets associated to $\mathcal{A}$. In the collection of such sets there is a minimal element, denoted by $\mathcal{F}_{\mathcal{A}}$, and a maximal element, which is $\mathcal{C}_{\mathcal{A}}$.

There are natural projection maps among the associated De Concini-Procesi models: if the building sets satisfy $\mathcal{B}_1 \subset \mathcal{B}_2$ then there is a projection of $Y_{\mathcal{B}_2}$ onto $Y_{\mathcal{B}_1}$.

$$\begin{tikzcd} Y_{\mathcal{B}_2} \arrow[d] & \subset & V \times \prod_{A \in \mathcal{B}_2} \mathbb{P}\left(V/A^{\perp}\right) \arrow[d] \\ Y_{\mathcal{B}_1} & \subset & V \times \prod_{A \in \mathcal{B}_1} \mathbb{P}\left(V/A^{\perp}\right) \end{tikzcd}$$

The example of root arrangements

Let us consider a root system Φ in a euclidean vector space V with finite Coxeter group W, and a basis of *simple roots* $\Delta = \{\alpha_1, ..., \alpha_n\}$ for Φ. Let $\mathcal{A}_\Phi$ be the corresponding root hyperplane arrangement.

Then

- $\mathcal{C}_{\mathcal{A}_\Phi} = \mathcal{C}_\Phi$ is the building set of all the subspaces that can be generated as the span of some of the roots in Φ.
- $\mathcal{F}_{\mathcal{A}_\Phi} = \mathcal{F}_\Phi$ is the building set made by all the subspaces which are spanned by the irreducible root subsystems of Φ.

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If $\Phi = A_{n-1}$ (the braid arrangement in $V = \mathbb{R}^n/\mathbb{R}(1, 1, ..., 1)$), the roots are $x_i - x_j$.
A subspace A in V^* belongs to $\mathcal{F}_{A_{n-1}}$ if and only if $A^\perp$ is of type $A^\perp = \{\mathbf{x} \in V \,|\, x_{i_1} = x_{i_2} = \cdots = x_{i_r}\}$.

Therefore we can represent subspaces in $\mathcal{F}_{A_{n-1}}$ by subsets of $\{1, 2, ..., n\}$ of cardinality ≥ 2. For instance:

$$A^\perp = \{\mathbf{x} \in V \,|\, x_2 = x_4 = x_5\} \longleftrightarrow \{2, 4, 5\}$$

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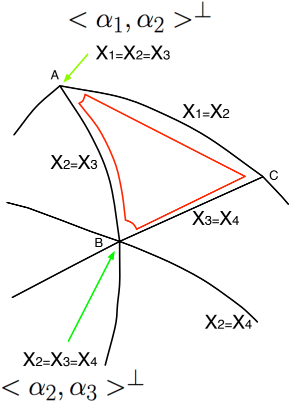
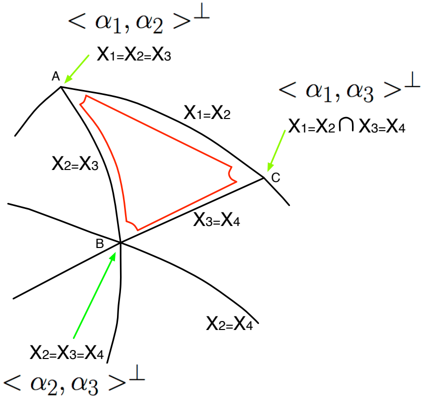


Case A_3, building set of irreducibles $\mathcal{F}_{A_3}$.



Case A_3, maximal building set $\mathcal{C}_{A_3}$.

Nested sets

Definition

Let $\mathcal{G}$ be a building set of subspaces of V^*. A subset $\mathcal{S} \subset \mathcal{G}$ is called $\mathcal{G}$**-nested** if and only if for every subset $\{A_1, \cdots, A_k\} \subset \mathcal{S}$ ($k \geq 2$) of pairwise non comparable elements of $\mathcal{S}$ the subspace $A = A_1 + \cdots + A_k$ does not belong to $\mathcal{G}$.

For instance in the case of the building set $\mathcal{F}_{A_{n-1}}$ this means that the subsets of $\{1, 2, ..., n\}$ that represent the elements of $\mathcal{S}$ are pairwise disjoint or one included into the other.

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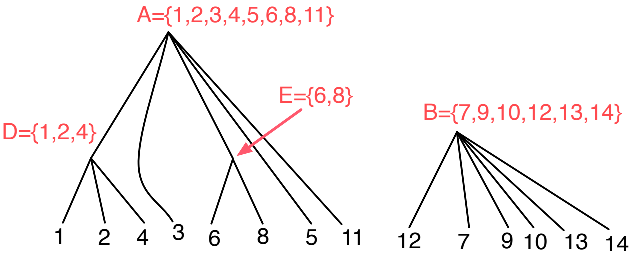
Example of a nested set for the minimal model, case A_{13}.



Nested set: {A,B,D,E}

In $Y_{\mathcal{G}}$ one "adds" to the complement $M(\mathcal{G})$ of the subspace arrangement the union $\mathcal{D}$ of smooth irreducible divisors D_A indexed by the elements $A \in \mathcal{G}$.
Given some divisors $D_{A_1}, \ldots, D_{A_n}$ ($A_j \in \mathcal{G}$), their intersection is non empty if and only if $\mathcal{S} = \{A_1, \ldots, A_n\}$ is $\mathcal{G}$-nested. In this case their intersection is transversal and gives rise to a smooth irreducible variety $\mathcal{D}_{\mathcal{S}} = \bigcap_i D_{A_i}$.

Information on Cohomology

- a presentation for the integer cohomology ring of complex models of subspace arrangements $Y_{\mathcal{G}}$ was provided by De Concini and Procesi (1995), then a basis was given by Yuzvinski (1997), G. (1997).
- cohomology of real models of subspace arrangements was computed by Rains (2010) (in the braid case by Etingof, Henriques, Kamnitzer, Rains 2010)
- in the case of complex reflection groups $G(r, 1, n)$ a formula for the character of action on the cohomology of the minimal models was computed by Henderson (2004).

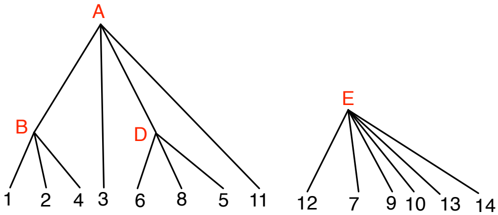
Bases of the cohomology of complex models

The integer cohomology rings of complex De Concini-Procesi models are torsion free. They can be presented as

$$\frac{\mathbb{Z}[c_A]_{A\in\mathcal{G}}}{I}$$

where c_A is the Chern class of the divisor D_A.

We can explicitly describe $\mathbb{Z}$-bases of the cohomology made by monomials. Example: how to obtain a monomial of the basis of $H^*(Y_{\mathcal{F}_{A_{13}}}, \mathbb{Z})$. Start with a nested set $\{A, B, D, E\}$.



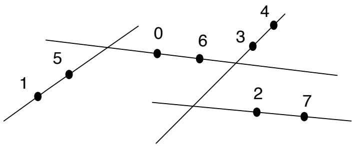
Basis monomial:

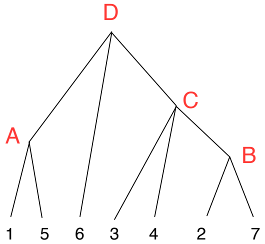
$$c_A^{1 \text{ or } 2} \; c_B^{1} \; c_D^{1} \; c_E^{1 \text{ or } 2 \text{ or } 3 \text{ or } 4}$$

The geometric extended action on the compact model $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$

There is a well know S_{n+1} action on the De Concini-Procesi minimal compact model $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$: it comes from the isomorphism with the moduli space $\overline{M}_{0,n+1}$.

Example of the correspondence between two representations of the boundary of $\overline{M}_{0,8} = \overline{Y}_{\mathcal{F}_{A_6}}$:





Nested set:

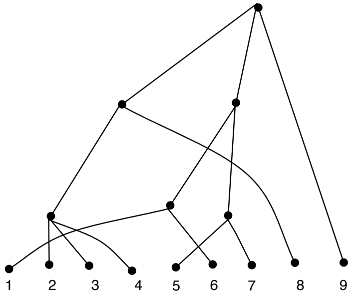
A={1,5} B={2,7} C={3,4} D={1,2,3,4,5,6,7}

The combinatorial S_{n+k} action on the strata of $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$

Let us denote by F^k the set of k-codimensional irreducible strata of $\overline{Y}_{\mathcal{F}_{A_{n-1}}} \cong \overline{M}_{0,n+1}$. These are indexed by nested sets with $k+1$ elements (including V^*).
There is a S_{n+k} action on F^k.
This comes from an explicit bijection between F^k and the set of unordered partitions of $\{1, 2, ..., n+k\}$ into $k+1$ parts of cardinality greater than or equal to 2.

For instance, when $n = 9, k = 5$:

$$S = \{ \{2,3,4\}, \{1,6\}, \{5,7\}, \{2,3,4,8\}, \{1,5,6,7\}, \{1,2,3,4,5,6,7,8,9\} \}$$



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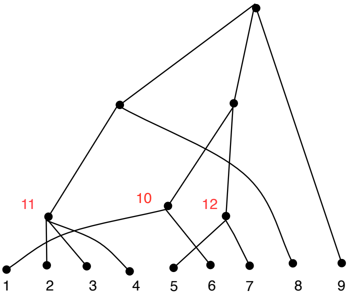

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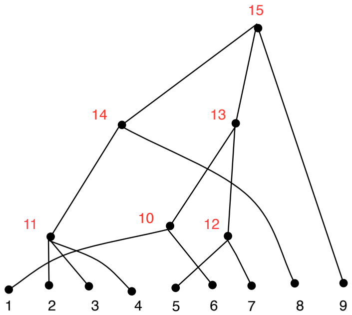

{1,6} {2,3,4} {5,7} {8,11} {10,12} {9,13,14}

This S_{n+k} action on F^k is not geometric, i.e. it is not compatible with the natural S_n action on $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ (neither with the extended S_{n+1}).
Nevertheless it induces an action that permutes the monomials of the basis of the integer cohomology.
The restriction to S_n of the resulting representation on the cohomology module is not isomorphic to the natural S_n representation.

Let $n = 7$ and let us consider the monomial $c_{A_1}^2 c_{A_2} c_{A_3}$ in the basis of $H^8(\overline{Y}_{\mathcal{F}_{A_7}}, \mathbb{Z})$, where

$$A_1 = \{1,2,3,4,5,6,7,8\}, A_2 = \{1,2,3\}, A_3 = \{4,6,7\}.$$

We associate to the nested set $\{A_1, A_2, A_3\}$ the following partition of the set $\{1,2,..,10\}$:

$$\{1,2,3\}\{4,6,7\}\{5,8,9,10\}$$

Finally we associate to $c_{A_1}^2 c_{A_2} c_{A_3}$ the following *labelled partition* of $\{1,2,..,10\}$:

$$\{1,2,3\}^1\{4,6,7\}^1\{5,8,9,10\}^2$$

Let us denote by $\Psi(q,t,z)$ the following exponential generating series:

$$\Psi(q,t,z) = 1 + \sum_{\substack{n \geq 2, \\ \mathcal{S} \text{ nested set of } \mathcal{F}_{A_{n-1}}}} P(S) z^{|\mathcal{S}|} \frac{t^{n+|\mathcal{S}|-1}}{(n+|\mathcal{S}|-1)!}$$

where, for every $n \geq 2$,

- $\mathcal{S}$ ranges over all the nested sets of the building set $\mathcal{F}_{A_{n-1}}$;
- $P(\mathcal{S})$ is the polynomial, in the variable q, that expresses the contribution to the Poincaré polynomial of $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ provided by all the monomials m_f in the basis whose support is $\mathcal{S}$.

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We observe that the series $\Psi(q,t,z)$ encodes the same information that is encoded by the Poincaré series. In particular, for a fixed n, the Poincaré polynomial of the model $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ can be read from the coefficients of the monomials whose z,t component is $t^k z^s$ with $k-s=n-1$.

Proposition (Callegaro-G. 2014)

We have the following formula for the series $\Psi(q,t,z)$:

$$\Psi(q,t,z) = e^t \prod_{i\geq 3} e^{zq[i-2]_q \frac{t^i}{i!}}$$

where $[j]_q$ denotes the q-analog of j: $[j]_q = 1+q+\cdots+q^{j-1}$.

Example

If one wants to compute the Poincaré polynomial of $\overline{Y}_{\mathcal{F}_{A_4}}$ one has to single out all the monomials in Ψ whose z, t component is $t^k z^s$ with $k - s = 4$. A product of the exponential functions that appear in the formula gives:

$$\frac{t^4}{4!}[1] + \frac{t^5}{5!}z[16q + 6q^2 + q^3] + \frac{t^6}{6!}z^2[10q^2]$$

Therefore the Poincaré polynomial of $\overline{Y}_{\mathcal{F}_{A_4}} \cong \overline{M}_{0,6}$ is $1 + 16q + 16q^2 + q^3$.

This extends to the case of complex reflection groups $G(r,r,n)$:

Corollary

We have the following formula for the series $\Psi(q,t,z)$ of the models $Y_{G(r,r,n)}$:

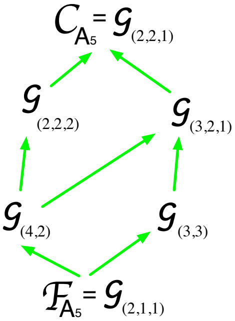
$$\Psi(q,t,z) = e^{tr} \prod_{i\geq 3} e^{\frac{z}{r} q[i-2]_q \frac{(rt)^i}{i!}}$$

where $[j]_q$ denotes the q-analog of j: $[j]_q = 1 + q + \cdots + q^{j-1}$.

A problem of lacking symmetry

Let us consider again the S_{n+1} action on the De Concini-Procesi compact model $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ induced from the isomorphism with $\overline{M}_{0,n+1}$.
This action does not extend to the other models (non minimal) of type A_{n-1}, in particular, it does not extend to the maximal model. Why ?

A picture of the S_6-invariant building sets associated to the arrangement of type A_5.



Let us denote by $\mathcal{B}(n-1)$ the set of strata of $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$. It is indexed by the nested sets of $\mathcal{F}_{A_{n-1}}$ that contain V^*.
We can construct the model $\overline{Y}_{\mathcal{B}(n-1)}$ starting from $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ and blowing up all the strata.

We notice that $\mathcal{B}(n-1) \cup \emptyset$ is a simplicial complex. There is a combinatorial notion of *building set* of a simplicial complex, due to Feichtner and Kozlov (2004).
Let us consider the family $\mathcal{T}_{n-1}$ of all the combinatorial building subsets of $\mathcal{B}(n-1)$. The maximum element in $\mathcal{T}_{n-1}$ is $\mathcal{B}(n-1)$ itself.
For every $K \in \mathcal{T}_{n-1}$ we can construct the model $\overline{Y}_K$ starting from $\overline{Y}_{\mathcal{F}_{A_{n-1}}}$ and blowing up all the strata that appear in K (see for instance MacPherson-Procesi (1998) or Li (2009)).

It turns out that the maximal De Concini-Procesi models is "too small" to admit the S_{n+1} action:

Theorem (Callegaro-G. 2014)

The model $\overline{Y}_{\mathcal{B}(n-1)}$ is the only one model in $\{\overline{Y}_K \mid K \in \mathcal{T}_{n-1}\}$ that admits the extended S_{n+1} action and also admits a birational projection onto the maximal De Concini-Procesi model $\overline{Y}_{\mathcal{C}_{A_{n-1}}}$.

Theorem (Callegaro-G. 2014)

A basis of the integer cohomology of the complex model $\overline{Y}_{\mathcal{B}(n-1)}$:

$$\eta \, c_{\mathcal{S}_1}^{\delta_1} c_{\mathcal{S}_2}^{\delta_2} \cdots c_{\mathcal{S}_k}^{\delta_k}$$

1. $\mathcal{S}_1 \subsetneq \mathcal{S}_2 \subsetneq \cdots \subsetneq \mathcal{S}_k$ *is a chain of* $\mathcal{F}_{A_{n-1}}$*-nested sets (possibly empty, i.e.* $k = 0$*);*
2. *the element* $c_{\mathcal{S}_i}$ *is the Chern class of the normal bundle of* $L_{\mathcal{S}_i}$ *(the proper transform of* $D_{\mathcal{S}_i}$*) in* $\overline{Y}_{\mathcal{B}(n-1)}$*;*
3. *the exponents* δ_i *satisfy the inequalities:* $1 \leq \delta_i \leq |\mathcal{S}_i| - |\mathcal{S}_{i-1}| - 1$*;*
4. η *is a monomial in a basis of* $H^*(D_{\mathcal{S}_1})$ *(if* $k \geq 1$*) or to* $H^*(\overline{Y}_{\mathcal{F}})$ *(if* $k = 0$*).*

The inertia around divisors

Let W be an irreducible complex reflection group. Let $\mathcal{A}_W$ be the corresponding arrangement, in the complex vector space V. and let $M(\mathcal{A}_W)$ be its complement. We write P_W for the fundamental group $\pi_1(M(\mathcal{A}_W))$, which is the pure braid group of W.
We call $Y_{\mathcal{F}_W}$ the minimal (not compact) wonderful model associated with the arrangement $\mathcal{A}_W$.

The minimal building set is made by the subspaces $A \in \mathcal{C}_{\mathcal{A}_W}$ such that the parabolic subgroup

$$W_A := \{w \in W \mid w \text{ fixes } A^{\perp} \text{ pointwise}\}.$$

is irreducible.

Definition

We denote by $j_{D_A} \in P_W$ the inertia around the divisor D_A. This is the homotopy class of a counterclockwise loop in the big open part $Y_{\mathcal{F}_W} \setminus (\bigcup_{B \in \mathcal{F}_W} D_B)$ around D_A, that is identified with a loop in $M(\mathcal{A}_W)$.

Proposition

The inertia j_{D_V} generates the center of P_W. When $A \neq V$, the inertia j_{D_A} around the divisor D_A in $Y_{\mathcal{F}_W}$ is a generator of the center of the corresponding parabolic subgroup P_{W_A} of P_W.

The model $Y_{\mathcal{F}_W}$ can be constructed by a suitable series of blowups of strata of non-decreasing dimension.
In particular, let Y_0 be the first step in this blowup process, that is the blowup of the space V in the origin O and let $D_V^0 \subset Y_0$ be the corresponding exceptional divisor.

We identify the complement in Y_0 of the proper transforms of the hyperplanes in $\mathcal{A}_W$ and of D_V^0 with the space $M(\mathcal{A}_W)$:

$$M(\mathcal{A}_W) \simeq Y_0 \setminus \left(D_V^0 \cup \bigcup_{H \in \mathcal{A}_W} \widetilde{H} \right).$$

Definition

We denote by $j_{D_V^0} \in P_W$ (or simply j) the inertia around the divisor D_V^0. This is represented by a counterclockwise loop in $Y_0 \setminus (\bigcup_{H \in \mathcal{A}_W} \widetilde{H})$ around D_V^0, that is identified with a loop in $M(\mathcal{A}_W)$.

We notice that j is also the inertia around the divisor D_V in $Y_{\mathcal{F}_W}$: in fact we are looking at the homotopy class of a loop in the big open part of Y_0, which is identified with the big open part of $Y_{\mathcal{F}_W}$ (and both are identified with $M(\mathcal{A}_W)$).

Since Y_0 is the closure of the image of the map

$$V \setminus \{O\} \to V \times \mathbb{P}(V)$$

there is a well defined projection π of Y_0 onto $\mathbb{P}(V)$ and hence we can consider the fibration

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & Y_0 \\ & & \big\downarrow \pi \\ & & \mathbb{P}(V) \end{array}$$

where the 0-section is the divisor D_V^0. This fibration is the normal bundle of D_V^0 in Y_0 and hence we can choose a representative of j as a loop avoiding 0 in the fiber of a generic point.

We now consider the restriction of the previous fibration to $\mathbb{P}(M(\mathcal{A}_W))$:

$$\begin{tikzcd} \mathbb{C} \arrow[r] & Y_0 \arrow[d, "\pi"] \\ & \mathbb{P}(V) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & Y_0 \setminus (\bigcup_{H\in\mathcal{A}_W} \widetilde{H}) \arrow[d, "\pi"] \\ & \mathbb{P}(M(\mathcal{A}_W)). \end{tikzcd}$$

We can fix an hyperplane $H_0 \in \mathcal{A}_W$. We can identify the fiber over any point with a line in V and a translation of H_0 in V will intersect in a point the line that is identified with the fiber over any $[v] \in \mathbb{P}(V) \setminus \mathbb{P}(H_0)$. Hence there exists a non-zero section and we have the trivial sub-fibration

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] & M(\mathcal{A}_W) \arrow[d, "\pi"] \\ & \mathbb{P}(M(\mathcal{A}_W)). \end{tikzcd}$$

We can factor $M(\mathcal{A}_W)$ as a product $\mathbb{C}^* \times \mathbb{P}(M(\mathcal{A}_W))$ where the inertia j is represented by a loop along the factor $\mathbb{C}^*$.