Computing toric degenerations of flag varieties

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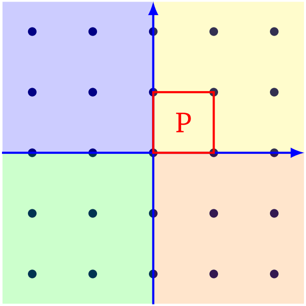
(arXiv 1702.05505)

Compute Gröbner toric degenerations of $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$

Compare them with the degenerations obtained using representation theory techniques (Littelman(1998),Berenstein-Zelevinsky(2001), Caldero(2002),Alexeev-Brion (2005)).

Why toric degenerations ?

Toric varieties give a powerful dictionary which translates combinatorial properties to algebraic and geometric properties.



$$\longrightarrow \quad \begin{array}{c} \mathbb{P}^1 \times \mathbb{P}^1 \\ \Big\downarrow \scriptstyle{P} \\ \mathbb{P}^3 \end{array}$$

Why toric degenerations?

$\Longrightarrow$ Extend this dictionary to a larger class of varieties.

Use a **toric degeneration**, i.e a flat family $\varphi : \mathcal{F} \to \mathbb{A}^1$ for which the fibre over 0 is a toric variety and all the other fibres are isomorphic to the variety $\mathcal{F}\ell_n$.

Why flag varieties?

Let $\mathbb{k}$ be any field.

Definition

The set of all complete flags

$$\mathcal{V}: \ \{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{k}^n$$

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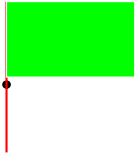

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$\mathcal{F}\ell_n$ can be embedded in $\mathrm{Gr}(1, \mathbb{k}^n) \times \cdots \times \mathrm{Gr}(n-1, \mathbb{k}^n)$.
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$\Longrightarrow$ Flag varieties are a good toy model because of their additional structures.

Plücker embedding

$\mathcal{F}\ell_n := \{\mathcal{V} : \{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \Bbbk^n\}$

$\mathcal{F}\ell_n \subset \mathrm{Gr}(1, \Bbbk^n) \times \cdots \times \mathrm{Gr}(n-1, \Bbbk^n)$

Using Plücker embeddings $\mathcal{F}\ell_n$ becomes a subvariety of $\mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ and it has defining ideal

$$I_n \subset \Bbbk[p_J : \varnothing \neq J \subsetneq \{1, \ldots, n\}].$$

Example: $\mathcal{F}\ell_3$

Let $n = 3$ then

$$\mathcal{F}\ell_3 = \{(\ell, H) \in \mathrm{Gr}(1, \mathbb{k}^3) \times \mathrm{Gr}(2, \mathbb{k}^3) : \ell \subset H\}.$$

It is a subvariety of $\mathrm{Gr}(1, \mathbb{k}^3) \times \mathrm{Gr}(2, \mathbb{k}^3) \cong \mathbb{P}^2 \times \mathbb{P}^2$.

It is defined in $\mathbb{k}[p_1, p_2, p_3, p_{12}, p_{13}, p_{23}]$ by the ideal $I_3 = \langle p_3 p_{12} - p_2 p_{13} + p_1 p_{23} \rangle$.

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Toric degenerations

We are looking for a flat family $\varphi : \mathcal{F} \to \mathbb{A}^1$ for which the fibre over 0 is a toric variety and all the other fibres are isomorphic to the variety $\mathcal{F}\ell_n$.

After the embedding we have $\mathcal{F}\ell_n \subset \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ and $\mathcal{F}\ell_n = V(I_n)$.

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$\Longrightarrow$ Consider *Gröbner degenerations.*

Gröbner toric degenerations

Definition

Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}}$ with $\mathbf{u} \in \mathbb{Z}^n$ be a polynomial in $\Bbbk[x_1, \ldots, x_n]$. For each $\mathbf{w} \in \mathbb{R}^n$ we define its *initial form* to be

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{w}\cdot\mathbf{u} \text{ is minimal}} a_{\mathbf{u}} x^{\mathbf{u}}.$$

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$$= p^{(0,0,1,1,0,0)} - p^{(0,1,0,0,1,0)} + p^{(1,0,0,0,0,1)}$$

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then $\quad \mathrm{in}_{(1,0,0,0,0,0)}(f) = p_3 p_{12} - p_2 p_{13}$

Definition

If I is an ideal in S, then its *initial ideal* with respect to $\mathbf{w}$ is

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I \rangle.$$

There exists a flat family $\varphi : \mathcal{F} \to \mathbb{A}^1$ for which the fibre over 0 is isomorphic to $V(\mathrm{in}_{\mathbf{w}}(I))$ and all the other fibres are isomorphic to the variety $V(I)$. This is called a *Gröbner degeneration* of $V(I)$.

Example: $\mathcal{F}\ell_3$

For $\mathcal{F}\ell_3$ the defining ideal is $I_3 = \langle p_3 p_{12} - p_2 p_{13} + p_1 p_{23} \rangle$. If $\mathbf{w} = (1, 0, 0, 0, 0, 0)$ then

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Find embedded (possibly not normal) toric degenerations of $V(I_n)$.

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Algebraic reformulation

Find toric initial ideals of I_n.

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Algebraic reformulation

Find toric initial ideals of I_n.

Consider the *tropicalization* of X.

Tropicalization

Let $I \subset \Bbbk[x_1, \ldots, x_n]$ and $X = V(I)$.

Definition

The tropicalization $\mathrm{trop}(X)$ of X is defined to be

$$\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) \text{ does not contain monomials}\}$$

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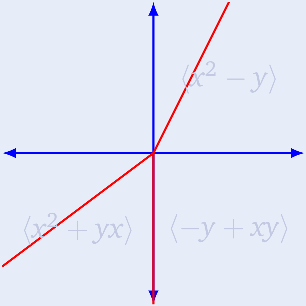
The tropical variety $\mathrm{trop}(X)$ has a fan structure such that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$ for all $\mathbf{w}', \mathbf{w}$ in the relative interior of a cone $C \in \mathrm{trop}(X)$.
Each cone C corresponds to a different initial ideal.

Example

Let X be $V(x^2 - y + yx)$. Then $\mathrm{trop}(X) \subset \mathbb{R}^2$.



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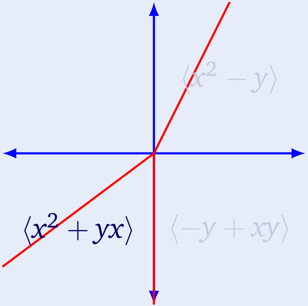

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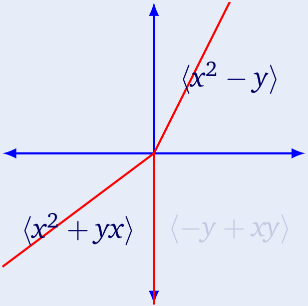

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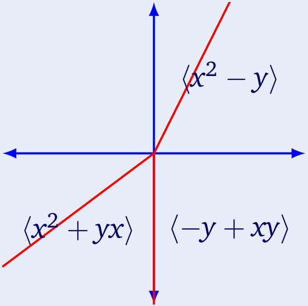

Example: $\mathrm{trop}(\mathcal{F}\ell_3)$

The tropicalization of $\mathcal{F}\ell_3$ has 3 maximal cones. The three toric initial ideals are:

$$\langle p_3p_{12} - p_2p_{13}\rangle$$
$$\langle p_3p_{12} + p_1p_{23}\rangle$$
$$\langle -p_2p_{13} + p_1p_{23}\rangle.$$

The three corresponding toric varieties are all isomorphic.

Compute Gröbner toric degenerations of $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$

Compare them with the degenerations associated to the string polytopes for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$
(Littelman(1998), Berenstein-Zelevinsky (2001),Caldero (2002), Alexeev-Brion (2004))

Results

Theorem (Bossinger,Lamboglia,Mincheva,Mohammadi)

There are 4 non isomorphic Gröbner toric degeneration of the flag variety $\mathcal{F}\ell_4$. Among these 4 there is one not isomorphic to any of the degenerations coming from string polytopes.

A similar result holds for $\mathcal{F}\ell_5$ where we find 180 toric degenerations and 168 are new.

The tropicalization trop($\mathcal{F}\ell_4$) has 78 maximal cones grouped in five $S_4 \rtimes \mathbb{Z}_2$-orbits.

Orbit	Size	Prime	F-vector of associated polytope
1	24	Yes	(42, 141, 202, 153, 63, 13)
2	12	Yes	(40, 132, 186, 139, 57, 12)
3	12	Yes	(42, 141, 202, 153, 63, 13)
4	24	Yes	(43, 146, 212, 163, 68, 14)
5	6	No	

Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	-
	String 4

Results

The tropicalization $\mathrm{trop}(\mathcal{F}\ell_5)$ has 69780 maximal cones grouped in 536 $S_5 \rtimes \mathbb{Z}_2$-orbits.

$\Longrightarrow$ 180 of them give rise to toric initial ideals which define 180 non-isomorphic toric degenerations.

$\Longrightarrow$ 168 of the 180 are not isomorphic to any toric degenerations constructed from representation theory techniques.

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Proposition

For $\mathcal{F}\ell_4$ the procedure gives rise to three new toric degenerations. The polytopes associated to two of them are combinatorially equivalent to the String 4 polytope.

Example

Let $X = V(I) \subset \mathbb{P}^2$ with $I = \langle xz + xy + yz \rangle$. Then the toric variety has three maximal cones and the initial ideals are

$$\langle xy + yz \rangle \qquad \langle xy + xz \rangle \qquad \langle zy + zx \rangle$$

which are all non prime.

Re-embedding procedure

Input: Non prime initial ideal $\mathrm{in}_C(I) = \langle xy + yz \rangle$.

1. Compute the primary decomposition of $\mathrm{in}_C(I)$
   $\Longrightarrow \langle y \rangle \cdot \langle x + z \rangle$;
2. Compute the binomials that generate $\langle x + y \rangle$ but are not in $\mathrm{in}_C(I)$
   $\Longrightarrow x + y$;
3. Let $I' \in \mathbb{C}[x, y, z, u]$ be the ideal $I + \langle u - x - y \rangle$. Then $V(I) \cong V(I')$.
4. Tropicalize $V(I')$ and check if there are toric initial ideals such that $\mathrm{in}_C(I) \subset \mathrm{in}_{C'}(I') \cap \mathbb{C}[x, y, z]$
   $\Longrightarrow \mathrm{in}_{C'}(I') = \langle x + y, y^2 - zu \rangle$.