A tropical approach to a generalized Hodge conjecture for positive currents

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February 20, 2017 - Toblach

Are all positive currents with Hodge classes approximable by positive sums of integration currents? (Demailly 1982)

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No! (Joint work with June Huh)

Currents

X complex smooth manifold of complex dimension n.

- $\mathcal{D}^k(X) :=$ Space of smooth differential forms of degree k, with compact support $=$ test forms
- $\mathcal{D}'_k(X) =$ Space of currents of dimension k $:=$ Topological dual to $\mathcal{D}^k(X)$
- $\langle T, \varphi \rangle \in \mathbb{C}$ (linear continuous action)
- $T \in \mathcal{D}'_k(X)$ current is **closed** ($=$ d-closed), $\langle dT, \varphi \rangle := (-1)^{k+1} \langle T, d\varphi \rangle = 0$, $\forall \varphi \in \mathcal{D}^{k-1}(X)$

- $\mathcal{D}^{p,q}(X)$: Smooth (p,q)-forms with compact support
- $\mathcal{D}'_{p,q}(X) := \left(\mathcal{D}^{p,q}(X)\right)'$
- For currents (p,q)-bidimension $=$ $(n-p, n-q)$-bidegree

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- For currents (p,q)-bidimension $= (n-p, n-q)$-bidegree
- $T_j \to T$ in weak limit, if $\langle T_j, \varphi\rangle \to \langle T, \varphi\rangle \in \mathbb{C}$

Integration currents

Example

Let $Z \subset X$ a smooth submanifold of dimension p, define the *integration current along* Z, denoted by $[Z] \in D'_{p,p}(X)$

$$\langle [Z], \varphi \rangle := \int_Z \varphi, \quad \varphi \in \mathcal{D}^{p,p}(X).$$

This definition extends to analytic subsets Z, by integrating over the smooth locus.

Positivity

Definition

A smooth differential (p, p)-form φ is *positive* if $\varphi(x)|_S$ is a nonnegative volume form for all p-planes $S \subset T_x X$ and $x \in X$.

Definition

A current $T \in \mathcal{D}'_{p,p}(X)$ is called *positive* if

$$\langle T, \varphi \rangle \geq 0$$

for every positive test form $\varphi \in \mathcal{D}_{p,p}(X)$.

Examples of positive currents

- An integration current on an analytic subset is a positive current, with support equal to Z
- Convex sum of positive currents

The generalized Hodge conjecture for positive currents (HC^+)

Question/Conjecture: Are all the positive closed currents approximable by a convex sum of integration currents along analytic cycles?

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On a smooth projective variety X, and

$$\{\mathscr{T}^+\} \in \mathbb{R} \otimes_{\mathbb{Z}} \big(H^{2q}(X,\mathbb{Z})/\mathrm{tors} \ \cap H^{q,q}(X)\big),$$

where $q = n - p$.

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where $q = n - p$.

Demailly, the superhero, 1982: True for $p = 0, n-1, n$.

The Hodge conjecture (HC)

The Hodge conjecture: The group

$$\mathbb{Q} \otimes_{\mathbb{Z}} \big(H^{2q}(X, \mathbb{Z})/\mathrm{tors} \ \cap H^{q,q}(X)\big),$$

consists of classes of p-dimensional algebraic cycles with rational coefficients.

Demailly 1982: $\mathrm{HC}^+ \implies \mathrm{HC}$.

Hodge conjecture for real currents (HC')

If $\mathscr{T}$ is a (p,p)-dimensional real closed current on X with cohomology class

$$\{\mathscr{T}\} \in \mathbb{R} \otimes_{\mathbb{Z}} \big(H^{2q}(X,\mathbb{Z})/\mathrm{tors}\ \cap H^{q,q}(X)\big),$$

then $\mathscr{T}$ is a weak limit of the form

$$\mathscr{T} \underset{i}{\longleftarrow} \sum_j \lambda_{ij}[Z_{ij}],$$

where λ_{ij} are real numbers and Z_{ij} are p-dimensional subvarieties of X.

Demailly 2012: $\mathrm{HC}' \iff \mathrm{HC}$

HC^+ not true in general!

Theorem (B - Huh)

There is a 4-dimensional smooth projective toric variety X and a $(2,2)$-dimensional positive closed current $\mathscr{T}^+$ on X with the following properties:

(1) *The cohomology class of $\mathscr{T}^+$ satisfies*

$$\{\mathscr{T}^+\} \in H^4(X,\mathbb{Z})/tors \cap H^{2,2}(X).$$

(2) *The current $\mathscr{T}^+$ is not a weak limit of the form*

$$\mathscr{T}^+ \underset{i}{\longleftarrow} \sum_j \lambda_{ij}^+ [Z_{ij}],$$

where $\lambda_{ij}^+ > 0$, Z_{ij} are algebraic surfaces in X.

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Extremality in the cone of closed positive currents

Definition

A (p,p)-closed positive current T is called extremal if for any decomposition $T = T_1 + T_2$, there exist $\lambda_1, \lambda_2 \geq 0$ such that $T = \lambda_1 T_1$ and $T = \lambda_2 T_2$. (T_i closed, positive and same bidimension).

Extremality reduces the problem to sequences

Lemma

X an algebraic variety, $\mathscr{T}^+$ be a (p,p)-dimensional current on X of the form

$$\mathscr{T}^+ \underset{i}{\longleftarrow} \sum_j \lambda_{ij}^+ [Z_{ij}],$$

where $\lambda_{ij}^+ > 0$, Z_{ij} are p-dimensional irreducible analytic subsets of X. If $\mathscr{T}$ is extremal then

$$\mathscr{T}^+ \underset{i}{\longleftarrow} \lambda_i^+ [Z_i].$$

for some $\lambda_i^+ > 0$ and Z_i irreducible analytic sets.

Obstruction by the Hodge index theorem in dimension 4

Proposition

Let $\{\mathscr{T}\}$ be a $(2,2)$ cohomology class on the 4 dimensional smooth projective toric variety X. If there are nonnegative real numbers λ_i and 2-dimensional irreducible subvarieties $Z_i \subset X$ such that

$$\{\mathscr{T}\} = \lim_{i\to\infty} \{\lambda_i [Z_i]\},$$

then the matrix

$$[L_{ij}]_{\{\mathscr{T}\}} = -\{\mathscr{T}\}.D_{\rho_i}.D_{\rho_j},$$

has at most one negative eigenvalue.

Our goal

A $(2,2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed
- Positive
- Extremal, and
- Its intersection form has more than one negative eigenvalues

Tropical currents

$$\begin{array}{rcl} \mathrm{Log} : (\mathbb{C}^*)^n & \to & \mathbb{R}^n \\ (z_1, \ldots, z_n) & \mapsto & (-\log|z_1|, \ldots, -\log|z_n|) \end{array}$$

- $\mathrm{Log}^{-1}(\{pt\}) \simeq (S^1)^n$,
- $\dim_{\mathbb{R}} \mathrm{Log}^{-1}(\text{rational} p\text{-plane}) = n + p$
- $\mathrm{Log}^{-1}(\text{rational } p\text{-plane})$ has a natural fiberation over $(S^1)^{n-p}$ with fibers of complex dimension p
- Similarly for any p-cell σ, $\mathrm{Log}^{-1}(\sigma)$ has a natural fiberation over $(S^1)^{n-p}$

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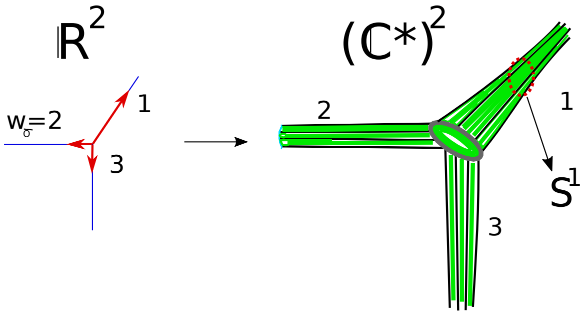
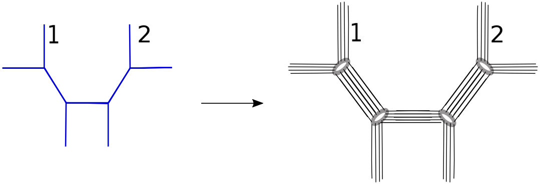
$n = 2, p = 1$

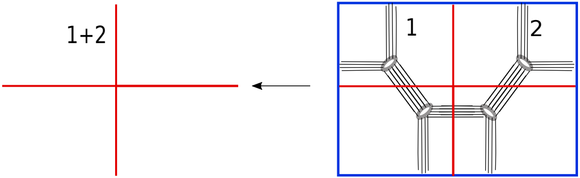


Support $\mathscr{T}_{\mathscr{C}} = \mathrm{Log}^{-1}(\mathscr{C})$, $\mathscr{T}_{\mathscr{C}} = \sum_\sigma w_\sigma \int_{S^{n-p}}$ [fibers of $\mathrm{Log}^{-1}(\sigma)$] $d\mu$

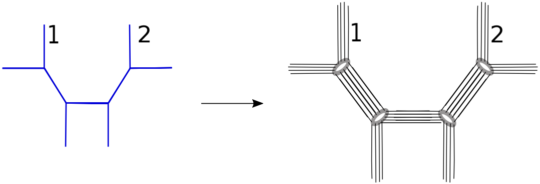
Dimension n



$\mathscr{C} \subset \mathbb{R}^n, \dim(\mathscr{C}) = p$ $\qquad$ $\mathscr{T}_{\mathscr{C}} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, Support $\mathscr{T}_{\mathscr{C}} = \mathrm{Log}^{-1}(\mathscr{C})$



$\{\overline{\mathscr{T}}_{\mathscr{C}}\} = \mathrm{rec}(\mathscr{C}) \in H^{n-p,n-p}(X_\Sigma)$ $\qquad$ $\overline{\mathscr{T}}_{\mathscr{C}} \in \mathcal{D}'_{p,p}(X_\Sigma)$



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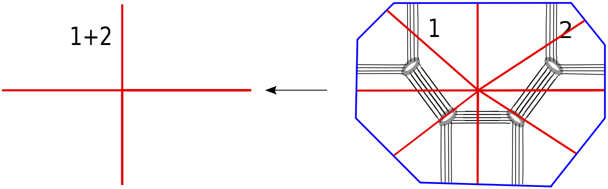


$\{\overline{\mathscr{T}_{\mathscr{C}}}\} = \mathrm{rec}(\mathscr{C}) \in H^{n-p,n-p}(X_\Sigma)$ $\overline{\mathscr{T}_{\mathscr{C}}} \in \mathcal{D}'_{p,p}(X_\Sigma)$

A $(2,2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed

  Balanced complex

- Positive

  Positive weights

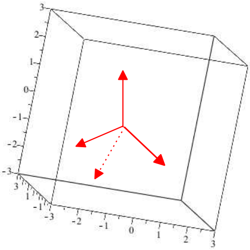
- Extremal

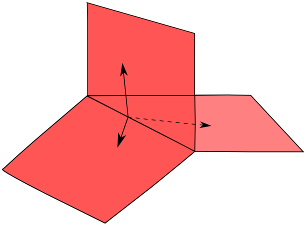
  ?

- Its intersection form has more than one negative eigenvalues

  ?

Extremality of tropical currents in any dimension/codimension





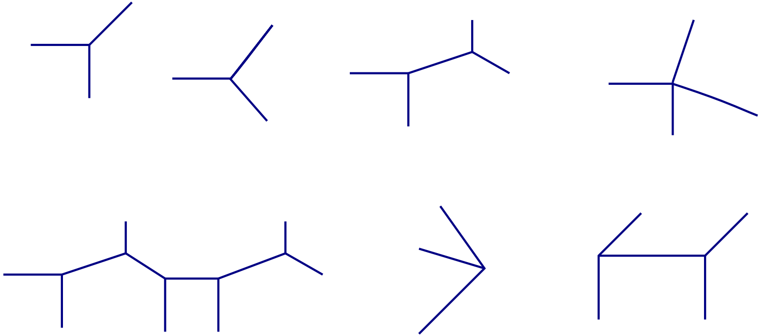
Weights unique up to a multiple + Not contained in any proper affine subspace

Examples of extremal currents

Lelong 1973: Integration currents along irreducible analytic subsets are extremal. Is that all?
Demailly 1982: $\frac{i}{\pi}\partial\bar{\partial}\log\max\{|z_0|,|z_1|,|z_2|\}$ is extremal on $\mathbb{P}^2$, and its support has real dimension 3, thus cannot be an integration current along any analytic set.

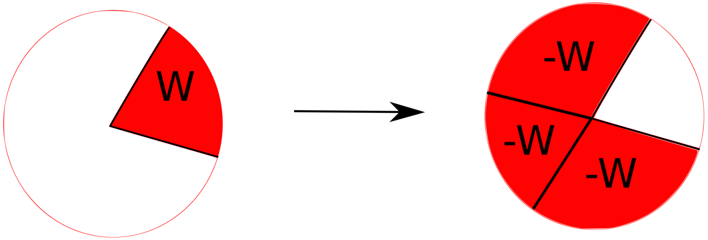
Dynamical systems (usually with fractal supports, thus non-analytic):
Codimension 1: Bedford and Smillie 1992, Fornaess and Sibony 1992, Sibony 1999, Cantat 2001, Diller and Favre 2001, Guedj 2002...
Higher Codimension: Dinh and Sibony 2005, Guedj 2005, Dinh and Sibony 2013

Complicated structures, easily seen to be approximable!



Extremal if: weights unique up to a multiple + Not contained in any proper affine subspace

Manipulation of signatures for 2-cells in dimension 4



The operation $F \longmapsto F_{ij}^{-}$ produces one new positive and one new negative eigenvalue for its intersection matrix

A $(2,2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed

  Balanced complex

- Positive

  Positive weights

- Extremal

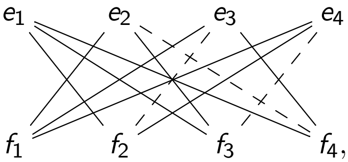
  Non-degenerate + weights unique up to a multiple

- Its intersection form has more than one negative eigenvalues

  The operation on two cells provides one new negative and one new positive eigenvalue

A concrete example

Consider $G \subseteq \mathbb{R}^4 \setminus \{0\}$



where e_1, e_2, e_3, e_4 are the standard basis vectors of $\mathbb{R}^4$ and f_1, f_2, f_3, f_4 the rows of

$$M := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

The weights of solid (resp. dashed) edges are $+1$ (resp. -1).

Thank you for your attention, indeed!