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Maximal Space Regularity for Abstract Linear Non-autonomous Parabolic Equations

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The linear non-autonomous evolution equation $u'(t) - A(t)\,u(t) = f(t)$, $t \in [0, T]$, with the initial datum $u(0) = x$, in the space $C([0, T], E)$, where E is a Banach space and $\{A(t)\}$ is a family of infinitesimal generators of bounded analytic semi-groups is considered; the domains $D(A(t))$ are supposed constant in t and possibly not dense in E. Maximal regularity of the strict and classical solutions, i.e., regularity of u' and $A(\cdot)u(\cdot)$ with values in the interpolation spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$ between $D(A(0))$ and E, is studied. A characterization of such spaces in a concrete case is also given.

Let E be a Banach space, and $\{A(t)\}_{t \in [0,T]}$ a family of closed linear operators on E. We consider the linear non-autonomous Cauchy problem

$$\begin{aligned} &u'(t) - A(t)\,u(t) = f(t), \qquad t \in [0, T], \\ &u(0) = x, \\ &x \in E, \qquad f \in C([0, T], E) \text{ prescribed}, \end{aligned} \tag{P}$$

where $C([0, T], E)$ is the space of continuous functions $[0, T] \to E$. We are concerned here with the parabolic case: in other words, we suppose that for each $t \in [0, T]$ $A(t)$ is the infinitesimal generator of an analytic semi-group $\{e^{\xi A(t)}\}_{\xi > 0}$ (not necessarily strongly continuous at 0), and its domain $D(A(t))$ does not depend on t and is possibly not dense in E. In a recent paper (Acquistapace and Terreni [2]) existence of strict and classical (i.e., continuously differentiable) solutions of (P) is proved under the same hypotheses of Tanabe [14] and Sobolevskii [13], provided the data x, f are

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sufficiently regular. That paper also contains several time regularity results "of maximal type" for such solutions, namely, if f belongs to a suitable subspace of $C([0, T], E)$ (i.e., f is Hölder continuous), then u'—and consequently $A(\cdot)\,u(\cdot)$—belongs to the same subspace.

The aim of the present paper is to prove, under the same hypotheses, existence and "maximal" space regularity results for strict and classical solutions of (P), by making use of the intermediate spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$ between $D(A(0))$ and E: namely, if f is continuous or bounded with values in any of such spaces, then the same holds for u' and $A(\cdot)\,u(\cdot)$.

Results of this kind are known in the autonomous case, i.e., $A(t) \equiv A$ (see Sinestrari [12]). In the non-autonomous case, other space regularity results had been proved in Sobolevskii's classical paper [13] by using the domains of the fractional powers of $-A(t)$. More recently Da Prato and Grisvard [5] proved results similar to ours under the stronger assumption that $D_{A(t)}(\theta+1) =$ constant for some $\theta \in \,]0, 1[$ (here $D_{A(t)}(\theta+1) = \{x \in D(A(t)):$ $A(t)\,x \in D_{A(t)}(\theta)\}$); in addition Sobolevskii's condition, namely Hölder continuity of $t \to A(t)$ with values in the space $\mathscr{L}(D(A(0)), E)$ of bounded linear operators from $D(A(0))$ into E, is replaced in [5] by the assumption that $t \to A(t)$ is continuous with values in $\mathscr{L}(D_{A(0)}(\theta+1), D_{A(0)}(\theta))$.

As in [2], our method does not require the construction of the fundamental solution; it is based instead upon a representation formula for the solutions of (P), and all our results are obtained by a careful analysis of it. The formula that we used in [2] is different from the present one: the former required suitable time regularity assumptions on f, while the latter is meaningful provided f has suitable space regularity properties. Of course if f is assumed to be regular both in time and in space, then both formulas apply and in fact they coincide.

Our representation formula can be derived by the following heuristic argument: if u solves (P), fix $t \in \,]0, T]$ and consider the function

$$v(s) = e^{(t-s)A(t)}u(s), \qquad s \in [0, t];$$

the first derivative of $v(s)$ is

$$\begin{aligned} v'(s) &= -A(t)\,e^{(t-s)A(t)}u(s) + e^{(t-s)A(t)}(A(s)\,u(s) + f(s)) \\ &= e^{(t-s)A(t)}(A(s) - A(t))\,u(s) + e^{(t-s)A(t)}f(s). \end{aligned}$$

Integrating over $[0, t]$ we get

$$u(t) - e^{tA(t)}x = \int_0^t e^{(t-s)A(t)}(1 - A(t)A(s)^{-1})\,A(s)\,u(s)\,ds + \int_0^t e^{(t-s)A(t)}f(s)\,ds,$$

and applying $A(t)$ we obtain an integral equation for $A(t)\,u(t)$:

$$A(t)\,u(t) - \int_0^t A(t)\,e^{(t-s)A(t)}(1 - A(t)\,A(s)^{-1})\,A(s)\,u(s)\,ds$$

$$= A(t)\,e^{tA(t)}x + \int_0^t A(t)\,e^{(t-s)A(t)}f(s)\,ds.$$

Denoting by H the integral operator

$$Hg(t) = \int_0^t A(t)\,e^{(t-s)A(t)}(1 - A(t)\,A(s)^{-1})\,g(s)\,ds, \qquad t \in [0, T],$$

we check

$$A(t)\,u(t) = (1-H)^{-1}\left(A(t)\,e^{tA(t)}x + \int_0^t A(t)\,e^{(t-s)A(t)}f(s)\,ds\right),$$

or

$$u(t) = A(t)^{-1}((1-H)^{-1}(L(\cdot, 0)\,x + Lf)(t)), \tag{0.1}$$

where

$$L(t,s) = A(t)\,e^{(t-s)A(t)}, \qquad 0 \leqslant s < t \leqslant T,$$

$$Lg(t) = \int_0^t L(t,s)\,g(s)\,ds, \qquad t \in [0, T].$$

This procedure is quite heuristic and we need to give some sense to it. We will see that the integral operator H is of Volterra type with integrable kernel, and that the operator L is well defined on the space of bounded functions with values in some $D_{A(0)}(\theta, \infty)$. Thus if we take f in such a space and x suitably regular, formula (0.1) will turn out to be perfectly meaningful and will give the desired representation of the solutions of (P).

Let us describe now the subjects of the next sections. Section 1 contains a list of notations, definitions and assumptions; in Section 2 we establish some preliminary results. In Section 3 we derive the basic technical background which is needed to prove our main theorems. In Section 4 we discuss the existence of strict and classical solutions. Section 5 is devoted to space regularity results. Finally in Section 6 we illustrate an example where an explicit characterization of $D_A(\theta, \infty)$ and $D_A(\theta)$ is given, when A is a second order ordinary differential operator with Neumann boundary conditions in the space of continuous functions. A similar characterization in the case of several variables and more general boundary conditions will be given in a forthcoming paper.

1. Notations and Assumptions

Let us list some notations. If A is a linear operator on a Banach space E, we set:

$$D(A) = \text{domain of } A, \qquad R(A) = \text{range of } A,$$

$$\rho(A) = \text{resolvent set of } A, \qquad \sigma(A) = \text{spectrum of } A,$$

$$R(\lambda, A) = (\lambda - A)^{-1} \quad \text{for each } \lambda \in \rho(A).$$

If X, Y are Banach spaces, we denote by $\mathscr{L}(X, Y)$ (or simply $\mathscr{L}(X)$ when $Y = X$) the Banach space of bounded linear operators with domain X and range contained in Y, with the usual norm

$$\|B\|_{\mathscr{L}(X,Y)} = \sup_{x \in X - \{0\}} \frac{\|Bx\|_Y}{\|x\|_X}.$$

Let Y, E two Banach spaces, with Y continuously imbedded into E. We shall make use of the following Banach spaces of functions:

(a) $B_\beta(0, T, Y) = \{u:]0, T] \to Y: u$ is strongly E-measurable and $\sup_{t \in]0,T]} \|t^\beta u(t)\|_Y < \infty\}$, with norm

$$\|u\|_{B_\beta(0,T,Y)} = \sup_{t \in]0,T]} \|t^\beta u(t)\|_Y;$$

when $\beta = 0$ we shall simply write $B(0, T, Y)$ instead of $B_0(0, T, Y)$.

(b) $C_\beta(]0, T], Y) = \{u \in B_\beta(0, T, Y): u:]0, T] \to Y$ is continuous$\}$, $\beta \in [0, 1[$, with the norm of $B_\beta(0, T, Y)$, and its closed subspace $C_\beta([0, T], Y) = \{u \in C_\beta(]0, T], Y): \exists \lim_{t \to 0^+} t^\beta u(t) \in Y\}$; when $\beta = 0$ we shall write $C([0, T], Y)$ instead of $C_0([0, T], Y)$.

(c) $C^\theta([0, T], Y) = \{u \in C([0, T], Y): \|u(t) - u(r)\|_Y = O(|t - r|^\theta)$ as $|t - r| \to 0^+\}$, $\theta \in]0, 1[$, with norm

$$\|u\|_{C^\theta([0,T],Y)} = \|u\|_{C([0,T],Y)} + \sup_{t \neq r} \frac{\|u(t) - u(r)\|_Y}{|t - r|^\theta},$$

and its closed subspace $h^\theta([0, T], Y) = \{u \in C^\theta([0, T], Y): \|u(t) - u(r)\|_Y = o(|t - r|^\theta)$ as $|t - r| \to 0^+\}$.

(d) $\mathrm{Lip}([0, T], Y) = \{u \in C([0, T], Y): \|u(t) - u(r)\|_Y = O(|t - r|)$ as $|t - r| \to 0^+\}$, with norm

$$\|u\|_{\mathrm{Lip}([0,T],Y)} = \|u\|_{C([0,T],Y)} + \sup_{t \neq r} \frac{\|u(t) - u(r)\|_Y}{|t - r|}.$$

(e) $C^1([0,T],Y)=\{u\in C([0,T],Y): u$ is differentiable and $u'\in C([0,T],Y)\}$, with norm

$$\|u\|_{C^1([0,T],Y)}=\|u\|_{C([0,T],Y)}+\|u'\|_{C([0,T],Y)}.$$

We shall also consider the following spaces of functions:

(f) $B(0^+,T,Y)=\bigcap_{\varepsilon\in]0,T]}B(\varepsilon,T,Y)$ and

(g) $C([0,T],Y)$, $C^\theta(]0,T],Y)$, $h^\theta(]0,T],Y)(\theta\in]0,1[)$, $\mathrm{Lip}(]0,T],Y)$, $C^1(]0,T],Y)$,

which are defined similarly. We observe explicitly that $C(]0,T],Y)$ and $C_0(]0,T],Y)$ are different spaces.

Now we list our assumptions.

HYPOTHESIS I. For each $t\in[0,T]$ $A(t)$ is a closed linear operator on the Banach space E, with domain $D(A(t))\equiv D(A(0))$ independent of t, which is the infinitesimal generator of a bounded analytic semi-group $\{e^{\xi A(t)}\}_{\xi\geqslant 0}$. More precisely:

(i) there exists $\theta_0\in]\pi/2,\pi[$ such that

$$\rho(A(t))\subseteq\sum_{\theta_0}=\{z\in\mathbb{C}:|\arg z|<\theta_0\}\cup\{0\},\qquad\forall t\in[0,T];$$

(ii) there exists $M>0$ such that

$$\|R(\lambda,A(t))\|_{\mathscr{L}(E)}\leqslant\frac{M}{|\lambda|}\ \forall\lambda\in\Sigma_{\theta_0}-\{0\},\qquad\|A(t)^{-1}\|_{\mathscr{L}(E)}\leqslant M\ \ \forall t\in[0,T].$$

HYPOTHESIS II. There exist $\alpha\in]0,1[$ and $K>0$ such that

$$\|1-A(t)A(r)^{-1}\|_{\mathscr{L}(E)}\leqslant K|t-r|^\alpha,\qquad\forall t,r\in[0,T].$$

Remark 1.1. $D(A(0))$ is not supposed to be dense in E; however, if Hypothesis I holds and E is locally sequentially weakly compact (e.g., E is reflexive) then necessarily $\overline{D(A(0))}=E$ (see Kato [6]).

Let us recall our definitions of solutions of Problem (P) (see [2]):

DEFINITION 1.2. Let $f\in C([0,T],E)$, $x\in E$; a function $u\in C([0,T],E)$ is a *strict solution* of (P) if $u\in C^1([0,T],E)$, $u(t)\in D(A(0))\ \forall t\in[0,T]$ and

$$u'(t)-A(t)\,u(t)=f(t)\qquad\text{in }[0,T],\qquad u(0)=x.$$

DEFINITION 1.3. Let $f\in C(]0,T],E)$, $x\in E$; a function $u\in C([0,T],E)$

is a *classical solution* of (P) if $u \in C^1(]0, T], E)$, $u(t) \in D(A(0))\ \forall t \in\ ]0, T]$ and

$$u'(t) - A(t)\,u(t) = f(t) \qquad \text{in }]0, T], \qquad u(0) = x.$$

In [2] a weaker type of solution is also considered, namely the strong solution. We will not study such solutions here: we just recall their definition and some related properties.

DEFINITION 1.4. Let $f \in C([0, T], E)$, $x \in E$; a function $u \in C([0, T], E)$ is a *strong solution* of (P) if there exists $\{u_n\}_{n \in \mathbf{N}} \subseteq C^1([0, T], E)$ such that $u_n(t) \in D(A(0))\ \forall t \in [0, T]$ and $n \in \mathbf{N}$, and

$$\begin{aligned} &u_n \to u && \text{in } C([0, T], E) \\ &u_n' - A(\cdot)\,u_n(\cdot) \equiv f_n \to f && \text{in } C([0, T], E) \\ &u_n(0) = x_n \to x && \text{in } E. \end{aligned}$$

Remark 1.5. By definition it is clear that a strict solution is a classical and a strong one. It can be seen also that a classical solution is a strong one, provided $f \in C([0, T], E)$ and Hypotheses I, II hold ([2, Remark 6.7]). In [2] the following necessary conditions are proved under Hypotheses I and II:

(a) if Problem (P) has a strict solution, then $x \in D(A(0))$ and $A(0)\,x + f(0) \in \overline{D(A(0))}$;

(b) if Problem (P) has a classical (resp. strong) solution, then $x \in \overline{D(A(0))}$. In addition, the strict (or classical, or strong) solution is unique.

About existence, under Hypotheses I, II the following properties are known ([2]):

(c) if $f \in C^\theta([0, T], E)$, $x \in D(A(0))$ and $A(0)\,x + f(0) \in \overline{D(A(0))}$, then a strict solution exists (Theorem 4.3);

(d) if $f \in C_\sigma(]0, T], E) \cap C^\theta(]0, T], E)$ and $x \in \overline{D(A(0))}$, then a classical solution exists (Theorem 5.4);

(e) if $f \in C([0, T], E)$ and $x \in \overline{D(A(0))}$, then a strong solution exists (Theorem 6.4).

2. PRELIMINARIES

Let A be a closed linear operator on the Banach space E, satisfying Hypothesis I; then the bounded analytic semi-group $\{e^{\xi A(t)}\}_{\xi \geqslant 0}$ can be represented by a Dunford integral

$$e^{\xi A} = \frac{1}{2\pi i} \int_\gamma e^{\xi\lambda} R(\lambda, A)\, d\lambda, \qquad \xi > 0,$$

where $\gamma = \gamma^0 \cup \gamma^+ \cup \gamma^-$, and

$$\gamma^0 = \{z \in \mathbb{C}: |z| = 1, |\arg z| \leqslant \theta\},$$
$$\gamma^\pm = \{z \in \mathbb{C}: \arg z = \pm\theta, |z| \geqslant 1\},$$

with $\theta \in \,]\pi/2, \theta_0[$. The operator $e^{\xi A}$ maps E into $\bigcap_{n \in \mathbb{N}} D(A^n)$ for each $\xi > 0$ and

$$A^n e^{\xi A} = \frac{1}{2\pi i} \int_\gamma \lambda^n e^{\xi\lambda} R(\lambda, A)\, d\lambda \qquad \forall n \in \mathbb{N}, \qquad \xi > 0,$$

the integrals being absolutely convergent.

If A is a closed linear operator on E, then its domain $D(A)$, equipped with the graph norm, is itself a Banach space continuously imbedded into E. It is then possible to construct the interpolation spaces $(D(A), E)_{\sigma,\infty}$ and $(D(A), E)_\sigma$, $\sigma \in \,]0, 1[$, as follows (see Lions [7], Lions and Peetre [8], Butzer and Berens [3]):

DEFINITION 2.1. Let $x \in E$; we say that $x \in (D(A), E)_{\sigma,\infty}$ (resp. $(D(A), E)_\sigma$) if there exists $u:]0, 1] \to D(A)$ having first derivative (in the sense of distributions) $u':]0, 1] \to E$, such that

(i) $u', Au \in C_\sigma(]0, 1], E)$ (resp. $C_\sigma([0, 1], E)$ with $\lim_{t\to 0^+} \|t^\sigma u'(t)\|_E = \lim_{t\to 0^+} \|t^\sigma Au(t)\|_E = 0$)

(ii) $u(0) = x$.

Condition (ii) is meaningful because from (i) we easily deduce that $u \in C^{1-\sigma}([0, 1], E)$.

Clearly

$$D(A) \subseteq (D(A), E)_\sigma \subseteq (D(A), E)_{\sigma,\infty} \subseteq (D(A), E)_\beta \subseteq \overline{D(A)} \quad \text{if } 0 < \beta < \sigma < 1.$$

If in addition A generates a bounded analytic semi-group, the spaces $(D(A), E)_{1-\theta,\infty}$ and $(D(A), E)_{1-\theta}$, $\theta \in \,]0, 1[$, are denoted by $D_A(\theta, \infty)$ and $D_A(\theta)$, and can be characterized in several ways (see Butzer and Berens [3] for the case $\overline{D(A)} = E$ and Acquistapace and Terreni [1] for the general case), namely:

$$\begin{aligned} D_A(\theta, \infty) &= \left\{ x \in E: \sup_{t>0} \left\| \frac{e^{tA} - 1}{t^\theta} x \right\|_E < \infty \right\} \\ &= \left\{ x \in E: \sup_{t>0} \|t^{1-\theta} A e^{tA} x\|_E < \infty \right\} \\ &= \left\{ x \in E: \sup_{\lambda \in \rho(A)} \||\lambda|^\theta A R(\lambda, A) x\|_E < \infty \right\}, \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
D_A(\theta) &= \left\{ x \in D_A(\theta, \infty) \colon \lim_{t \to 0^+} \left\| \frac{e^{tA} - 1}{t^\theta} x \right\|_E = 0 \right\} \\
&= \left\{ x \in D_A(\theta, \infty) \colon \lim_{t \to 0^+} \| t^{1-\theta} A e^{tA} x \|_E = 0 \right\} \\
&= \left\{ x \in D_A(\theta, \infty) \colon \lim_{\substack{\lambda \in \rho(A) \\ |\lambda| \to +\infty}} \| |\lambda|^\theta A R(\lambda, A) x \|_E = 0 \right\}.
\end{aligned} \tag{2.2}
$$

$D_A(\theta, \infty)$ becomes a Banach space with the norm

$$
\|x\|_{D_A(\theta,\infty)} = \|x\|_E + \sup_{t>0} \left\| \frac{e^{tA} - 1}{t^\theta} x \right\|_E, \tag{2.3}
$$

which is equivalent to the usual norm of $(D(A(0)), E)_{1-\theta,\infty}$ (see [3]). It can be seen that there exist c^0, c', c'' such that

$$
\begin{aligned}
\sup_{\lambda \in \rho(A)} \| |\lambda|^\theta A R(\lambda, A) x \|_E &\leqslant c^0 \sup_{t>0} \left\| \frac{e^{tA} - 1}{t^\theta} x \right\|_E \leqslant c' \sup_{t>0} \| t^{1-\theta} A e^{tA} x \|_E \\
&\leqslant c'' \sup_{\lambda \in \rho(A)} \| |\lambda|^\theta A R(\lambda, A) x \|_E.
\end{aligned} \tag{2.4}
$$

Hence the quantities in (2.4) are equivalent semi-norms on $D_A(\theta, \infty)$. If, in addition, $0 \in \rho(A)$, the semi-norms in (2.4) become equivalent norms on $D_A(\theta, \infty)$.

If $0 < \beta < \theta < 1$ we easily get that the inclusions $D(A) \subseteq D_A(\theta, \infty) \subseteq D_A(\beta, \infty) \subseteq \overline{D(A)}$ are continuous, i.e.,

$$
\begin{aligned}
\|x\|_E &\leqslant C \|x\|_{D_A(\beta,\infty)} && \forall x \in D_A(\beta, \infty), \\
\|x\|_{D_A(\beta,\infty)} &\leqslant C \|x\|_{D_A(\theta,\infty)} && \forall x \in D_A(\theta, \infty), \\
\|x\|_{D_A(\theta,\infty)} &\leqslant C \|x\|_{D(A)} && \forall x \in D(A).
\end{aligned} \tag{2.5}
$$

Remark 2.2. It is useful to observe that if $0 \in \rho(A)$, $x \in E$, $r > 0$ and

$$
\sup_{\mu \in [r,\infty[} \mu^\theta \|A R(\mu, A) x\|_E < \infty \tag{2.6}
$$

then $x \in D_A(\theta, \infty)$. Indeed, if (2.6) holds, then it is easily seen that

$$
\begin{aligned}
\sup_{\lambda \in \rho(A)} |\lambda|^\theta \|A R(\lambda, A) x\|_E &\leqslant C \sup_{\mu \in [0,\infty[} \mu^\theta \|A R(\mu, A) x\|_E \\
&\leqslant C(r) \sup_{\mu \in [r,\infty[} \mu^\theta \|A R(\mu, A) x\|_E.
\end{aligned}
$$

Therefore we will sometimes consider the quantity

$$\sup_{\mu\in[r,\infty[} \mu^\theta \|AR(\mu,A)x\|_E, \qquad x\in D_A(\theta,\infty) \tag{2.7}$$

with $r>0$; under the above assumptions, it defines a norm on $D_A(\theta,\infty)$ equivalent to (2.3).

It can be verified that for each $\theta\in\,]0,1[$ $D_A(\theta)$ is a closed subspace of $D_A(\theta,\infty)$ which coincides with the closure of $D(A)$ in the norm of $D_A(\theta,\infty)$ (see Butzer and Berens [3, Chap. III, Proposition 3.16], or Sinestrari [12, Proposition 1.8]).

Let us assume, from now on, Hypotheses I and II. In the rest of this section, unless otherwise specified, such assumptions will always be supposed to hold. We will state some results of general character, whose proofs, when omitted, can be found in [2], Martin [10], and Sinestrari [12].

LEMMA 2.3. *There exists $C>0$ such that*

$$\|R(\lambda,A(t))-R(\lambda,A(r))\|_{\mathscr{L}(E)}\leqslant C\frac{|t-r|^\alpha}{|\lambda|} \qquad \forall\lambda\in\Sigma_{\theta_0}-\{0\},\ \forall t,r\in[0,T].$$

Proof. It is sufficient to note that

$$R(\lambda,A(t))-R(\lambda,A(r))=R(\lambda,A(t))(A(t)A(r)^{-1}-1)A(r)R(\lambda,A(r)). \qquad \blacksquare$$

As a direct consequence of the fact that $D(A(t))$ does not depend on t, we have:

PROPOSITION 2.4. *For each $t\in[0,T]$ and $\theta\in\,]0,1[$ we have:*

$$D_{A(t)}(\theta,\infty)=D_{A(0)}(\theta,\infty), \qquad D_{A(t)}(\theta)=D_{A(0)}(\theta);$$

in addition there exist $C_1,C_2>0$ such that if $x\in D_{A(0)}(\theta,\infty)$

$$\sup_{\xi>0}\left\|\frac{e^{\xi A(0)}-1}{\xi^\theta}x\right\|_E\leqslant C_1\sup_{\xi>0}\left\|\frac{e^{\xi A(t)}-1}{\xi^\theta}x\right\|_E\leqslant C_2\sup_{\xi>0}\left\|\frac{e^{\xi A(0)}-1}{\xi^\theta}x\right\|_E.$$

Proof. See Lemma 2.3 in [2]. $\blacksquare$

From now on we shall simply write

$$\|x\|_\theta=\|x\|_{D_{A(0)}(\theta,\infty)} \qquad \forall\theta\in\,]0,1[,\ \forall x\in D_{A(0)}(\theta,\infty). \tag{2.8}$$

PROPOSITION 2.5. *Let $\theta\in\,]0,1[$.*

(i) *The following statements are equivalent*:

(a) $x \in D_{A(0)}(\theta)$;

(b) $\lim_{|\lambda| \to \infty, \lambda \in \Sigma_{\theta_0}} |\lambda|^\theta \|A(t) R(\lambda, A(t)) x\|_E = 0$ *uniformly in* $t \in [0, T]$;

(c) $\lim_{\xi \to 0^+} \xi^{1-\theta} \|A(t) e^{\xi A(t)} x\|_E = 0$ *uniformly in* $t \in [0, T]$;

(d) $\lim_{\xi \to 0^+} \|e^{\xi A(t)} - 1/\xi^\theta x\|_E = 0$ *uniformly in* $t \in [0, T]$.

(ii) *The following statements are equivalent*:

(a) $g \in C([0, T], D_{A(0)}(\theta))$;

(b) $\lim_{|\lambda| \to \infty, \lambda \in \Sigma_{\theta_0}} |\lambda|^\theta \|A(t) R(\lambda, A(t)) g(s)\|_E = 0$ *uniformly in* $s, t \in [0, T]$;

(c) $\lim_{\xi \to 0^+} \xi^{1-\theta} \|A(t) e^{\xi A(t)} g(s)\|_E = 0$ *uniformly in* $s, t \in [0, T]$;

(d) $\lim_{\xi \to 0^+} \|((e^{\xi A(t)} - 1)/\xi^\theta) g(s)\|_E = 0$ *uniformly in* $s, t \in [0, T]$.

Proof. (i) (a) $\Rightarrow$ (b). Let $x \in D_{A(0)}(\theta)$ and let $\varepsilon > 0$. By (2.2) there exists $M_\varepsilon > 0$ such that

$$\lambda \in \Sigma_{\theta_0}, \qquad |\lambda| > M_\varepsilon \Rightarrow |\lambda|^\theta \|A(0) R(\lambda, A(0)) x\|_E \leqslant \varepsilon.$$

Since

$$\begin{aligned} &A(t) R(\lambda, A(t)) - A(0) R(\lambda, A(0)) \\ &\quad = \lambda R(\lambda, A(t))(A(t) A(0)^{-1} - 1) A(0) R(\lambda, A(0)) \end{aligned}$$

it follows that if $\lambda \in \Sigma_{\theta_0}$ and $|\lambda| > M_\varepsilon$

$$\begin{aligned} |\lambda|^\theta \|A(t) R(\lambda, A(t)) x\|_E &\leqslant (1 + CT^\alpha) |\lambda|^\theta \|A(0) R(\lambda, A(0)) x\|_E \\ &\leqslant C\varepsilon \qquad \forall t \in [0, T]. \end{aligned}$$

(b) $\Rightarrow$ (c). Let $\varepsilon > 0$. By hypothesis, there exists $M_\varepsilon > 0$ such that

$$\lambda \in \Sigma_{\theta_0}, \qquad |\lambda| > M_\varepsilon \Rightarrow |\lambda|^\theta \|A(t) R(\lambda, A(t)) x\|_E < \varepsilon \qquad \forall t \in [0, T].$$

Now if $\xi > 0$ we have

$$\begin{aligned} \xi^{1-\theta} A(t) e^{\xi A(t)} x &= \frac{1}{2\pi i} \int_\gamma \xi^{1-\theta} e^{\xi\lambda} A(t) R(\lambda, A(t) x d\lambda \\ &= \frac{1}{2\pi i} \int_{\{z : z/\xi \in \gamma\}} \frac{e^z}{\xi^\theta} A(t) R\left(\frac{z}{\xi}, A(t)\right) x dz \\ &= \frac{1}{2\pi i} \int_\gamma \frac{e^z}{\xi^\theta} A(t) R\left(\frac{z}{\xi}, A(t)\right) x dz, \end{aligned}$$

so that

$$\xi^{1-\theta}\,\|A(t)\,e^{\xi A(t)}x\|_E \leqslant C\int_\gamma \frac{1}{|z|^\theta}\,e^{\operatorname{Re} z}\left|\frac{z}{\xi}\right|^\theta \|A(t)\,R\left(\frac{z}{\xi},A(t)\right)x\|_E\,|dz|;$$

hence if $\xi < 1/M_\varepsilon$ we get

$$\xi^{1-\theta}\|A(t)\,e^{\xi A(t)}x\|_E \leqslant C\varepsilon\int_\gamma \frac{e^{\operatorname{Re} z}}{|z|^\theta}\,|dz| \leqslant C\varepsilon \qquad \forall t\in[0,T].$$

(c) $\Rightarrow$ (*d*). Let $\varepsilon>0$. By assumption, there exists $\delta_\varepsilon>0$ such that

$$\xi<\delta_\varepsilon \Rightarrow \xi^{1-\theta}\,\|A(t)\,e^{\xi A(t)}\,x\|_E<\varepsilon \qquad \forall t\in[0,T].$$

Thus if $\xi<\delta_\varepsilon$ we get

$$\left\|\frac{e^{\xi A(t)}-1}{\xi^\theta}\,x\right\|_E = \left\|\frac{1}{\xi^\theta}\int_0^\xi A(t)\,e^{sA(t)}x\,ds\right\|_E \leqslant \frac{1}{\xi^\theta}\int_0^\xi \frac{ds}{s^{1-\theta}} = \frac{\varepsilon}{\theta} \qquad \forall t\in[0,T].$$

(d) $\Rightarrow$ (a). Obvious.

(ii) (a) $\Rightarrow$ (b). If $g\in C([0,T],D_{A(0)}(\theta))$, for each $\varepsilon>0$ there exist $s_1,\dots,s_{n_\varepsilon}\in[0,T]$ such that

$$\min_{1\leqslant i\leqslant n_\varepsilon}\|\,g(s)-g(s_i)\|_\theta<\varepsilon \qquad \forall s\in[0,T].$$

For each i, $1\leqslant i\leqslant n_\varepsilon$, we have by (i)

$$\lim_{\substack{|\lambda|\to\infty\\ \lambda\in\Sigma_{\theta_0}}} |\lambda|^\theta\,\|A(t)\,R(\lambda,A(t))\,g(s)\|_E=0 \text{ uniformly in } t\in[0,T];$$

hence there exists $M_\varepsilon>0$ such that if $\lambda\in\Sigma_{\theta_0}$ and $|\lambda|>M_\varepsilon$ we have

$$\begin{aligned}\lim_{\substack{|\lambda|\to\infty\\ \lambda\in\Sigma_{\theta_0}}} |\lambda|^\theta\,\|A(t)\,R(\lambda,A(t))\,g(s)\|_E &\leqslant \varepsilon + \lim_{\substack{|\lambda|\to\infty\\ \lambda\in\Sigma_{\theta_0}}} |\lambda|^\theta\,\|A(t)\,R(\lambda,A(t))\,g(s_{i_0})\|_E\\ &\leqslant 2\varepsilon \qquad \forall s\in[0,T]\end{aligned}$$

provided $i_0=i_0(s)$ is such that $\|\,g(s)-g(s_{i_0})\|_\theta<\varepsilon$.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). These implications are proved similarly. ∎

PROPOSITION 2.6. *Let* $\theta\in\,]0,1[$.

(i) *If* $x\in D_{A(0)}(\theta)$ *then* $\lim_{t\to0^+}\|(e^{tA(s)}-1)\,x\|_\theta=0$ *uniformly in* $s\in[0,T]$.

(ii) *If* $g \in C([0,T], D_{A(0)}(\theta))$ *then* $\lim_{t\to 0^+} \|(e^{tA(s)}-1)\, g(r)\|_\theta = 0$ *uniformly in* $s, r \in [0,T]$.

Proof. (i) Let $\varepsilon > 0$; by Proposition 2.5(i) there exists $\delta_\varepsilon > 0$ such that

$$\|\xi^{1-\theta} A(s)\, e^{\xi A(s)} x\|_E < \varepsilon, \qquad \left\|\frac{e^{\xi A(s)}-1}{\xi^\theta}\, x\right\|_E < \varepsilon \qquad \forall \xi \in\,]0, \delta_\varepsilon[,\ \forall s \in [0,T].$$

Hence for each $t \in\,]0, \delta_\varepsilon[$ we have $\forall s \in [0,T]$

$$\begin{aligned}
&\|\xi^{1-\theta} A(s)\, e^{\xi A(s)}(e^{tA(s)}-1)\, x\|_E \\
&\quad \leqslant \|e^{tA(s)}-1\|_{\mathscr{L}(E)}\, \|\xi^{1-\theta} A(s)\, e^{\xi A(s)} x\|_E \leqslant C\varepsilon \qquad \text{if} \quad \xi < t \\
&\quad \leqslant \|\xi A(s)\, e^{\xi A(s)}\|_{\mathscr{L}(E)} \left\|\frac{e^{tA(s)}-1}{\xi^\theta}\, x\right\|_E \leqslant C\varepsilon \qquad \text{if} \quad \xi \geqslant t
\end{aligned}$$

and the result follows.

(ii) For each $\varepsilon > 0$ there exist $r_1, \ldots, r_{n_\varepsilon} \in [0,T]$ such that

$$\min_{1 \leqslant i \leqslant n_\varepsilon} \|g(r) - g(r_i)\|_\theta < \varepsilon \qquad \forall r \in [0,T].$$

Taking into account (i), the result follows easily. ∎

PROPOSITION 2.7. *We have*:

(i) $\overline{D(A(0))} = \{x \in E: t \to e^{tA(s)} x \in C([0,T], E)\}$
$= \{x \in E: \lim_{t\to 0^+} \|e^{tA(s)}x - x\|_E = 0\}\ \forall s \in [0,T]$.

(ii) $D(A(0)) \subseteq \{x \in E: t \mapsto ((e^{tA(s)}-1)/t)\, x \in B(0,T,E)\}$
$= \{x \in E: t \to A(s)\, e^{tA(s)} x \in B(0,T,E)\}\ \forall s \in [0,T]$.

(iii) *Define* $D = \{x \in D(A(0)): A(0)\, x \in \overline{D(A(0))}\}$; *then*

$$\begin{aligned}
D &= \left\{x \in E: t \to \frac{e^{tA(0)}-1}{t}\, x \in C([0,T], E)\right\} \\
&= \left\{x \in D(A(0)): \lim_{t\to 0^+} \left\|\frac{e^{tA(0)}-1}{t} - A(0)\, x\right\|_E = 0\right\} \\
&= \{x \in E: t \to A(0)\, e^{tA(0)} x \in C([0,T], E)\} \\
&= \left\{x \in D(A(0)): \lim_{t\to 0^+} \|A(0)\, e^{tA(0)} x - A(0)\, x\|_E = 0\right\}.
\end{aligned}$$

Proof. See Lemma 2.5 of [2]. ∎

By Proposition 2.7 we see that the definitions of the spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$ are meaningful for $\theta = 0$ and $\theta = 1$ also: when $\theta = 0$ we get $D_{A(0)}(0, \infty) = E$, $D_{A(0)}(0) = \overline{D(A(0))}$; when $\theta = 1$ we get $D_{A(0)}(1) = \ker A(0) = \{0\}$, while the space $D_{A(0)}(1, \infty)$ is intermediate between $D(A(0))$ and $\bigcap_{\theta \in]0,1[} D_{A(0)}(\theta)$.

However, in order to simplify notations and statements, from now on we shall adopt the following convention:

CONVENTION 2.8. $D_{A(0)}(0, \infty) = E$, $D_{A(0)}(0) = \overline{D(A(0))}$, $D_{A(0)}(1, \infty) = D(A(0))$, and $D_{A(0)}(1) = D = \{x \in D(A(0)) \colon A(0)\, x \in \overline{D(A(0))}\}$.

The following definition is useful, too:

DEFINITION 2.9. For each $\theta \in]0, 1[$ we set

$$D_{A(0)}(\theta + 1, \infty) = \{x \in D(A(0)) \colon A(0)\, x \in D_{A(0)}(\theta, \infty)\},$$
$$D_{A(0)}(\theta + 1) = \{x \in D(A(0)) \colon A(0)\, x \in D_{A(0)}(\theta)\}.$$

LEMMA 2.10. *We have*:

(i) $\|A(s)^n e^{tA(s)}\|_{\mathscr{L}(E)} \leqslant C_n/t^n$ $\forall s \in [0, T]$, $\forall t \in]0, T]$, $\forall n \in \mathbf{N}$.

(ii) $\|A(r)\, e^{tA(s)}\|_{\mathscr{L}(D_{A(0)}(\beta, \infty), E)} \leqslant C/t^{1-\beta}$ $\forall r, s \in [0, T]$, $\forall t \in]0, T]$, $\forall \beta \in [0, 1]$.

(iii) $\|A(s)^n e^{tA(s)}\|_{\mathscr{L}(D_{A(0)}(\beta, \infty), E)} \leqslant C_n/t^{n-\beta}$ $\forall s \in [0, T]$, $\forall t \in]0, T]$, $\forall \beta i [0, 1]$, $\forall n \in \mathbf{N}$.

Proof (i)–(ii). See [2, 10, 12].

(iii) If $x \in D_{A(0)}(\beta, \infty)$, $\beta \in [0, 1]$, we can write

$$\|A(s)^n e^{tA(s)} x\|_E \leqslant \|A(s)^{n-1} e^{tA(s)/2}\|_{\mathscr{L}(E)} \|A(s)\, e^{tA(s)/2}\|_{\mathscr{L}(D_{A(0)}(\beta, \infty), E)} \|x\|_\beta$$
$$\leqslant \frac{C_n}{(\frac{1}{2}t)^{n-1}} \frac{C_n}{(\frac{1}{2}t)^{1-\beta}} \|x\|_\beta \leqslant \frac{C_n}{t^{n-\beta}} \|x\|_\beta. \quad \blacksquare$$

LEMMA 2.11. *For each $t, r \in [0, T]$ and $s \in]0, T]$ we have*:

(i) $\| e^{sA(t)} - e^{sA(r)} \|_{\mathscr{L}(D_{A(0)}(\beta, \infty), D_{A(0)}(\theta, \infty))} \leqslant C(|t - r|^\alpha / s^{\theta - \beta})$ $\forall \theta, \beta \in [0, 1]$,

(ii) $\|A(t)\, e^{sA(t)} - A(r)\, e^{sA(r)}\|_{\mathscr{L}(D_{A(0)}(\beta, \infty), D_{A(0)}(\theta, \infty))}$
$\leqslant C(|t - r|^\alpha / s^{1+\theta-\beta})$ $\forall \theta, \beta \in [0, 1]$.

Proof. We will just prove (ii) in the case $\theta, \beta \in]0, 1[$, since (i) and the other cases of (ii) are quite similar (and even simpler).

Let $\theta, \beta \in]0,1[$ and take $x \in D_{A(0)}(\beta, \infty)$. Then obviously $(A(t)\, e^{sA(t)} - A(r)\, e^{sA(r)})\, x \in D(A(0)) \subseteq D_{A(0)}(\theta)$; taking into account Remark 2.2, we have to estimate the quantity

$$\sup_{\mu \in [2,\infty]} \mu^\theta \|A(t)\, R(\mu, A(t))(A(t)\, e^{sA(t)} - A(r)\, e^{sA(r)})\, x\|_E.$$

Now if $\mu \in [2, \infty[$

$$\begin{aligned}
&\mu^\theta \,\|A(t)\, R(\mu, A(t))(A(t)\, e^{sA(t)} - A(r)\, e^{sA(r)})\, x\|_E \\
&\quad = \mu^\theta \left\| \frac{1}{2\pi i} \int_\gamma \lambda e^{s\lambda} A(t)\, R(\mu, A(t))(R(\lambda, A(t)) - R(\lambda, A(r)))\, x d\lambda \right\|_E \\
&\quad = \mu^\theta \left\| \frac{1}{2\pi i} \int_\gamma A(t) \frac{R(\mu, A(t)) - R(\lambda, A(t))}{\lambda - \mu} \lambda e^{s\lambda} (A(t)\, A(r)^{-1} - 1) \right. \\
&\quad \quad \left. \times \frac{1}{|\lambda|^\beta} |\lambda|^\beta A(r)\, R(\lambda, A(r))\, x d\lambda \right\|_E \\
&\quad \leqslant C \int_\gamma \mu^\theta \frac{|\lambda|}{|\lambda - \mu|} e^{s \operatorname{Re} \lambda} |t - r|^\alpha \frac{1}{|\lambda|^\beta} |d\lambda| \, \|x\|_\beta \\
&\quad \leqslant C \int_\gamma |\lambda|^{\theta - \beta} e^{s \operatorname{Re} \lambda} |t - r|^\alpha \, |d\lambda| \, \|x\|_\beta \leqslant C \frac{|t - r|^\alpha}{s^{1 + \theta - \beta}} \|x\|_\beta,
\end{aligned}$$

where we have used the estimate

$$\frac{\mu^\theta}{|\lambda - \mu|} \leqslant \frac{C}{|\lambda|^{1-\theta}} \qquad \forall \lambda \in \gamma, \forall \mu \in [2, \infty[; \tag{2.9}$$

to prove (2.9) observe that if $\lambda \in \gamma$ and $\mu > 2$ we have $|\lambda - \mu| \geqslant (\mu - 1) \vee |\lambda|$, so that

$$\frac{\mu^\theta}{|\lambda - \mu|} = \frac{\mu^\theta}{|\lambda - \mu|^\theta} \frac{1}{|\lambda - \mu|^{1-\theta}} \leqslant \left(\frac{\mu}{\mu - 1} \right)^\theta \frac{1}{|\lambda|^{1-\theta}} \leqslant \frac{C}{|\lambda|^{1-\theta}}.$$

Hence we have

$$\|(A(t)\, e^{sA(t)} - A(r)\, e^{sA(r)})\, x\|_\theta \leqslant C \frac{|t - r|^\alpha}{s^{1+\theta-\beta}} \|x\|_\beta \; \forall t, r \in [0, T], \forall s \in]0, T]$$

and the result follows. ∎

Finally we have the following inclusion property:

PROPOSITION 2.12. *For each* $\beta \in]0,1]$ *and* $\theta \in [0,\beta[$

$$C([0,T],E) \cap B(0,T,D_{A(0)}(\beta,\infty)) \subseteq C([0,T],D_{A(0)}(\theta))$$

with continuous inclusion.

Proof. Let $g \in C([0,T],E) \cap B(0,T,D_{A(0)}(\beta,\infty))$ and take $\varepsilon > 0$. There exists $\delta_\varepsilon > 0$ such that $\|g(t)-g(r)\|_E < \varepsilon$ provided $|t-r| < \delta_\varepsilon$. Hence if $|t-r| < \delta_\varepsilon$ we get

$$\begin{aligned} \|\xi^{1-\theta}A(0)\,e^{\xi A(0)}(g(t)-g(r))\|_E &\leqslant \frac{C}{\xi^\theta}\cdot\varepsilon \leqslant C\varepsilon^{1-\theta} && \text{if } \xi \geqslant \varepsilon \\ &\leqslant 2\xi^{\beta-\theta}\,\|g\|_{B(0,T,D_{A(0)}(\beta,\infty))} \leqslant C\varepsilon^{\beta-\theta} && \text{if } \xi < \varepsilon, \end{aligned}$$

which implies

$$\|g(t)-g(r)\|_\theta = o(1) \qquad \text{as} \quad |t-r| \to 0. \quad \blacksquare$$

Remark 2.13. In the next sections we shall use the following property: If $u \in B_\beta(]0,T],\ D_{A(0)}(\theta,\infty))$, $\beta \in [0,1[$, $\theta \in]0,1[$, then the function $t \to \|u(t)\|_\theta$ can always be assumed Lebesgue measurable (and therefore integrable over $]0,T]$). Indeed, the real function $t \to \|u(t)\|_E$ is obviously Lebesgue measurable; on the other hand, it is easily seen that another equivalent norm in $D_{A(0)}(\theta,\infty)$ is given by

$$\|x\|^* = \sup_{n\in\mathbb{N}} n^\theta\,\|A(0)\,R(n,A(0))\,x\|_E, \qquad x \in D_{A(0)}(\theta,\infty):$$

now $t \to \|u(t)\|^*$ is a measurable function, since it is the supremum of a countable family of real measurable functions.

3. BASIC LEMMATA

This section contains a list of technical results which analyze in detail the operators and functions appearing in the integral equation (0.2) and in the representation formula (0.3) of the Introduction. We follow the same lines of Section 3 in [1, 2], where a similar sequence of statements is given.

Throughout this section, unless otherwise specified, Hypotheses I and II are assumed. We also recall Convention 2.8 about the symbols $D_{A(0)}(\theta,\infty)$ and $D_{A(0)}(\theta)$ for $\theta = 0, 1$.

(a) *The Function* $t \to L(t,0)\,x = A(t)\,e^{tA(t)}x$

LEMMA 3.1. *We have*:

(i) *If* $x \in E$, *then* $L(\cdot, 0)\, x \in C^{\alpha}(]0, T], D(A(0)))$;

(ii) $L(\cdot, 0) \in \mathscr{L}(D_{A(0)}(\theta, \infty),\ B_{1-\theta+\sigma}(0, T, D_{A(0)}(\sigma, \infty))$ $\forall \sigma \in [0, \theta]$, $\forall \theta \in [0, 1]$;

(iii) $L(\cdot, 0) \in \mathscr{L}(D, C([0, T], E))$ *and in particular* $L(0, 0)x = A(0)x$ $\forall x \in D$;

(iv) $L(\cdot, 0) \in \mathscr{L}(D_{A(0)}(\theta + 1, \infty), B(0, T, D_{A(0)}(\theta)))$ $\forall \theta \in [0, \alpha]$;

(v) $L(\cdot, 0) \in \mathscr{L}(D_{A(0)}(\theta + 1), C([0, T], D_{A(0)}(\theta)))$ $\forall \theta \in]0, \alpha[$;

(vi) $L(\cdot, 0) \in \mathscr{L}(D_{A(0)}(\theta + 1, \infty), C^{\theta}([0, T], E))$ $\forall \theta \in]0, \alpha]$;

(vii) $L(\cdot, 0) \in \mathscr{L}(D_{A(0)}(\theta + 1), h^{\theta}([0, T], E))$ $\forall \theta \in]0, \alpha[$.

Proof. (i) If $x \in E$ and $t \in]0, T]$ then evidently $L(t, 0)\, x \in D(A(0))$. Moreover if $0 < \varepsilon \leqslant r < t \leqslant T$ we have:

$$\begin{aligned}
&\|L(t, 0)\, x - L(r, 0)\, x\|_{D(A(0))} \\
&\quad \leqslant \|A(t)\, e^{tA(t)}x - A(r)\, e^{tA(r)}x\|_{D(A(0))} \\
&\qquad + \|A(r)\, e^{tA(r)}x - A(r)\, e^{rA(r)}\, x\|_{D(A(0))} \\
&\quad \leqslant \|A(t)\, e^{tA(t)} - A(r)\, e^{tA(r)}\|_{\mathscr{L}(E, D(A(0)))}\, \|x\|_E \\
&\qquad + \left\| A(0) \int_r^t A(r)^2\, e^{sA(r)}\, x\, ds \right\|_E \qquad (3.1)\\
&\quad \leqslant \frac{C}{t^2}\, (t - r)^{\alpha}\, \|x\|_E + \|A(0)\, A(r)^{-1}\|_{\mathscr{L}(E)} \left\| \int_r^t A(r)^3\, e^{sA(r)}\, x\, ds \right\|_E \\
&\quad \leqslant \frac{C}{t^2}\, (t - r)^{\alpha}\, \|x\|_E + C \int_r^t \frac{ds}{s^3}\, \|x\|_E \leqslant C \left\{ \frac{1}{\varepsilon^2}\, (t - r)^{\alpha} + \frac{1}{\varepsilon^3}\, (t - r) \right\} \|x\|_E,
\end{aligned}$$

where we have used Lemma 2.11(ii) in the case $\theta = 1, \beta = 0$.

(ii) We just consider the case $\theta \in]0, 1[$, $\sigma \in]0, \theta]$, since the others are even simpler. By (i), $L(t, 0)\, x \in D_{A(0)}(\sigma)$ for each $t \in]0, T]$ and in addition

$$\begin{aligned}
t^{1-\theta+\sigma}\, \|L(t, 0)\, x\|_{\sigma} &\leqslant \sup_{\xi > 0} t^{1-\theta+\sigma} \xi^{1-\theta}\, \|A(t)^2\, e^{(\xi+t)A(t)}\, x\|_E \\
&\leqslant \sup_{\xi > 0} \frac{t^{1-\theta+\sigma}\, \xi^{1-\sigma}}{(\xi + t)^{2-\theta}}\, \|x\|_{\theta} \leqslant C\, \|x\|_{\theta}.
\end{aligned}$$

(iii) Let $x \in D$. By (i) and (ii) (with $\theta = 1$, $\sigma = 0$) we only have to show that $\lim_{t\to 0^+} \|L(t,0)x - A(0)x\|_E = 0$; by Lemma 2.11(ii) (case $\theta = 0, \beta = 1$) and Proposition 2.7(iii) we have as $t \to 0^+$

$$\|A(t)e^{tA(t)}x - A(0)x\|_E \leqslant \|A(t)e^{tA(t)}x - A(0)e^{tA(0)}x\|_{\mathscr{L}(D(A(0)),E)} \|x\|_{D(A(0))} + \|(e^{tA(0)} - 1)A(0)x\|_E \leqslant Ct^\alpha \|A(0)x\|_E + o(1).$$

(iv) Again, we omit the (obvious) case $\theta = 0$. Let $x \in D_{A(0)}(\theta + 1, \infty)$, $\theta \in]0, \alpha]$. For each $t \in [0, T]$

$$\begin{aligned}\|A(t)e^{tA(t)}x\|_\theta &\leqslant \|e^{tA(t)}(A(t)A(0)^{-1} - 1)A(0)x\|_\theta + \|e^{tA(t)}A(0)x\|_\theta \\ &\leqslant \sup_{\xi>0} \|\xi^{1-\theta}A(t)e^{(\xi+t)A(t)}(A(t)A(0)^{-1} - 1)A(0)x\|_E \\ &\quad + \sup_{\xi>0} \|\xi^{1-\theta}A(t)e^{(\xi+t)A(t)}A(0)x\|_E \\ &\leqslant C\frac{\xi^{1-\theta}}{\xi+t} t^\alpha \|A(0)x\|_E + C\|A(0)x\|_\theta \leqslant C\|A(0)x\|_\theta.\end{aligned}$$

(v) Let $\theta \in]0, \alpha[$. By (i) and (iv) we only need to prove that $\lim_{t\to 0^+} \|L(t,0)x - A(0)x\|_\theta = 0$. By Lemma 2.11(ii) (case $\theta \in [0, \alpha], \beta = 1$) and Proposition 2.6(i) we get as $t \to 0^+$:

$$\|A(t)e^{tA(t)}x - A(0)x\|_\theta \leqslant \|A(t)e^{tA(t)} - A(0)e^{tA(0)}\|_{\mathscr{L}(D(A(0)),D_{A(0)}(\theta))} \|x\|_{D(A(0))} + \|(e^{tA(0)} - 1)A(0)x\|_\theta \leqslant Ct^{\alpha-\theta} \|A(0)x\|_E + o(1).$$

(vi) Let $x \in D_{A(0)}(\theta + 1, \infty)$, $\theta \in]0, \alpha]$. If $0 \leqslant r < t \leqslant T$ we have

$$\begin{aligned}&\|A(t)e^{tA(t)}x - A(r)e^{rA(r)}x\|_E \\ &\leqslant \|A(t)e^{tA(t)}x - A(r)e^{tA(r)}x\|_E + \left\|\int_r^t A(r)^2 e^{sA(r)}x\,ds\right\|_E \\ &\leqslant \|A(t)e^{tA(t)} - A(r)e^{tA(r)}\|_{\mathscr{L}(D(A(0)),E)} \|x\|_{D(A(0))} \\ &\quad + \left\|\int_r^t A(r)^2 e^{sA(r)}(A(r)A(0)^{-1} - 1)A(0)x\,ds\right\|_E \\ &\quad + \left\|\int_r^t A(r)e^{sA(r)}A(0)x\,ds\right\|_E \leqslant C\left\{(t-r)^\alpha + \int_r^t \frac{r^\alpha}{s}ds\right\} \|A(0)x\|_E \\ &\quad + C\int_r^t \frac{ds}{s^{1-\theta}} \|A(0)x\|_\theta \leqslant C(t-r)^\alpha \|A(0)x\|_E + C(t-r)^\alpha \|A(0)x\|_\theta.\end{aligned} \tag{3.2}$$

(vii) Let $x \in D_{A(0)}(\theta+1)$, $\theta \in]0, \alpha[$. By (i) and (vi) we have to show that if $0 \leqslant r < t \leqslant T$, then $\|L(t,0)x - L(r,0)x\|_E = o(t-r)^\theta$ as $t - r \to 0^+$. As in (3.2) we check

$$\|A(t)\,e^{tA(t)}x - A(r)\,e^{rA(r)}x\|_E \leqslant C(t-t)^\alpha\,\|A(0)\,x\|_E + \left\|\int_r^t A(r)\,e^{sA(r)}A(0)\,xds\right\|_E;$$

now by Proposition 2.5(i) we have as $t - r \to 0^+$:

$$\left\|\int_r^t A(r)\,e^{sA(r)}A(0)\,xds\right\|_E = \left\|e^{rA(r)}\int_r^t (s-r)^{1-\theta}A(r)\,e^{(s-r)A(r)}A(0)\,x\frac{ds}{(s-r)^{1-\theta}}\right\|_E \leqslant c\int_r^t \frac{ds}{(s-r)^{1-\theta}}\,o(1) = o(t-r)^\theta,$$

and the result follows. ∎

(b) *The Operator* $Lg(t) = \int_0^t A(t)\,e^{(t-s)A(t)}\,g(s)\,ds$

LEMMA 3.2. *We have*:

(i) $L \in \mathscr{L}(B_\beta(0, T, D_{A(0)}(\theta, \infty)))$ $\forall\beta \in [0, 1[$, $\forall\theta \in]0, 1[$;

(ii) $L \in \mathscr{L}(C_\beta(]0, T], D_{A(0)}(\theta)))$ $\forall\beta \in [0, 1[$, $\forall\theta \in]0, 1[$;

(iii) $L \in \mathscr{L}(C([0, T], D_{A(0)}(\theta)))$ $\forall\theta \in]0, 1[$; *in particular* $Lg(0) = 0$ $\forall g \in C([0, T], D_{A(0)}(\theta))$;

(iv) $L \in \mathscr{L}(B(0, T, D_{A(0)}(\theta, \infty)), C^\theta([0, T], E))$ $\forall\theta \in]0, \alpha]$;

(v) $L \in \mathscr{L}(C([0, T], D_{A(0)}(\theta)), h^\theta([0, T], E))$ $\forall\theta \in]0, \alpha[$;

(vi) *If* $g \in B_\beta(0, T, D_{A(0)}(\theta, \infty))$, $\beta \in]0, 1[$, $\theta \in]0, \alpha]$, *then* $Lg \in C^\theta(]0, T], E)$;

(vii) *If* $g \in C_\beta(]0, T]$, $D_{A(0)}(\theta))$, $\beta \in [0, 1[$, $\theta \in]0, \alpha[$, *then* $Lg \in h^\theta(]0, T], E)$.

Proof (i) For each $\xi > 0$ and $t \in]0, T]$ we have

$$\|t^\beta \xi^{1-\theta} A(t)\,e^{\xi A(t)} Lg(t)\|_E = \left\|t^\beta \xi^{1-\theta}\int_0^t A(t)^2\,e^{(\xi+t-s)A(t)}\,g(s)\,ds\right\|_E$$

$$\leqslant Ct^\beta \xi^{1-\theta} \int_0^t \frac{ds}{(\xi+t-s)^{2-\theta} s^\beta} \| g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))}$$

$$\leqslant Ct^\beta \xi^{1-\theta} \left[\frac{1}{(\xi+\frac{1}{2}t)^{2-\theta}} \int_0^{t/2} \frac{ds}{s^\beta} + \frac{1}{(\frac{1}{2}t)^\beta} \int_{t/2}^t \frac{ds}{(\xi+t-s)^{2-\theta}} \right]$$

$$\times \| g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))}$$

$$\leqslant C \| g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))}.$$

(ii) By (i) we have only to show that $Lg \in C(]0,T], D_{A(0)}(\theta))$ whenever $g \in C_\beta(]0,T], D_{A(0)}(\theta))$. Let us first prove that $Lg(t) \in D_{A(0)}(\theta)$ for each $t \in]0,T]$; we have

$$\lim_{\xi\to 0^+} \|\xi^{1-\theta} A(t)\, e^{\xi A(t)} Lg(t)\|_E$$

$$\leqslant \lim_{\xi\to 0^+} \left\| \int_0^t \xi^{1-\theta} A(t)^2 e^{(\xi+t-s)A(t)} (g(s)-g(t))\, ds \right\|_E$$

$$+ \lim_{\xi\to 0^+} \|\xi^{1-\theta} A(t)\, e^{\xi A(t)} (e^{tA(t)} - 1)\, g(t)\|_E.$$

The second term on the right-hand side is zero since $g(t) \in D_{A(0)}(\theta)$; let us verify that the first term also vanishes. Take $\varepsilon > 0$; for each $\eta \in]0,t[$ there exists $\delta(\varepsilon,\eta) \in]0, t-\eta[$ such that $\| g(s) - g(t)\|_\theta < \varepsilon$ provided $s, t \in [\eta, T]$ and $|t-s| < \delta$. Hence

$$\left\| \int_0^t \xi^{1-\theta} A(t)^2 e^{(\xi+t-s)A(t)} (g(s)-g(t))\, ds \right\|_E$$

$$\leqslant C \left\{ \int_0^\eta \frac{\xi^{1-\theta}}{(\xi+t-s)^{2-\theta}} \| g(s)-g(t)\|_\theta\, ds \right.$$

$$+ \int_\eta^{t-\delta} \frac{\xi^{1-\theta}}{(\xi+t-s)^{2-\theta}} \| g\|_{C([\eta,T],D_{A(0)}(\theta))} ds$$

$$\left. + \int_{t-\delta}^t \frac{\xi^{1-\theta}}{(\xi+t-s)^{2-\theta}} \|g(s)-g(t)\|_\theta\, ds \right\}$$

$$\leqslant C \frac{\xi^{1-\theta}}{(\xi+t-\eta)^{2-\theta}} \int_0^\eta \frac{ds}{s^\beta} \| g\|_{C_\beta(]0,T],D_{A(0)}(\theta))}$$

$$+ C \frac{\xi^{1-\theta}}{(\xi+\delta)^{2-\theta}} (t-\delta-\eta) \| g\|_{C([\eta,T],D_{A(0)}(\theta))}$$

$$+ C\varepsilon \xi^{1-\theta} \left[\frac{1}{\xi^{1-\theta}} - \frac{1}{(\xi+\delta)^{1-\theta}} \right],$$

which implies

$$\limsup_{\xi\to 0^+}\left\|\int_0^t \xi^{1-\theta}A(t)^2 e^{(\xi+t-s)A(t)}(g(s)-g(t))\,ds\right\|_E \leqslant C\varepsilon \qquad \forall\varepsilon>0.$$

Next, we have to show that if $0<\eta\leqslant r<t\leqslant T$

$$\sup_{\xi>0}\|\xi^{1-\theta}A(t)\,e^{\xi A(t)}(Lg(t)-Lg(r))\|_E=o(1) \qquad \text{as} \quad t-r\to 0^+.$$

We write

$$\begin{aligned}
Lg(t)-Lg(r)=&\int_r^t A(t)\,e^{(t-s)A(t)}(g(s)-g(t))\,ds+(e^{(t-r)A(t)}-1)\,g(t)\\
&+\int_0^r (A(t)\,e^{(t-s)A(t)}-A(r)\,e^{(t-s)A(r)})\,g(s)\,ds\\
&+\int_0^r\int_{r-s}^{t-s} A(r)^2\,e^{qA(r)}(g(s)-g(r))\,dqds\\
&+(e^{rA(r)}-1)(e^{(t-r)A(r)}-1)\,g(r)=A_1+A_2+A_3+A_4+A_5.
\end{aligned}$$

We estimate each term as follows:

$$\begin{aligned}
\|A_1\|_\theta &\leqslant \sup_{\xi>0}\left\|\xi^{1-\theta}\int_r^t A(t)^2\,e^{(\xi+t-s)A(t)}(g(s)-g(t))\,ds\right\|_E\\
&\leqslant C\sup_{\xi>0}\int_r^t \frac{\xi^{1-\theta}}{(\xi+t-s)^{2-\theta}}\,\|g(s)-g(t)\|_\theta ds=o(1) \qquad \text{as } t-r\to 0^+,
\end{aligned}$$

since $g\in C([\eta,T],D_{A(0)}(\theta))$;

$$\|A_2\|_\theta=o(1) \qquad \text{as } t-r\to 0^+ \qquad \text{(Proposition 2.6(i))};$$

$$\begin{aligned}
\|A_3\|_\theta &\leqslant \int_0^r \|A(t)\,e^{(t-s)A(t)}-A(r)\,e^{(t-s)A(r)}\|_{\mathscr{L}(D_{A(0)}(\theta,\infty),D_{A(0)}(\theta))}\,\|g(s)\|_\theta ds\\
&\leqslant C\int_0^r \frac{(t-r)^\alpha}{t-s}\,\frac{1}{s^\beta}\,ds\,\|g\|_{C_\beta(]0,T],D_{A(0)}(\theta))}\\
&= C\left\{\int_0^{\eta/2}\cdots+\int_{\eta/2}^r\cdots ds\right\}\|g\|_{C_\beta(]0,T],D_{A(0)}(\theta))}\\
&\leqslant C\left\{\frac{1}{\frac{1}{2}\eta}(t-r)^\alpha\int_0^{\eta/2}\frac{ds}{s^\beta}+\frac{1}{(\frac{1}{2}\eta)^\beta}(t-r)^\alpha\log\left(1+\frac{T}{t-r}\right)\right\}\\
&\quad\times\|g\|_{C_\beta(]0,T],D_{A(0)}(\theta))}=o(1) \qquad \text{as } t-r\to 0^+,
\end{aligned}$$

where we have used Lemma 2.11(ii);

$$\|A_5\|_\theta \leqslant \sup_{\xi>0} \left\| \frac{e^{\xi A(r)}-1}{\xi^\theta} (e^{rA(r)}-1)(e^{(t-r)A(r)}-1)\, g(r) \right\|_E$$
$$\leqslant C \| (e^{(t-r)A(r)}-1)\, g(r)\|_\theta = o(1) \qquad \text{as } t-r\to 0^+$$

by Proposition 2.6(ii).

Finally to estimate A_4 let $\varepsilon>0$ and choose $\delta(\varepsilon,\eta)\in\,]0,\tfrac{1}{2}\eta[$ such that $\| g(s)-g(r)\|_\theta<\varepsilon$ provided $s\in[\eta,T]$ and $|r-s|<\delta$. Then we have

$$\|A_4\|_\theta \leqslant \sup_{\xi>0} \left\| \xi^{1-\theta} \int_0^r \int_{r-s}^{t-s} A(r)^3 e^{(q+\xi)A(r)} (g(s)-g(r))\, dq\, ds \right\|_E$$
$$\leqslant C \sup_{\xi>0} \xi^{1-\theta} \int_0^r \int_{r-s}^{t-s} \frac{1}{(q+\xi)^{3-\theta}} \| g(s)-g(r)\|_\theta \, dq\, ds$$
$$= C \sup_{\xi>0} \left\{ \xi^{1-\theta} \left[\int_0^{\eta/2} \cdots + \int_{\eta/2}^{r-\delta} \cdots + \int_{r-\delta}^{r} \cdots ds \right] \right\}$$
$$\leqslant C \sup_{\xi>0} \left\{ C(\eta) \frac{\xi^{1-\theta}(t-r)}{(\frac{1}{2}\eta+\xi)^{3-\theta}} \| g\|_{C_\beta(]0,T],D_{A(0)}(\theta))} \right.$$
$$+ C \frac{\xi^{1-\theta}(t-r)}{(\delta+\xi)^{3-\theta}} \| g\|_{C([\eta/2,T],D_{A(0)}(\theta))}$$
$$\left. + C\varepsilon \xi^{1-\theta} \int_{r-\delta}^{r} \left[\frac{1}{(r-s+\xi)^{2-\theta}} - \frac{1}{(t-s+\xi)^{2-\theta}} \right] ds \right\}$$
$$\leqslant C(\eta,\varepsilon)(t-r) + C\varepsilon,$$

and consequently $\|A_4\|_\theta = o(1)$ as $t-r\to 0^+$.

This shows that $Lg\in C([\eta,T], D_{A(0)}(\theta))$ and the result follows.

(iii) The proof is similar to the proof of (ii), but simpler, and we omit it.

(iv) Let $0\leqslant r<t\leqslant T$. Then

$$\|Lg(t)-Lg(r)\|_E \leqslant \left\| \int_r^t A(t)\, e^{(t-s)A(t)} g(s)\, ds \right\|_E$$
$$+ \left\| \int_0^r (A(t)\, e^{(t-s)A(t)} - A(r)\, e^{(t-s)A(r)})\, g(s)\, ds \right\|_E$$
$$+ \left\| \int_0^r \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} g(s)\, dq\, ds \right\|_E, \tag{3.3}$$

which implies

$$\begin{aligned}
&\|Lg(t)-Lg(r)\|_E \\
&\quad\leqslant C\int_r^t \frac{1}{(t-s)^{1-\theta}}\,\|g(s)\|_\theta\,ds \\
&\qquad+\int_0^r \|A(t)\,e^{(t-s)A(t)}-A(r)\,e^{(t-s)A(r)}\|_{\mathscr{L}(D_{A(0)}(\theta,\infty),E)}\,\|g(s)\|_\theta\,ds \\
&\qquad+C\int_0^r\int_{r-s}^{t-s}\frac{dq}{q^{2-\theta}}\,\|g(s)\|_\theta\,ds\leqslant C\left\{(t-r)^\theta+(t-r)^\alpha\int_0^r\frac{ds}{(t-s)^{1-\theta}}\right. \\
&\qquad\left.+\int_0^r\left[\frac{1}{(r-s)^{1-\theta}}-\frac{1}{(t-s)^{1-\theta}}\right]ds\right\}\,\|g\|_{B(0,T,D_{A(0)}(\theta,\infty))} \\
&\quad\leqslant C(t-r)^\theta\,\|g\|_{B(0,T,D_{A(0)}(\theta,\infty))}.
\end{aligned}$$

(v) As in (iv) we get for $0\leqslant r<t\leqslant T$

$$\begin{aligned}
\|Lg(t)-Lg(r)\|_E\leqslant&\left\|\int_r^t A(t)\,e^{(t-s)A(t)}g(s)\,ds\right\|_E \\
&+C(t-r)^\alpha\int_0^r\frac{ds}{(t-s)^{1-\theta}}\,\|g\|_{B(0,T,D_{A(0)}(\theta,\infty))} \\
&+\left\|\int_0^r\int_{r-s}^{t-s}A(r)^2\,e^{qA(r)}g(s)\,dq\,ds\right\|_E. \qquad (3.4)
\end{aligned}$$

The first term can be rewritten as

$$\left\|\int_r^t A(t)\,e^{(t-s)A(t)}(g(s)-g(t))\,ds+(e^{(t-r)A(t)}-1)\,g(t)\right\|_E; \qquad (3.5)$$

now as $t-r\to 0^+$

$$\begin{aligned}
\left\|\int_r^t A(t)\,e^{(t-s)A(t)}(g(s)-g(t))\,ds\right\|_E&\leqslant\int_r^t\frac{1}{(t-s)^{1-\theta}}\,\|g(s)-g(t)\|_\theta\,ds \\
&=o(t-r)^\theta,
\end{aligned}$$

and by Proposition 2.5(ii) we have $\|(e^{(t-r)A(t)}-1)g(t)\|_E=o(t-r)^\theta$ as $t-r\to 0^+$, so that

$$\left\|\int_r^t A(t)\,e^{(t-s)A(t)}g(s)\,ds\right\|_E=o(t-r)^\theta \qquad \text{as } t-r\to 0^+. \qquad (3.6)$$

Finally the third term in (3.4) becomes

$$\left\| \int_0^{r-\delta} \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} g(s)\, dq ds + \int_{r-\delta}^{r} \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} (g(s) - g(r))\, dq ds \right.$$
$$\left. + (e^{\delta A(r)} - 1)(e^{(t-r)A(r)} - 1)\, g(r) \right\|_E, \quad (3.7)$$

where $\delta = \delta(\varepsilon)$ is such that $\| g(s) - g(r) \|_\theta < \varepsilon$ as $|s-r| < \delta$. Now we have

$$\begin{aligned}
&\left\| \int_0^{r-\delta} \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} g(s)\, dq ds \right\|_E \\
&\quad = \left\| \int_0^{r-\delta} A(r)\, e^{(r-s)A(r)} \int_{r-s}^{t-s} A(r)\, e^{(q-r+s)A(r)} g(s)\, dq ds \right\|_E \\
&\quad \leqslant C \int_0^{r-\delta} \frac{1}{r-s} \int_{r-s}^{t-s} \frac{1}{(q-r+s)^{1-\theta}} \| (q-r+s)^{1-\theta} A(r) \\
&\qquad \times e^{(q-r+s)A(r)} g(s) \|_E \, dq ds \\
&\quad \leqslant C \frac{T}{\delta} o(t-r)^\theta = o(t-r)^\theta \qquad \text{as } t-r \to 0^+, \\
&\left\| \int_{r-\delta}^{r} \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} (g(s) - g(r))\, dq ds \right\|_E \\
&\quad \leqslant \int_{r-\delta}^{r} \int_{r-s}^{t-s} \frac{dq}{q^{2-\theta}} \| g(s) - g(r) \|_\theta \, ds \leqslant C\varepsilon (t-r)^\theta \qquad \text{if } t-r < \delta,
\end{aligned}$$

and by Proposition 2.5(ii)

$$\| (e^{\delta A(r)} - 1)(e^{(t-r)A(r)} - 1)\, g(r) \|_E = o(t-r)^\theta \qquad \text{as } t-r \to 0^+.$$

Hence we have

$$\left\| \int_0^{r} \int_{r-s}^{t-s} A(r)^2 e^{qA(r)} g(s)\, dq ds \right\|_E = o(t-r)^\theta \qquad \text{as } t-r \to 0^+. \quad (3.8)$$

By (3.4), (3.6) and (3.8) we finally check

$$\| Lg(t) - Lgr(r) \|_E = o(t-r)^\theta \qquad \text{as } t-r \to 0^+.$$

(vi) Let $\varepsilon \leqslant r < t \leqslant T$. As in (iv) we get (3.3); from (3.3) we derive

$$\begin{aligned}
\|Lg(t)-Lg(r)\|_E &\leqslant C\int_r^t \frac{1}{(t-s)^{1-\theta}}\|g(s)\|_\theta\,ds + C(t-r)^\alpha \\
&\quad\times\int_0^r \frac{1}{(t-s)^{1-\theta}}\|g(s)\|_\theta\,ds + C\int_0^r\int_{r-s}^{t-s}\frac{1}{q^{2-\theta}}\|g(s)\|_\theta\,ds \\
&\leqslant C(t-r)^\theta\,\|g\|_{B(\epsilon,T,D_{A(0)}(\theta,\infty))} \\
&\quad + C(t-r)^\alpha\int_0^r \frac{ds}{(t-s)^{1-\theta}s^\beta}\|g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))} \\
&\quad + C\frac{t-r}{(\frac{1}{2}\varepsilon)^{2-\theta}}\int_0^{\epsilon/2}\frac{ds}{s^\beta}\|g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))} \\
&\quad + C\int_{\epsilon/2}^r\left[\frac{1}{(r-s)^{1-\theta}}-\frac{1}{(t-s)^{1-\theta}}\right]\|g\|_{B(\epsilon/2,T,D_{A(0)}(\theta,\infty))} \\
&\leqslant C(\varepsilon)(t-r)^\theta.
\end{aligned}$$

(vii) Let $\eta \leqslant r < t \leqslant T$. We start from (3.3); since $g \in C([\eta, T], D_{A(0)}(\theta))$, the first term on the right-hand side of (3.3) can be estimated as in (v), so that (3.6) holds. The second term can be treated as in (vi), obtaining

$$\begin{aligned}
&\left\|\int_0^r (A(t)\,e^{(t-s)A(t)} - A(r)\,e^{(t-s)A(r)})\,g(s)\,ds\right\|_E \\
&\qquad\leqslant C(t-r)^\alpha\int_0^r\frac{ds}{(t-s)^{1-\theta}s^\beta}\|g\|_{C_\beta(]0,T],D_{A(0)}(\theta))} \\
&\qquad= o(t-r)^\theta \qquad \text{as } t-r\to 0^+. \qquad (3.9)
\end{aligned}$$

To estimate the third term in (3.3), we split it as in (3.7), where now $\delta=\delta(\varepsilon,\eta)\in\,]0,\frac{1}{2}\eta[$ is such that $\|g(s)-g(r)\|_\theta<\varepsilon$ as $|s-r|<\delta$ (this is possible because $g\in C([\frac{1}{2}\eta, T], D_{A(0)}(\theta)))$. The three terms in (3.7) are estimated as follows:

$$\begin{aligned}
&\left\|\int_0^{r-\delta}\int_{r-s}^{t-s} A(r)^2\,e^{qA(r)}g(s)\,dq ds\right\|_E \\
&\quad\leqslant C\int_0^{r-\delta}\frac{t-r}{(\frac{1}{2}\eta)^{2-\theta}}\frac{ds}{s^\beta}\|g\|_{C_\beta(]0,T],D_{A(0)}(\theta))} = o(t-r)^\theta \qquad \text{as}\quad t-r\to 0^+,
\end{aligned}$$

$$\left\|\int_{r-\delta}^{r}\int_{r-s}^{t-s} A(r)^2\, e^{qA(r)}(g(s)-g(r))\,dq\,ds\right\|_E$$

$$\leqslant \int_{r-\delta}^{r}\int_{r-s}^{t-s}\frac{dq}{q^{2-\theta}}\|g(s)-g(r)\|_\theta\,ds\leqslant C\varepsilon(t-r)^\theta \quad \text{if} \quad t-r<\delta,$$

$$\|(e^{\delta A(r)}-1)(e^{(t-r)A(r)}-1)\,g(r)\|_E=o(t-r)^\theta \quad \text{as} \quad t-r\to 0^+.$$

This gives (3.8). By (3.6), (3.9) and (3.8) we get the result. ∎

(c) *The Operator* $Hg(t)=\int_0^t A(t)\,e^{(t-s)A(t)}(1-A(t)\,A(s)^{-1})\,g(s)\,ds$

LEMMA 3.3. *We have*:

(i) $H\in\mathscr{L}(B_\beta(0,T,E),B_\beta(0,T,D_{A(0)}(\alpha,\infty)))\ \forall\beta\in[0,1[$;

(ii) *If* $g\in B_\beta(0,T,E)$, $\beta\in[0,1[$, *then* $Hg\in C^\sigma(]0,T],E)$ $\forall\sigma\in\,]0,\alpha[$;

(iii) *If* $g\in B_\beta(0,T,E)\cap C^\delta(]0,T],E)$, $\beta\in[0,1[$, $\delta\in\,]0,1[$, *then* $Hg\in C^\alpha(]0,T],E)$;

(iv) $H\in\mathscr{L}(C_\beta(]0,T],E),C_\beta(]0,T],D_{A(0)}(\theta)))$ $\forall\beta\in[0,1[$, $\forall\theta\in[0,\alpha[$;

(v) $H\in\mathscr{L}(B(0,T,E),\ C^\sigma([0,T],E))\ \forall\sigma\in\,]0,\alpha[$; *in particular* $Hg(0|=0\ \forall g\in B(0,T,E)$;

(vi) $H\in\mathscr{L}(C^\delta([0,T],E),C^\alpha([0,T],E))\ \forall\delta\in\,]0,1[$;

(vii) $H\in\mathscr{L}(C([0,T],E),C([0,T],D_{A(0)}(\theta)))\ \forall\theta\in[0,\alpha[$.

Proof. (i) For each $\xi>0$ and $t\in\,]0,T]$ we have

$$\|t^\beta\xi^{1-\alpha}A(t)\,e^{\xi A(t)}Hg(t)\|_E$$

$$=\left\|t^\beta\xi^{1-\alpha}\int_0^t A(t)^2\,e^{(\xi+t-s)A(t)}(1-A(t)\,A(s)^{-1})\,g(s)\,ds\right\|_E$$

$$\leqslant C\,t^\beta\xi^{1-\alpha}\int_0^t\frac{(t-s)^\alpha}{(\xi+t-s)^2}\|g(s)\|_E\,ds$$

$$\leqslant C\,t^\beta\xi^{1-\alpha}\int_0^t\frac{ds}{(\xi+t-s)^{2-\alpha}s^\beta}\|g\|_{B_\beta(0,T,E)}$$

$$=C t^\beta\xi^{1-\alpha}\left[\int_0^{t/2}\cdots+\int_{t/2}^t\cdots ds\right]\|g\|_{B_\beta(0,T,E)}\leqslant C\,\|g\|_{B_\beta(0,T,E)}.$$

(ii) We have to show that if $g\in B_\beta(0,T,E)$ and $\varepsilon\in\,]0,T[$ then $Hg\in C^\sigma([\varepsilon,T],E)$ for each $\sigma\in\,]0,\alpha[$. Now if $\varepsilon\leqslant r<t\leqslant T$ we have

$$\begin{aligned}
&\|Hg(t)-Hg(r)\|_E\\
&\quad\leqslant \left\|\int_r^t A(t)\,e^{(t-s)A(t)}(1-A(t)\,A(s)^{-1})\,g(s)\,ds\right\|_E\\
&\qquad+\left\|\int_0^r A(t)\,e^{(t-s)A(t)}(1-A(t)\,A(r)^{-1})\,A(r)\,A(s)^{-1}\,g(s)\,ds\right\|_E\\
&\qquad+\left\|\int_0^r (A(t)\,e^{(t-s)A(t)}-A(r)\,e^{(t-s)A(r)})(1-A(r)\,A(s)^{-1})\,g(s)\,ds\right\|_E\\
&\qquad+\left\|\int_0^r\int_{r-s}^{t-s} A(r)^2\,e^{qA(r)}(1-A(r)\,A(s)^{-1})\,g(s)\,ds\right\|_E.
\end{aligned}\tag{3.10}$$

Hence, using Lemma 2.11(ii) we get

$$\begin{aligned}
&\|Hg(t)-Hg(r)\|_E\\
&\leqslant C\int_r^t \frac{ds}{(t-s)^{1-\alpha}}\,\|g\|_{B(\epsilon,T,E)}+C(t-r)^\alpha\int_0^r\frac{ds}{(t-s)\,s^\beta}\,\|g\|_{B_\beta(0,T,E)}\\
&\quad+C(t-r)^\alpha\int_0^r\frac{(r-s)^\alpha\,ds}{(t-s)\,s^\beta}\,\|g\|_{B_\beta(0,T,E)}\\
&\quad+C\int_0^r\int_{r-s}^{t-s}\frac{(r-s)^\alpha\,ds}{q^2\,s^\beta}\,\|g\|_{B_\beta(0,T,E)}\\
&\leqslant C(\varepsilon)\left\{(t-r)^\alpha+(t-r)^\alpha\log\left(1+\frac{T}{t-r}\right)\right\}=O(t-r)^\sigma\qquad\text{as } t-r\to 0^+.
\end{aligned}$$

(iii) Let $\varepsilon\leqslant r<t\leqslant T$. As in (ii) we obtain (3.10) and similarly we deduce that

$$\begin{aligned}
&\|Hg(t)-Hg(r)\|_E\\
&\quad\leqslant C\int_r^t\frac{ds}{(t-s)^{1-\alpha}}\,\|g\|_{B(\epsilon,T,E)}\\
&\qquad+\left\|\int_0^r A(t)\,e^{(t-s)A(t)}(1-A(t)\,A(r)^{-1})\,A(r)\,A(s)^{-1}g(s)\,ds\right\|_E\\
&\qquad+C(t-r)^\alpha\int_0^r\frac{(r-s)^\alpha\,ds}{(t-s)\,s^\beta}\,\|g\|_{B_\beta(0,T,E)}\\
&\qquad+C\int_0^r\int_{r-s}^{t-s}\frac{(r-s)^\alpha\,dq ds}{q^2\,s^\beta}\,\|g\|_{B_\beta(0,T,E)}
\end{aligned}$$

$$\leqslant C(\varepsilon)(t-r)^{\alpha}$$

$$+\left\|\int_0^r A(t)\, e^{(r-s)A(t)}(1-A(t)\,A(r)^{-1})\,A(r)\,A(s)^{-1}(g(s)-g(r))\,ds\right\|_E$$

$$+\left\|\int_0^r A(t)\, e^{(t-s)A(t)}(1-A(t)\,A(r)^{-1})(A(r)\,A(s)^{-1}-1)\,g(r)\,ds\right\|_E$$

$$+\|(e^{tA(t)}-e^{(t-r)A(t)})(1-A(t)\,A(r)^{-1})\,g(r)\|_E$$

$$\leqslant C(\varepsilon)(t-r)^{\alpha}+\left\|\left\{\int_0^{\varepsilon/2}+\int_{\varepsilon/2}^r\right\} A(t)\, e^{(t-s)A(t)}(1-A(t)\,A(r)^{-1})\right.$$

$$\left.\times A(r)\,A(s)^{-1}(g(s)-g(r))\,ds\right\|_E$$

$$+\,C(t-r)^{\alpha}\int_0^r \frac{(r-s)^{\alpha}}{t-s}\,ds\,\|g\|_{B(\varepsilon,T,E)}+C(t-r)^{\alpha}\,\|g\|_{B(\varepsilon,T,E)}$$

$$\leqslant C(\varepsilon)(t-r)^{\alpha}+C(t-r)^{\alpha}\left\{\int_0^{\varepsilon/2}\frac{ds}{(t-s)\,s^{\beta}}\,\|g\|_{B_\beta(0,T,E)}\right.$$

$$\left.+\int_{\varepsilon/2}^r \frac{(r-s)^{\delta}\,ds}{t-s}\,\|g\|_{C^{\delta}([\varepsilon/2,T],E)}\right\}$$

$$=O(t-r)^{\alpha}\qquad \text{as } t-r\to 0^+.$$

(iv) Since $B(\varepsilon,T,D_{A(0)}(\alpha,\infty))\cap C([\varepsilon,T],E)\subseteq C([\varepsilon,T],D_{A(0)}(\theta))$ with continuous inclusion for each $\theta\in\,]0,\alpha[$ (Proposition 2.12), the result follows by (i) and (ii).

(v) The proof is similar to the proof of (ii) (but simpler) and we omit it.

(vi) Similar to (iii).

(vii) The result follows by (i) (with $\beta=0$), (v) and Proposition 2.12.

(d) *The Operator* $(1-H)^{-1}$

LEMMA 3.4. *We have*:

(i) $(1-H)^{-1}\in\mathscr{L}(B_\beta(0,T,D_{A(0)}(\theta,\infty)))\ \forall\beta\in[0,1[,\ \forall\theta\in[0,\alpha]$;

(ii) $(1-H)^{-1}\in\mathscr{L}(C_\beta([0,T],D_{A(0)}(\theta)))\ \forall\beta\in[0,1[,\ \forall\theta\in[0,\alpha[$;

(iii) $(1-H)^{-1}\in\mathscr{L}(C([0,T],E))$; *in particular* $(1-H)^{-1}\,g(0)=g(0)$ $\forall g\in C([0,T],E)$;

(iv) $(1-H)^{-1}\in\mathscr{L}(C([0,T],D_{A(0)}(\theta)))\ \forall\theta\in[0,\alpha[$;

(v) $(1-H)^{-1}\in\mathscr{L}(C^{\theta}([0,T],E))\ \forall\theta\in\,]0,\alpha]$;

(vi) $(1-H)^{-1}\in\mathscr{L}(h^{\theta}([0,T],E))\ \forall\theta\in\,]0,\alpha[$.

Proof. (i) Suppose first $\theta = 0$. Define

$$\|g\|_b = \sup_{t \in]0,T]} \|t^\beta e^{-bt} g(t)\|_E, \qquad g \in B_\beta(0, T, E), b \geqslant 0;$$

clearly

$$e^{-bT} \|g\|_{B_\beta(0,T,E)} \leqslant \|g\|_b \leqslant \|g\|_{B_\beta(0,T,E)} \qquad \forall g \in B_\beta(0, T, E),$$

and in addition

$$\|Hg\|_b = \sup_{t \in]0,T]} \left\| t^\beta e^{-bt} \int_0^t A(t)\, e^{(t-s)A(t)} (1 - A(t) A(s)^{-1})\, g(s)\, ds \right\|_E$$
$$\leqslant C \sup_{t \in]0,T]} \int_0^t \frac{e^{-b(t-s)}\, ds}{(t-s)^{1-\alpha} s^\beta} \|g\|_b.$$

By Lebesgue's theorem we deduce that $\|Hg\|_b \leqslant \frac{1}{2}\|g\|_b$ for large b, and this implies that $(1-H)^{-1} \in \mathscr{L}(B_\beta(0, T, E))$.

Suppose now $\theta \in]0, \alpha]$. If $h = (1-H)^{-1} g$, we know that, in particular, $h \in B_\beta(0, T, E)$ and

$$h - Hh = g. \tag{3.11}$$

By Lemma 3.3(i), $Hh \in B_\beta(0, T, D_{A(0)}(\alpha, \infty))$ and $\|Hh\|_{B_\beta(0,T,D_{A(0)}(\alpha,\infty))} \leqslant C\|h\|_{B_\beta(0,T,E)}$, and by (3.11) we conclude that $h = Hh - g \in B_\beta(0, T, D_{A(0)}(\theta, \infty))$ and

$$\|h\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))} \leqslant C\, \|Hh\|_{B_\beta(0,T,D_{A(0)}(\alpha,\infty))} + \|g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))}$$
$$\leqslant C\, \|g\|_{B_\beta(0,T,D_{A(0)}(\theta,\infty))}.$$

(ii) As in (i) it is easily seen that $(1-H)^{-1} \in \mathscr{L}(C_\beta(]0, T], E)$. Thus if $\theta \in]0, \alpha[$ and $g \in C_\beta(]0, T], D_{A(0)}(\theta))$ we have $h = (1-H)^{-1} g \in C_\beta(]0, T], E)$; hence by Lemma 3.3(iv) we obtain $Hh \in C_\beta(]0, T), D_{A(0)}(\theta))$, and (3.11) yields the result.

(iii) Quite similar to (i).

(iv) If $g \in C([0, T], D_{A(0)}(\theta))$ we have $h \in C([0, T], E)$ by (iii); thus Lemma 3.3(vii) and (3.11) lead to the result.

(v) Let $g \in C^\theta([0, T], E)$; as $h \in C([0, T], E)$, Lemma 3.3(v) gives $Hh \in C^\sigma([0, T], E)$ for each $\sigma \in]0,\alpha[$, which implies $h \in C^{\sigma \wedge \theta}([0, T], E)$ for each $\sigma \in]0,\alpha[$. By Lemma 3.3(vi) and (3.11) we get $h \in C^\theta([0, T], E)$. The estimate for $(1-H)^{-1}$ follows also easily.

(vi) Similar to (v). ∎

4. Strict and Classical Solutions

It is proved in [2] that under Hypotheses I, II Problem (P) has a unique strong solution u, provided $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$. We want to prove now that if x and f are more regular then u is a strict, or classical, solution of (P).

First of all we will show that formula (0.1) necessarily holds for a strict solution; next, we will verify that (0.1) really is the required solution. In this section, as usual, we will always assume that Hypotheses I and II are satisfied.

We recall the definition of the space D, given in Proposition 2.7(iii) (see also Convention 2.8): $D = \{x \in D(A(0)) : A(0)\, x \in \overline{D(A(0))}\}$.

We have:

THEOREM 4.1 (A priori representation formula). *Let* $x \in D$ *and* $f \in C([0, T], E) \cap B(0, T, D_{A(0)}(\theta, \infty))$, $\theta \in \,]0, \alpha]$. *Then if* u *is a strict solution of* (P) *the following formula holds*:

$$u(t) = A(t)^{-1}\{(1-H)^{-1}(Lf(\cdot) + L(\cdot, 0)\, x)(t)\}, \qquad t \in [0, T]. \quad (4.1)$$

Proof. We repeat here the argument used in the Introduction, which is now perfectly justified. Fix $t \in \,]0, T[$ and let

$$v(s) = e^{(t-s)A(t)}\, u(s), \qquad s \in [0, t].$$

As u is a strict solution, v is continuously differentiable on $[0, t]$ and

$$v'(s) = -A(t)\, e^{(t-s)A(t)} u(s) + e^{(t-s)A(t)}(A(s)\, u(s) + f(s)), \qquad s \in [0, t];$$

hence, integration over $[0, t]$ yields

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}(1 - A(t)\, A(s)^{-1})\, A(s)\, u(s)\, ds$$
$$+ \int_0^t e^{(t-s)A(t)} f(s)\, ds, \quad (4.2)$$

which implies

$$A(t)\, u(t) = A(t)\, e^{tA(t)}x + \int_0^t A(t)\, e^{(t-s)A(t)}(1 - A(t)\, A(s)^{-1})\, A(s)\, u(s)\, ds$$
$$+ \int_0^t A(t)\, e^{(t-s)A(t)} f(s)\, ds,$$

or

$$A(t)\,u(t) - H(A(\cdot)\,u(\cdot))(t) = Lf(t) + L(t, 0)\,x. \tag{4.3}$$

Now observe that $L(\cdot, 0)\,x \in C([0, T], E)$ by Lemma 3.1(iii), $Lf \in C^{\theta}([0, T], E)$ by Lemma 3.2(iv) and $H(A(\cdot)\,u(\cdot)) \in C^{\sigma}([0, T], E)$ for each $\sigma \in \,]0, \alpha[$ by Lemma 3.3(v); hence by Lemma 3.4(iii) we can write

$$A(t)\,u(t) = \{(1-H)^{-1}(Lf(\cdot) + L(\cdot, 0)\,x)\}(t), \qquad t \in [0, T],$$

and (4.1) follows. ∎

THEOREM 4.2 (Existence of the strict solution). *Let* $x \in D$ *and* $f \in C([0, T], E) \cap B(0, T, D_{A(0)}(\theta, \infty))$, $\theta \in \,]0, \alpha]$. *Then the function* u *given by* (4.1) *is the unique strict solution of* (P), *and moreover* $u' \in B(0^+, T, D_{A(0)}(\theta, \infty))$ *and* $A(\cdot)\,u(\cdot) \in B(0^+, T, D_{A(0)}(\theta, \infty)) \cap C^{\theta}(]0, T], E)$. *If, in addition,* $f \in C([0, T], D_{A(0)}(\theta))$, $\theta \in \,]0, \alpha[$, *then* $u' \in C(]0, T], D_{A(0)}(\theta))$ *and* $A(\cdot)\,u(\cdot) \in C(]0, T], D_{A(0)}(\theta)) \cap h^{\theta}(]0, T], E)$.

Proof. Uniqueness is proved in [2, Theorem 4.1].
Define f outside $[0, T]$ by

$$f(t) = f(0) \quad \text{if} \quad t < 0,$$
$$\quad = f(T) \quad \text{if} \quad t > T.$$

The convolutions

$$f_n(t) = (\varphi_n * f)(t) = \int_{-\infty}^{+\infty} \varphi_n(t-s)\,f(s)\,ds,$$

where $\varphi_n, n \in \mathbf{N}$, are mollifiers, belong to $\mathrm{Lip}([0, T], D_{A(0)}(\theta, \infty))$ and converge to f in $C([0, T], D_{A(0)}(\sigma))$ for each $\sigma \in \,]0, \theta[$ (since f belongs to such spaces by Proposition 2.12). Let v_n be the strict solution of

$$\begin{cases} v_n'(t) - A(t)\,v_n(t) = f_n(t), & t \in [0, T], \\ v_n(0) = x, \end{cases}$$

which exists by Theorem 4.3 of [2]. By Theorem 4.1 we have

$$v_n(t) = A(t)^{-1}\{(1-H)^{-1}(Lf_n(\cdot) + L(\cdot, 0)\,x)(t)\}, \qquad t \in [0, T],$$

and consequently (Lemma 3.2(iii) and Lemma 3.4(iii)) as $n \to \infty$ we get

$$v_n \to u = A(\cdot)^{-1}\{(1-H)^{-1}(Lf(\cdot) + L(\cdot, 0)\,x)(\cdot)\} \qquad \text{in } C([0, T], E);$$

in particular, $u \in C([0,T], D(A(0)))$ and $u(0) = A(0)^{-1}\{(1-H)^{-1}(Lf + L(\cdot,0)x)(0)\} = x$ (Lemmata 3.1(iii), 3.2(iii) and 3.4(iii)). Hence

$$\begin{aligned} v_n' &= A(\cdot)\, v_n(\cdot) + f_n \to (1-H)^{-1}(Lf + L(\cdot,0)\,x) + f \\ &\equiv A(\cdot)\, u(\cdot) + f \qquad \text{in } C([0,T], E). \end{aligned}$$

This shows that $u \in C^1([0,T], E)$ and $u'(t) = A(t)\, u(t) + f(t)$ $\forall t \in [0,T]$, i.e., u is a strict solution of (P).

To prove regularity we first note that $L(\cdot,0)\,x \in C([0,T], E) \cap B(0^+, T, D_{A(0)}(\sigma, \infty))$ for each $\sigma \in\,]0,1[$ by Lemma 3.1(iii)–(i); hence $h_1 \equiv (1-H)^{-1}(L(\cdot,0)\,x) \in C([0,T], E)$ (Lemma 3.4(iii)), and consequently $Hh_1 \in B(0, T, D_{A(0)}(\alpha, \infty))$ by Lemma 3.3(i). The equality $h_1 = Hh_1 + L(\cdot,0)\,x$ then gives $h_1 \in B(0^+, T, D_{A(0)}(\alpha, \infty))$. On the other hand we have $Lf \in B(0, T, D_{A(0)}(\theta, \infty))$ by Lemma 3.2(i), so that $h_2 \equiv (1-H)^{-1} Lf \in B(0, T, D_{A(0)}(\theta, \infty))$. As $A(\cdot)\, u(\cdot) = h_1 + h_2$, we get $A(\cdot)\, u(\cdot) \in B(0^+, T, D_{A(0)}(\theta, \infty))$; as $u' = A(\cdot)\, u(\cdot) + f$, the same holds for u'. If, in addition, $f \in C([0,T], D_{A(0)}(\theta))$, $\theta \in\,]0, \alpha[$, then by Lemmata 3.2(iii) and 3.4(iv) we have $h_2 \in C(]0,T], D_{A(0)}(\theta))$; the same is true for h_1 by Proposition 2.12, and thus we obtain $A(\cdot)\, u(\cdot), u' \in C(]0,T], D_{A(0)}(\theta))$. Next, as $L(\cdot,0)\,x \in C([0,T], E) \cap C^\alpha(]0,T], E)$ (Lemma 3.1(iii)–(i)), and $h_1 \in C([0,T], E)$ (Lemma 3.4(iii)), we get $Hh_1 \in C^\sigma([0,T], E)$ for each $\sigma \in\,]0, \alpha[$ by Lemma 3.3(v); since $h_1 = Hh_1 + L(\cdot,0)\,x$, we deduce $h_1 \in C([0,T], E) \cap C^\sigma(]0,T], E)$ for each $\sigma \in\,]0, \alpha[$, and consequently Lemma 3.3(iii) yields $Hh_1 \in C^\alpha(]0,T], E)$. Thus we check $h_1 \in C^\alpha(]0,T], E)$. On the other hand $Lf \in C^\theta([0,T], E)$ by Lemma 3.2(iv), so that Lemma 3.4(v) implies $h_2 \in C^\theta([0,T], E)$. This gives $A(\cdot)\, u(\cdot) \in C^\theta(]0,T], E)$. If, in addition, $f \in C([0,T], D_{A(0)}(\theta))$, $\theta \in\,]0,\alpha[$, then the same is true for h_2 by Lemma 3.2(v) and Lemma 3.4(vi), and as $h_1 \in C^\alpha(]0,T], E)$ this implies $A(\cdot)\, u(\cdot) \in h^\theta(]0,T], E)$. The proof is complete. ∎

Let us consider now classical solutions. About existence we have:

THEOREM 4.3 (Existence of the classical solution). *Let* $x \in \overline{D(A(0))}$ *and* $f \in C_\beta(]0,T], E) \cap B_\beta(0, T, D_{A(0)}(\theta, \infty))$, $\beta \in [0,1[$, $\theta \in\,]0,\alpha]$. *Then Problem* (P) *has a unique classical solution* u, *and moreover* $u' \in B(0^+, T, D_{A(0)}(\theta, \infty))$ *and* $A(\cdot)\, u(\cdot) \in B(0^+, T, D_{A(0)}(\theta, \infty)) \cap C^\theta(]0,T], E)$. *If, in addition,* $f \in C_\beta(]0,T], D_{A(0)}(\theta))$, $\beta \in [0,1[$, $\theta \in\,]0,\alpha[$, *then* $u' \in C(]0,T], D_{A(0)}(\theta))$ *and* $A(\cdot)\, u(\cdot) \in C(]0,T], D_{A(0)}(\theta)) \cap h^\theta(]0,T], E)$.

Proof. Uniqueness follows by Theorem 5.1 of [2].

Let v be the classical solution of

$$\begin{cases} v'(t) - A(t)\,v(t) = 0, & t \in]0, T], \\ v(0) = x, \end{cases} \tag{4.4}$$

which exists by Theorem 5.4 of [2]. If we are able to find the classical solution w of

$$\begin{cases} w'(t) - A(t)\,w(t) = f(t), & t \in]0, T], \\ w(0) = 0, \end{cases} \tag{4.5}$$

then the classical solution of (P) will be given by $u = v + w$. Hence it is enough to solve (4.5).

For each $n \in \mathbf{N}$ consider the functions

$$\begin{aligned} f_n(t) &= f(1/n) && \text{if} \quad 0 \leqslant t \leqslant 1/n \leqslant T, \\ &= f(t) && \text{if} \quad 1/n < t \leqslant T. \end{aligned}$$

Clearly $f_n \in C([0, T], E) \cap B(0, T, D_{A(0)}(\theta, \infty))$ for each $n \in \mathbf{N}$, and in addition it is easily seen that as $n \to \infty$

$$f_n \to f \qquad \text{in } C_\gamma(]0, T], E) \qquad \forall \gamma \in]\beta, 1[.$$

Let w_n be the strict solution of

$$\begin{cases} w_n'(t) - A(t)\,w_n(t) = f_n(t), & t \in [0, T], \\ w_n(0) = 0. \end{cases}$$

By Theorem 4.2 w_n exists and is given by

$$w_n(t) = A(t)^{-1}(1 - H)^{-1} L f_n(t), \qquad t \in [0, T].$$

By Lemmata 3.2(i) and 3.4(i) we get as $n \to \infty$

$$w_n \to A(\cdot)^{-1}(1 - H)^{-1} L f \qquad \text{in} \quad C_\gamma(]0, T], E) \qquad \forall \gamma \in]\beta, 1[\tag{4.6}$$

and

$$A(\cdot)\,w_n(\cdot) = (1 - H)^{-1} L f_n \to (1 - H)^{-1} L f \quad \text{in} \quad C_\gamma(]0, T], E) \qquad \forall \gamma \in]\beta, 1[.$$

This implies that

$$w_n' = A(\cdot)\,w_n(\cdot) + f_n \to (1 - H)^{-1} L f + f \qquad \text{in} \quad C_\gamma(]0, T], E) \qquad \forall \gamma \in]\beta, 1[. \tag{4.7}$$

By (4.6) and (4.7) it follows that the function $w = A(\cdot)^{-1}(1 - H)^{-1} L f(\cdot)$ is in $C^1(]0, T], E)$ and is the classical solution of (4.5).

Let us prove regularity. We have $u = v + w$, where v and u solve (4.4) and (4.5), respectively. In [2], Theorem 5.4, it is proved that $A(\cdot)\,v(\cdot) \in C^{\alpha}(]0,T],E)$, and we will show in Lemma 4.4 below that $A(\cdot)\,v(\cdot) = v' \in B(0^+, T, D_{A(0)}(\alpha,\infty))$; thus it suffices to verify the required regularity for w. We have $h \equiv A(\cdot)\,w(\cdot) = (1-H)^{-1}Lf$; now, by Lemma 3.2(i)–(vi) we get $Lf \in B_{\beta}(0,T,D_{A(0)}(\theta,\infty)) \cap C^{\theta}(]0,T],E)$, so that $h \in B_{\beta}(0,T,D_{A(0)}(\theta,\infty))$ (Lemma 3.4(i)). Lemma 3.3(ii) then yields $Hh \in C^{\sigma}(]0,T],E)$ for each $\sigma \in\,]0,\alpha[$; hence $h = Hh + Lf \in C^{\theta\wedge\sigma}(]0,T],E)$ for each $\sigma \in\,]0,\alpha[$. This gives $Hh \in C^{\alpha}(]0,T],E)$ (Lemma 3.3(iii)) and finally we conclude that $A(\cdot)\,w(\cdot) = h \in C^{\theta}(]0,T],E)$. If, in addition, $f \in C_{\beta}(]0,T],D_{A(0)}(\theta))$, $\theta \in\,]0,\alpha[$, then we have $Lf \in C_{\beta}(]0,T],D_{A(0)}(\theta)) \cap h^{\theta}(]0,T],E)$ (Lemma 3.2(ii)–(vii)) so that by Lemma 3.4(ii) we derive $h \in C_{\beta}(]0,T],D_{A(0)}(\theta))$ and, as before, $A(\cdot)\,w(\cdot) = h \in h^{\theta}(]0,T],E)$. The proof is complete (except for Lemma 4.4 below). ∎

LEMMA 4.4. *Let $x \in \overline{D(A(0))}$ and let v be the classical solution of* (4.4). *Then $v' \in B(0^+, T, D_{A(0)}(\alpha,\infty))$.*

Proof. Let $\varepsilon \leqslant t \leqslant T$. We proceed as in the proof of Theorem 4.1: the function $z(s) = e^{(t-s)A(t)}\,v(s)$ is in $C^1([\frac{1}{2}\varepsilon, t], E)$ and

$$z'(s) = e^{(t-s)A(t)}(1 - A(t)A(s)^{-1})\,A(s)\,v(s), \qquad s \in [\tfrac{1}{2}\varepsilon, t].$$

Integrating over $[\frac{1}{2}\varepsilon, t]$ and applying $A(t)$ to both members, we get

$$\begin{aligned} A(t)\,v(t) &= A(t)\,e^{(t-\varepsilon/2)A(t)}\,v(\tfrac{1}{2}\varepsilon) \\ &\quad + \int_{\varepsilon/2}^{t} A(t)\,e^{(t-s)A(t)}(1 - A(t)A(s)^{-1})\,A(s)\,v(s)\,ds, \quad t \in [\varepsilon, T]. \end{aligned}$$

Now we have

$$\begin{aligned} &\left\| A(t)\,e^{(t-\varepsilon/2)A(t)}\,v\left(\frac{1}{2}\varepsilon\right)\right\|_{\alpha} \\ &\leqslant \frac{C}{(t-\frac{1}{2}\varepsilon)^{\alpha}}\,\|A(\cdot)\,v(\cdot)\|_{C([\varepsilon/2,T],E)} \leqslant C(\varepsilon) \qquad \forall t \in [\varepsilon, T], \\ &\left\| \int_{\varepsilon/2}^{t} A(t)\,e^{(t-s)A(t)}(1 - A(t)A(s)^{-1})\,A(s)\,v(s)\,ds \right\|_{\alpha} \\ &\leqslant C \sup_{\xi>0} \int_{\varepsilon/2}^{t} \frac{\xi^{1-\alpha}(t-s)^{\alpha}}{(\xi+t-s)^2}\,ds\,\|A(\cdot)\,v(\cdot)\|_{C([\varepsilon/2,T],E)} \leqslant C(\varepsilon) \qquad \forall t \in [\varepsilon, T], \end{aligned}$$

so that $A(\cdot)\,v(\cdot) \in B(\varepsilon, T, D_{A(0)}(\alpha,\infty))$. As $v' = A(\cdot)\,v(\cdot)$ in $]0,T]$, the result follows. ∎

5. Global Space Regularity

In this section we will prove some global regularity results for strict solutions of (P). These results are quite similar to those of Sinestrari [12] relative to the autonomous case. As usual, Hypotheses I and II are always assumed to hold.

THEOREM 5.1. *Let* $x \in D_{A(0)}(\theta+1, \infty)$ *and* $f \in C([0,T],E) \cap B(0,T,D_{A(0)}(\theta,\infty))$, $\theta \in]0,\alpha]$. *Let u be the strict solution of (P); then*

(i) $u', A(\cdot)\,u(\cdot) \in B(0,T,D_{A(0)}(\theta,\infty))$;

(ii) $A(\cdot)\,u(\cdot) \in C^{\theta}([0,T],E)$.

Proof. We know that u is given by (4.1), and therefore

$$A(\cdot)\,u(\cdot) = H(A(\cdot)\,u(\cdot)) + Lf + L(\cdot,0)\,x. \tag{5.1}$$

As $x \in D_{A(0)}(\theta+1,\infty)$, by Lemma 3.1(iv)–(vi) we get $L(\cdot,0)\,x \in B(0,T,D_{A(0)}(\theta,\infty)) \cap C^{\theta}([0,T],E)$; as $f \in C([0,T],E) \cap B(0,T,D_{A(0)}(\theta,\infty))$, by Lemma 3.2(i)–(iv) we have $Lf \in B(0,T,D_{A(0)}(\theta,\infty)) \cap C^{\theta}([0,T]\,E)$. As $A(\cdot)\,u(\cdot) \in C([0,T],E)$, Lemma 3.3(v) yields $H(A(\cdot)\,u(\cdot)) \in C^{\sigma}([0,T],E)$ for each $\sigma \in]0,\alpha[$; thus from (5.1) we derive that $A(\cdot)\,u(\cdot) \in C^{\theta\wedge\sigma}([0,T],E)$ for each $\sigma \in]0,\alpha[$, and Lemma 3.3(vi) then implies that $H(A(\cdot)\,u(\cdot)) \in C^{\alpha}([0,T],E)$. On the other hand by Lemma 3.3(i) we know also that $H(A(\cdot)\,u(\cdot)) \in B(0,T,D_{A(0)}(\alpha,\infty))$. Again by (5.1) we finally obtain $A(\cdot)\,u(\cdot) \in B(0,T,D_{A(0)}(\theta,\infty)) \cap C^{\theta}([0,T],E)$.

To complete the proof we just need to observe that $u' = A(\cdot)\,u(\cdot) + f \in B(0,T,D_{A(0)}(\theta,\infty))$. ∎

THEOREM 5.2. *Let* $x \in D_{A(0)}(\theta+1)$, $f \in C([0,T],D_{A(0)}(\theta))$, $\theta \in]0,\alpha[$. *Let u be the strict solution of* (P); *then*

(i) $u', A(\cdot)\,u(\cdot) \in C([0,T],D_{A(0)}(\theta))$;

(ii) $A(\cdot)\,u(\cdot) \in h^{\theta}([0,T],E)$.

Proof. The proof is similar to the preceding one. The function u is given by (4.1), so that (5.1) holds. By Lemma 3.1(v)–(vii) $L(\cdot,0)\,x \in C([0,T],D_{A(0)}(\theta)) \cap h^{\theta}([0,T],E)$; by Lemma 3.2(iii)–(v) $Lf \in C([0,T],D_{A(0)}(\theta)) \cap h^{\theta}([0,T],E)$. On the other hand, as in the preceding proof, we obtain $H(A(\cdot)\,u(\cdot)) \in B(0,T,D_{A(0)}(\alpha,\infty)) \cap C^{\alpha}([0,T],E) \subseteq C([0,T],D_{A(0)}(\theta)) \cap h^{\theta}([0,T],E)$; hence by (5.1) we conclude that $A(\cdot)\,u(\cdot) \in C([0,T],D_{A(0)}(\theta)) \cap h^{\theta}([0,T],E)$.

Finally, $u' = A(\cdot)\,u(\cdot) + f \in C([0,T],D_{A(0)}(\theta))$. ∎

We finish this section with the following remark: With a slight change in the argument which leads to the representation formula (4.1), it is possible to check another existence and space regularity theorem, where it is not necessary that both the data x, f are regular: what is really needed is regularity of a suitable function of them. Namely, we have:

THEOREM 5.3. *Let* $x \in D(A(0))$ *and* $f \in C([0,T],E)$, *and suppose moreover that the function* $t \to A(0)x + f(t)$ *belongs to* $B(0,T,D_{A(0)}(\theta,\infty))$, $\theta \in]0,\alpha]$ (*resp.* $C([0,T],D_{A(0)}(\theta))$, $\theta \in]0,\alpha[$). *Then Problem* (P) *has a* (*unique*) *strict solution u, such that*

(i) u', $A(\cdot)u(\cdot) - A(0)x \in B(0,T,D_{A(0)}(\theta,\infty))$ (*resp.* $C([0,T], D_{A(0)}(\theta))$;

(ii) $A(\cdot)u(\cdot) \in C^\theta([0,T],E)$ (*resp.* $h^\theta([0,T],E)$).

If in the hypotheses we replace $A(0)x + f(\cdot)$ *with* $A(\cdot)x + f(\cdot)$, *then the same conclusion holds, with* $A(\cdot)u(\cdot) - A(0)x$ *replaced by* $A(\cdot)(u(\cdot) - x)$ *in* (i).

Proof. We proceed as in the proof of Theorems 4.1 and 4.2. Suppose first that a strict solution u of (P) does exist; then we easily get (4.2), which can be rewritten as

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}(1 - A(t)A(s)^{-1})A(s)u(s)\,ds$$
$$+ \int_0^t e^{(t-s)A(t)}(f(s) + A(0)x)\,ds - (e^{tA(t)} - 1)A(0)x.$$

Thus applying $A(t)$ to both members we obtain the integral equation

$$A(t)u(t) - H(A(\cdot)u(\cdot))(t)$$
$$= A(0)x + L(f + A(0)x)(t) + e^{tA(t)}(A(t)A(0)^{-1} - 1)A(0)x, \quad (5.2)$$

and consequently

$$u(t) = A(t)^{-1}\{(1-H)^{-1}(A(0)x + L(f + A(0)x)$$
$$+ e^{tA(t)}(A(t)A(0)^{-1} - 1)A(0)x)(t)\}. \quad (5.3)$$

Note that the function $t \to e^{tA(t)}(A(t)A(0)^{-1} - 1)A(0)x$ belongs to $C^\alpha([0,T],E) \cap B(0,T,D_{A(0)}(\alpha,\infty))$ since

$$\begin{aligned}
&\|e^{tA(t)}(A(t)A(0)^{-1}-1)A(0)x - e^{rA(r)}(A(r)A(0)^{-1}-1)A(0)x\|_E \\
&\quad \leqslant \|e^{tA(t)}(A(t)A(r)^{-1}-1)(A(r)A(0)^{-1})A(0)x\|_E \\
&\qquad + \|(e^{tA(t)}-e^{tA(r)})(A(r)A(0)^{-1}-1)A(0)x\|_E \\
&\qquad + \left\|\int_r^t A(r)\,e^{sA(r)}(A(r)A(0)^{-1}-1)A(0)x\,ds\right\|_E \\
&\quad \leqslant C(t-r)^\alpha \qquad \text{if } 0\leqslant r<t\leqslant T,
\end{aligned}$$

and

$$\sup_{\xi\to 0}\|\xi^{1-\alpha}A(t)\,e^{(\xi+t)A(t)}(A(t)A(0)^{-1}-1)A(0)x\|_E \leqslant C\sup_{\xi>0}\frac{\xi^{1-\alpha}t^\alpha}{\xi+t}\|A(0)x\|_E.$$

Next, we want to prove that the function (5.3) indeed is a strict solution of (P). As in the proof of Theorem 4.2, let v_n be the strict solution of

$$\begin{cases} v_n'(t)-A(t)\,v_n(t)=g_n(t)-A(0)\,x, & t\in[0,T],\\ v_n(0)=x \end{cases}$$

where $g_n\in \mathrm{Lip}([0,T],D_{A(0)}(\theta,\infty))$ and, as $n\to\infty$, $g_n\to f(\cdot)+A(0)x$ in $C([0,T],D_{A(0)}(\sigma))$ for each $\sigma\in\,]0,\theta[$. Such a strict solution exists by Theorem 4.3 of [2]. Then $v_n(t)$ is given by (5.3), i.e.,

$$v_n(t)=A(t)^{-1}\{(1-H)^{-1}(A(0)\,x+Lg_n+e^{tA(t)}(A(t)A(0)^{-1}-1)A(0)\,x)(t)\}.$$

As $n\to\infty$, we easily get that $v_n\to u$ and $A(\cdot)\,v_n(\cdot)\to A(\cdot)\,u(\cdot)$ with u given by (5.3), so that u is a strict solution of (P).

Regularity can be deduced by (5.2) as in the proofs of Theorems 5.1 and 5.2, since $t\to e^{tA(t)}(A(t)A(0)^{-1}-1)A(0)\,x$ belongs to $C^\alpha([0,T],E)\cap B(0,T,D_{A(0)}(\alpha,\infty))$.

Replace now in the hypotheses $f(\cdot)+A(0)\,x$ with $f(\cdot)+A(\cdot)\,x$. Then evidently u is a strict solution of (P) if and only if $v(t)=u(t)-x$ is a strict solution of

$$\begin{cases} v'(t)-A(t)\,v(t)=f(t)+A(t)\,x, & t\in[0,T],\\ v(0)=0. \end{cases}$$

Hence the conclusion follows by Theorems 4.2, 5.1 and 5.2. In particular, u is given by

$$u(t)=x+A(t)^{-1}\{(1-H)^{-1}L(f+A(\cdot)\,x)(t)\}. \quad \blacksquare$$

6. An Example

In this section, unless otherwise specified, all functions are assumed to be complex valued. We define

$$C^k([0,1]) = \{u \in C([0,1]) \colon u \text{ is } k \text{ times differentiable and } u^{(k)} \in C([0,1])\}, \qquad k \in \mathbf{N},$$

$$C^{k,\theta}([0,1]) = \{u \in C^k([0,1]) \colon u^{(k)} \in C^\theta([0,1])\}, \qquad k \in \mathbf{N}, \theta \in \,]0,1[,$$

$$h^{k,\theta}([0,1]) = \{u \in C^k([0,1]) \colon u^{(k)} \in h^\theta([0,1])\}, \qquad k \in \mathbf{N}, \theta \in \,]0,1[.$$

Set $E = C([0,1])$, $\|u\|_E = \sup_{x \in [0,1]} |u(x)|$, and define for each $t \in [0,T]$

$$\begin{cases} D(A(t)) = \{u \in C^2([0,1]) \colon \alpha_0 u(0) - \beta_0 u'(0) = \alpha_1 u(1) + \beta_1 u'(1) = 0\}, \\ A(t)u = a(\cdot,t)\,u'' + b(\cdot,t)\,u' + c(\cdot,t)\,u - \omega_0 u, \end{cases} \tag{6.1}$$

where

$$\alpha_i, \beta_i \geqslant 0, \qquad \alpha_i + \beta_i > 0, \qquad i = 0, 1, \tag{6.2}$$

and

$$\omega_0 \in \mathbf{R}, \qquad a, b, c \in C([0,1] \times [0,T], \mathbf{R}) \inf_{[0,1]\times[0,T]} a(x,t) > 0. \tag{6.3}$$

Obviously $D(A(t))$ does not depend on t; in addition we have:

PROPOSITION 6.1. *Let* $\{A(t)\}_{t \in [0,T]}$ *be defined by* (6.1) *and suppose* (6.2), (6.3) *hold. If* $\omega_0 > \max_{[0,1]\times[0,T]} |c(x,t)|$, *then*

(i) $\sigma(A(t)) \subseteq \,]-\infty, 0[\; \forall t \in [0,T]$,

(ii) *there exist* $M > 0$ *and* $\theta_0 \in \,]\frac{1}{2}\pi, \pi[$ *such that if* $\lambda \in \Sigma_{\theta_0}$

$$\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leqslant \frac{M}{|\lambda|}, \qquad \|A(t)^{-1}\|_{\mathscr{L}(E)} \leqslant M \;\forall t \in [0,T].$$

If, in addition, we assume that

$$|a(x,t) - a(x,r)| + |b(x,t) - b(x,r)| + |c(x,t) - c(x,r)| \leqslant B\,|t-r|^\alpha \qquad \forall t, r \in [0,T], \forall x \in [0,1]$$

for some constants $\alpha \in \,]0,1[$ *and* $B > 0$, *then there exists* $K > 0$ *such that*

(iii) $\|1 - A(t)A(r)^{-1}\|_{\mathscr{L}(E)} \leqslant K\,|t-r|^\alpha \;\forall t, r \in [0,T]$.

Proof. See [2, Remark 1.2 and Propositions 8.1, 8.3]. ∎

Thus we can apply the results of the preceding sections to the problem

$$\begin{aligned}
&u_t - a(x,t)\,u_{xx} - b(x,t)\,u_x - c(x,t)\,u + \lambda u = f(x,t), && (x,t) \in [0,1] \times [0,T],\\
&\alpha_0 u(0,t) - \beta_0 u_x(0,t) = \alpha_1 u(1,t) + \beta_1 u_x(1,t) = 0, && t \in [0,T],\\
&u(x,0) = g(x), && x \in [0,1]
\end{aligned}$$

where $\lambda \in \mathbb{C}, f \in C([0,1] \times [0,T]), g \in C([0,1])$.

We have only to characterize the spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$ in this concrete case. In the case of Dirichlet conditions, i.e., $\beta_0 = \beta_1 = 0$, it is known (see Da Prato and Grisvard [5] and Lunardi [9]) that

$$\begin{aligned}
D_{A(0)}(\theta, \infty) &= \{f \in C^{2\theta}([0,1]) : f(0) = f(1) = 0\} && \text{if } \theta \in \left]0, \tfrac{1}{2}\right[\\
&= \{f \in C^{1,2\theta-1}([0,1]) : f(0) = f(1) = 0\} && \text{if } \theta \in \left]\tfrac{1}{2}, 1\right[
\end{aligned}$$

and

$$\begin{aligned}
D_{A(0)}(\theta) &= \{f \in h^{2\theta}([0,1]) : f(0) = f(1) = 0\} && \text{if } \theta \in \left]0, \tfrac{1}{2}\right[\\
&= \{f \in h^{1,2\theta-1}([0,1]) : f(0) = f(1) = 0\} && \text{if } \theta \in \left]\tfrac{1}{2}, 1\right[.
\end{aligned}$$

Suppose now that $a, b, c \in C^\alpha([0,1] \times [0,T], \mathbf{R})$. Then in addition it is known that

$$\begin{aligned}
D_{A(0)}(\theta + 1, \infty) &= \{f \in C^{2,2\theta}([0,1]) : f(0) = f(1) = [A(0)f](0)\\
&= [A(0)f](1) = 0\} \qquad \forall \theta \in \left]0, \tfrac{1}{2}\alpha\right[,\\
D_{A(0)}(\theta + 1) &= \{f \in h^{2,2\theta}([0,1]) : f(0) = f(1) = [A(0)f](0)\\
&= [A(0)f](1) = 0\} \qquad \forall \theta \in \left]0, \tfrac{1}{2}\alpha\right[.
\end{aligned}$$

Let us consider now the general case of (6.2), i.e., $\beta_0 + \beta_1 > 0$. We have the following result:

THEOREM 6.2. *Let $\{A(t)\}_{t \in [0,T]}$ be defined by* (6.1) *and suppose* (6.2), (6.3) *hold. Then we have*:

$$\begin{aligned}
D_{A(0)}(\theta, \infty) &= C^{2\theta}([0,1]) && \text{if } \theta \in \left]0, \tfrac{1}{2}\right[\\
&= \{f \in C^{1,2\theta-1}([0,1]) : \alpha_0 f(0) - \beta_0 f'(0)\\
&= \alpha_1 f(1) + \beta_1 f'(1) = 0\} && \text{if } \theta \in \left]\tfrac{1}{2}, 1\right[,\\
D_{A(0)}(\theta) &= h^{2\theta}([0,1]) && \text{if } \theta \in \left]0, \tfrac{1}{2}\right[\\
&= \{f \in h^{1,2\theta-1}([0,1]) : \alpha_0 f(0) - \beta_0 f'(0)\\
&= \alpha_1 f(1) + \beta_1 f'(1) = 0\} && \text{if } \theta \in \left]\tfrac{1}{2}, 1\right[.
\end{aligned}$$

If, in addition, $a, b, c \in C^{\alpha}([0,1] \times [0,T], \mathbf{R})$, then

$$\begin{aligned} D_{A(0)}(\theta+1,\infty) &= \{f \in C^{2,2\theta}([0,1]) \colon \alpha_0 f(0) - \beta_0 f'(0) \\ &= \alpha_1 f(1) + \beta_1 f'(1) = 0\} \qquad \forall \theta \in \left]0, \tfrac{1}{2}\alpha\right[, \\ D_{A(0)}(\theta+1) &= \{f \in h^{2,2\theta}([0,1]) \colon \alpha_0 f(0) - \beta_0 f'(0) \\ &= \alpha_1 f(1) + \beta_1 f'(1) = 0\} \qquad \forall \theta \in \left]0, \tfrac{1}{2}\alpha\right[. \end{aligned}$$

Proof. We confine ourselves to the characterization of the spaces $D_{A(0)}(\theta,\infty)$ and $D_{A(0)}(\theta+1,\infty)$ since the proof in the remaining cases is quite analogous. Let f be a function such that

$$\begin{aligned} f &\in C^{2\theta}([0,1]) && \text{if } \theta \in \left]0, \tfrac{1}{2}\right[\\ &\in \{g \in C^{1,2\theta-1}([0,1]) \colon \alpha_0 g(0) - \beta_0 g'(0) && \\ &\qquad = \alpha_1 g(1) + \beta_1 g'(1) = 0\} && \text{if } \theta \in \left]\tfrac{1}{2}, 1\right[. \end{aligned} \tag{6.4}$$

We shall construct an extension of f to $\mathbf{R}$, in such a way that its regularity is preserved as well as the conditions at $y=0$ and $y=1$, if they exist. Set

$$\begin{aligned} F(y) &= 0 && \text{if } \ y \leqslant -1 \\ &= \eta(y) f(-y) - \frac{2\alpha_0}{\beta_0} \int_y^0 \exp\left(+\frac{\alpha_0}{\beta_0}(y-s)\right) f(-s)\,\eta(s)\,ds && \\ & && \text{if } \ -1 < y < 0 \\ &= f(y) && \text{if } \ 0 \leqslant y \leqslant 1 \\ &= \eta(y) f(2-y) - \frac{2\alpha_1}{\beta_1} \int_1^y \exp\left(-\frac{\alpha_1}{\beta_1}(y-s)\right) f(2-s)\,\eta(s)\,ds && \\ & && \text{if } \ 1 < y < 2 \\ &= 0 && \text{if } \ 2 \leqslant y. \end{aligned}$$

where $\eta \in C^{\infty}(\mathbf{R},\mathbf{R})$, $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in $[-\frac{1}{2}, \frac{3}{2}]$, and the support of η lies in $[-1,2]$. Note that the case $\beta_0 = 0$ (resp. $\beta_1 = 0$) is also covered: one has only to replace the corresponding integral by its limit as $\beta_0 \to 0^+$ (resp. $\beta_1 \to 0^+$), namely, $-2\eta(y) f(-y)$ (resp. $-2\eta(y) f(2-y)$). It is easy to verify that F has the same regularity as f, and that if

$$\alpha_0 f(0) - \beta_0 f'(0) = \alpha_1 f(1) + \beta_1 f'(1) = 0 \tag{6.5}$$

then the same holds for F.

We want to construct now a function $t \to u(t, \cdot)$ such that $u(0, \cdot) = f$ and

$$\begin{cases} u \in C_{1-\theta}(]0, \infty[, D(A(0))) \\ u' \in C_{1-\theta}(]0, \infty[, E). \end{cases} \tag{6.6}$$

First of all, let $\varphi \in C^\infty(\mathbf{R})$ be a non-negative, even function with support contained in $]-1, 1[$ and satisfying $\int_{\mathbf{R}} \varphi(y)\,dy = 1$, and let $\varphi_\varepsilon(y) = \varepsilon^{-1}\varphi(y \cdot \varepsilon^{-1})$, $\varepsilon > 0$, be the corresponding mollifiers. Let us consider the function $v(t, y)$ defined by the convolution between φ_ε and F, with parameter $\varepsilon = t^{1/2}$:

$$v(t, y) = \frac{1}{\sqrt{t}} \int_{\mathbf{R}} \varphi\left(\frac{y - x}{\sqrt{t}}\right) F(x)\,dx.$$

Finally, let $\psi \in C^\infty([0, \infty[, \mathbf{R})$ be such that $\psi(0) = 1$ and $0 \leqslant \psi \leqslant 1$, with support lying in $[0, \frac{1}{4}[$, and define the function

$$u(t, \cdot) = v(t, \cdot)|_{[0,1]} \cdot \psi(t), \qquad t \in [0, \infty[.$$

It is clear that $u(0, \cdot) = f$; we will prove that $t \to u(t, \cdot)$ satisfies (6.6): by Definition 2.1, this will imply $f \in D_{A(0)}(\theta, \infty)$.

We start with verifying that $u(t, \cdot) \in D(A(0))$ for each $t \in]0, \infty[$: as $u(t, \cdot) \in C^\infty([0, 1])$, we have only to show that $v(t, \cdot)$ satisfies (6.5). A tedious but easy calculation yields

$$\alpha_0 v(t, y) - \beta_0 v_y(t, y) = -\alpha_0 v(t, -y) + \beta_0 v_y(t, -y) \qquad \text{near } y = 0,$$

$$\alpha_1 v(t, y) + \beta_1 v_y(t, y) = -\alpha_1 v(t, 2 - y) - \beta_1 v_y(t, 2 - y) \quad \text{near } y = 1,$$

so that (6.5) is satisfied.

Next, we prove that $A(0)\,u(t, \cdot)$ and $u'(t, \cdot)$ belong to $C_{1-\theta}(]0, \infty[, E)$: since

$$\|A(0)\,u(t, \cdot)\|_E \leqslant C\{\|u_{yy}(t, \cdot)\|_E + \|u_y(t, \cdot)\|_E + \|u(t, \cdot)\|_E\},$$

it is enough to estimate the $C_{1-\theta}(]0, \infty[, E)$-norms of $u_t(t, \cdot)$, $u_{yy}(t, \cdot)$, $u_y(t, \cdot)$ and $u(t, \cdot)$, i.e., the $C_{1-\theta}(]0, \frac{1}{4}], E)$-norms of $v_t(t, \cdot)$, $v_{yy}(t, \cdot)$, $v_y(t, \cdot)$ and $v(t, \cdot)$.

Consider first the case $\theta \in]0, \frac{1}{2}[$. For each $y \in \mathbf{R}$ and $t \in]0, \frac{1}{4}]$ we have

$$|t^{1-\theta}\, v(t, y)| \leqslant |v(t, y)| \leqslant \sup_{y \in \mathbf{R}} |F(y)| \leqslant C\,\|f\|_{C([0,1])}, \tag{6.7}$$

$$|t^{1-\theta} v_y(t, y)| = \left| t^{1/2-\theta} \int_{\mathbf{R}} \varphi'(z)\, F(y - \sqrt{t}\, z)\,dz \right| \leqslant C\,\|f\|_{C([0,1])}, \tag{6.8}$$

$$\begin{aligned}|t^{1-\theta}v_{yy}(t,y)| &= \left| t^{-\theta}\int_{\mathbf{R}} \varphi''(z)\,F(y-\sqrt{t}\,z)\,dz \right| \\ &= \left| t^{-\theta}\int_{\mathbf{R}} \varphi''(z)(F(y-\sqrt{t}\,z)-F(y))\,dz \right| \\ &\leqslant C\int_{\mathbf{R}} |\varphi''(z)|\,|z|^{2\theta}\,dz \sup_{x\neq x'} \frac{|F(x)-F(x')|}{|x-x'|^{2\theta}} \leqslant C\,\|f\|_{C^{2\theta}([0,1])};\end{aligned} \tag{6.9}$$

finally

$$\begin{aligned}&|t^{1-\theta}v_t(t,y)| \\ &= \left| t^{1-\theta}\left[-\frac{1}{2}t^{-3/2}\int_{\mathbf{R}} \varphi\left(\frac{y-x}{\sqrt{t}}\right) F(x)\,dx \right.\right. \\ &\quad \left.\left. -\frac{1}{2}t^{-2}\int_{\mathbf{R}} \varphi'\left(\frac{y-x}{\sqrt{t}}\right)(y-x)\,F(x)\,dx \right]\right| \\ &= \left| \frac{1}{2}t^{-\theta}\left[\int_{\mathbf{R}} \varphi(z)\,F(y-\sqrt{t}\,z)\,dz + \int_{\mathbf{R}} \varphi'(z)\,zF(y-\sqrt{t}\,z)\,dz \right.\right| \qquad (6.10) \\ &= \left| \frac{1}{2}t^{-\theta}\int_{\mathbf{R}} (\varphi(z)+z\varphi'(z))(F(y-\sqrt{t}\,z)-F(y))\,dz \right| \leqslant C\,\|f\|_{C^{2\theta}([0,1])}.\end{aligned}$$

Let us see now the case $\theta \in]\frac{1}{2}, 1[$. The estimate (6.7) can be proved in the same manner. About $v_y(t,y)$ we have

$$\begin{aligned}|t^{1-\theta}v_y(t,y)| &= \left| t^{1-\theta}\int_{\mathbf{R}} \varphi'(z)\,F(y-\sqrt{t}\,z)\,dz \right| \\ &= \left| t^{1-\theta}\int_{\mathbf{R}} \varphi'(z)(F(y-\sqrt{t}\,z)-F(y))\,dz \right| \\ &\leqslant C\,\|f'\|_{C([0,1])}.\end{aligned}$$

Similarly

$$|t^{1-\theta}v_{yy}(t,y)| = \left| t^{1/2-\theta}\int_{\mathbf{R}} \varphi'(z)\,F'(y-\sqrt{t}\,z)\,dz \right| \in C\,\|f'\|_{C^{2\theta-1}([0,1])},$$

and finally, as $z \to z\varphi(z)$ is an odd function,

$$\begin{aligned}|t^{1-\theta}v_t(t,y)| &= \left| \frac{1}{2}t^{1/2-\theta}\int_{\mathbf{R}} \varphi(z)\,zF'(y-\sqrt{t}\,z)\,dz \right| \\ &= \left| \frac{1}{2}t^{1/2-\theta}\int_{\mathbf{R}} \varphi(z)\,z(F'(y-\sqrt{t}\,z)-F'(y))\,dz \right| \\ &\leqslant C\,\|f'\|_{C^{2\theta-1}([0,1])}.\end{aligned}$$

Thus we have shown that $f \in D_{A(0)}(\theta, \infty)$.

Let us verify now the reverse inclusion. Let $f \in D_{A(0)}(\theta, \infty)$: by definition, $f \in (D(A(0)), E)_{1-\theta,\infty} \subseteq (C^2([0,1]), C([0,1]))_{1-\theta,\infty}$; hence (see Triebel [15, Theorem 2.7.2/1(a)])

$$\begin{aligned} f &\in C^{2\theta}([0,1]) &&\text{if } \theta \in]0, \tfrac{1}{2}[\\ &\in C^{1,2\theta-1}([0,1]) &&\text{if } \theta \in]\tfrac{1}{2}, 1[. \end{aligned}$$

It remains to show that if $\theta \in]\frac{1}{2}, 1[$ then f satisfies (6.5). Choose $\beta \in]\frac{1}{2}, \theta[$; as $f \in D_{A(0)}(\theta, \infty) \subseteq D_{A(0)}(\beta)$, there exists $\{f_n\}_{n \in \mathbf{N}} \subseteq D(A(0))$ such that $\|f_n - f\|_\beta \to 0$ as $n \to \infty$ (see Section 2). For each $n \in \mathbf{N}$, f_n satisfies (6.5), and moreover

$$D_{A(0)}(\beta) \subseteq (C^2([0,1]), C([0,1]))_{1-\beta,\infty} = C^{1,2\beta-1}([0,1]) \subseteq C^1([0,1])$$

with continuous inclusions. Hence in particular we get $f_n \to f$ in $C^1([0,1])$ as $n \to \infty$, and therefore f satisfies (6.5), too.

The first part of Theorem 6.2 is proved. Suppose now that a, b, $c \in C^\alpha([0,1] \times [0,T], \mathbf{R})$. Then we have for each $\theta \in]0, \frac{1}{2}\alpha[$

$$f \in D_{A(0)}(\theta+1, \infty) \Leftrightarrow f \in D(A(0)), \qquad A(0)f \in C^{2\theta}([0,1]),$$

and as $f'' = a(\cdot,t)^{-1} \cdot (A(0)f - b(\cdot,t)f' - c(\cdot,t)f + \omega_0 f)$, we deduce

$f \in D_{A(0)}(\theta+1, \infty) \Leftrightarrow f \in D(A(0)), f'' \in C^{2\theta}([0,1]) \Leftrightarrow f \in C^{2,2\theta}([0,1])$ and (6.5) holds.

Theorem 6.2 is completely proved. ∎

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