On the Abstract Nonautonomous Parabolic Cauchy Problem in the Case of Constant Domains (*).

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Summary. – *We study existence, uniqueness and regularity of the strict, classical and strong solution* $u \in C([0, T], E)$ *of the non-autonomous evolution equation* $u'(t) - A(t)u(t) = f(t)$, *with the initial datum* $u(0) = x$, *in a Banach space* E, *where* $\{A(t)\}$ *is a family of infinitesimal generators of analytic semi-groups whose domains are constant in* t *and possibly not dense in* E. *We prove necessary and sufficient conditions for existence and Hölder regularity of the solutions and their first derivative.*

0. – Introduction.

Let E be a Banach space, $\{A(t)\}_{t\in[0,T]}$ a family of closed linear operators on E. Consider the following linear non-autonomous Cauchy problem:

$$\text{(P)} \qquad \begin{cases} u'(t) - A(t)u(t) = f(t)\,, \quad t \in [0, T] \\ u(0) = x \\ x \in E\,, \quad f \in C([0, T], E) \text{ prescribed,} \end{cases}$$

where $C([0, T], E)$ is the space of continuous functions $[0, T] \to E$. We treat here the parabolic case: in other words we suppose that for each $t \in [0, T]$ $A(t)$ is the infinitesimal generator of an analytic semi-group $\{\exp[\xi A(t)]\}_{\xi\geq 0}$ (not necessarily strongly continuous at 0), whose domain $D(A(t))$ does not depend on t and is possibly not dense in E.

Problem (P) in the parabolic non-autonomous case with constant domains was first studied by TANABE ([32], [33], [34]) and SOBOLEVSKII [31] (see also YOSIDA [37]). In all these papers a regularity assumption for the function $t \to A(t)A(0)^{-1}$ in the uniform topology is needed: in the final development of the theory (see [34] and [31]) this function is required to be Hölder continuous and this assumption has become a standard one in later advances. These authors assume that $D(A(t)) \equiv D(A(0))$ is dense in E and prove their results by constructing the so called fundamental solution; they find classical solutions, i.e. continuously differentiable solutions of the equation in $]0, T]$, for each $x \in E$, and strict solutions (i.e. differentiable solutions

(*) Entrata in Redazione il 19 ottobre 1983.

in $[0, T]\big)$ for each $x \in D\big(A(0)\big)$, provided, in either case, f is Hölder continuous. If, in addition, $x = f(0) = 0$ TANABE [33] and POULSEN [28] showed that the derivative of the strict solution is in fact Hölder continuous (with an exponent stricly less than the exponent of f).

A different approach was carried on by DA PRATO-GRISVARD [11], who studied Problem (P) as a particular case of their theory about sums of non-commuting linear operators. They treat evolution both in L^p-spaces and in spaces of continuous functions, finding only, in the latter case, strong solutions (see Definition 1.5 below) for each continuous f, provided $x = 0$. In addition they prove some regularity results «in the space» for strong solutions, by means of the intermediate spaces $D_{A(0)}(\theta, \infty)$ between $D\big(A(0)\big)$ and E, showing that such solutions are continuous with values in $D_{A(0)}(\theta, \infty)$ for any $\theta \in]0, 1[$. Other space regularity results had been proved by SOBOLEVSKII [31] for strict and classical solutions, by using instead the domains of the fractional powers of $-A(t)$.

Existence and regularity «in time» of strict and classical solutions of (P) was proved by DA PRATO-SINESTRARI [13] whenever f is Hölder continuous and $x \in D\big(A(0)\big)$, provided x and $f(0)$ satisfy a suitable compatibility condition which involves the intermediate spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$: their hypotheses essentially coincide with Tanabe's and Sobolevskii's, but their method does not require the construction of the fundamental solution, using instead some sharp regularity theorems for Problem (P) in the autonomous case (see SINESTRARI [30]), which prove existence of the strict solution u, and Hölder regularity of u' with the same exponent as f, thus improving the results of [33], [28]. This kind of time regularity (i.e. if f belongs to a suitable subspace of $C\big([0, T], E\big)$, then u'—and consequently $A(\cdot)u(\cdot)$—belongs to the same subspace) is called «of maximal type» and is an important tool in the study of nonlinear equations by linearization methods as well as integral or delay equations, etc.

In the present paper we assume the classical hypotheses of TANABE [34] and SOBOLEVSKII [31], without supposing $\overline{D\big(A(t)\big)} = E$, and prove existence, uniqueness and regularity of several kinds of solutions, namely strict, classical and strong solutions. In particular we find again Tanabe's and Sobolevskii's results, and generalize those of DA PRATO-GRISVARD [11] to the case $x \neq 0$. We also weaken the hypotheses of DA PRATO-SINESTRARI [13], where maximal time regularity of strict solutions is proved under a slightly stronger assumption on $t \to A(t)A(0)^{-1}$; in addition we show that if f is Hölder continuous then the compatibility condition of DA PRATO-SINESTRARI [13] on x and $f(0)$ is in fact necessary and sufficient in order that the strict solution u has Hölder continuous derivative with the same exponent as f. About space regularity, we do not consider here strict and classical solutions, whose related properties will be the object of a subsequent paper; results in this direction, under stronger assumptions, are due to DA PRATO-GRISVARD [12]. We only prove some sharp space regularity theorems for strong solutions, generalizing the results of [11], and an «a priori» estimate for $A(t)u(t)$ in the case of a classical solution u, which is of interest in the study of quasi-linear problems.

Our method is mainly inspired by the techniques of [11]: see Section 6 and, in particular, Theorem 6.7, where however a very general and abstract situation in considered (see also IANNELLI [17]). We do not construct the fundamental solution; we obtain a different representation formula for the solutions of (P) and derive all our results by a careful analysis of it. Our formula is closely related to a similar one which we used in a recent paper [1] where Problem (P) was studied in the case of variable domains. It can be heuristically obtained as follows: given f, we look for a function g, depending on f, such that the solution of (P) can be written as

$$u(t) = \exp[tA(0)]x + \int_0^t \exp[(t-s)A(s)]g(s)\,ds\,, \tag{0.1}$$

If $A(t) \equiv A$ this formula with $g = f$ gives the ordinary mild solution of (P) (see e.g. KATO [19], page 486); when $A(t)$ depends on t, it is natural to expect that g has to be suitably modified. Taking the formal derivative of (0.1) we get

$$u'(t) = A(0)\exp[tA(0)]x + g(t) + \int_0^t A(s)\exp[(t-s)A(s)]g(s)\,ds\,.$$

If we want that u is a solution of Problem (P) we need that

$$\begin{aligned} u'(t) - A(t)u(t) = [A(0) - A(t)]\exp[tA(0)]x + g(t) + \\ + \int_0^t [A(s) - A(t)]\exp[(t-s)A(s)]g(s)\,ds = f(t)\,. \end{aligned}$$

Hence g must solve the following integral equation:

$$g(t) + \int_0^t K(t,s)g(s)\,ds = f(t) - K(t,0)x\,, \qquad t \in [0,T]\,, \tag{0.2}$$

where

$$K(t,s) = [A(s) - A(t)]\exp[(t-s)A(s)]\,, \qquad 0 \leqslant s < t \leqslant T\,. \tag{0.3}$$

Denote by K the integral operator defined by

$$K\varphi(t) = \int_0^t K(t,s)\varphi(s)\,ds\,;$$

then the representation formula for a solution of (P) is formally given by

$$u(t) = \exp[tA(0)]x + \int_0^t \exp[(t-s)A(s)](1+K)^{-1}\big(f - K(\cdot,0)x\big)(s)\,ds\,, \qquad t \in [0,T]\,. \tag{F}$$

The analogous formula in the case of variable domains (see [1]) is different, since it has a different «kernel»: here we have $\exp[(t-s)A(s)]$, while in [1] the kernel is $\exp[(t-s)A(t)]$. Indeed, in both situations we need the basic property $u(t) \in D(A(t))$, which is guaranteed by the presence of the analytic semigroup $\exp[(t-s)A(t)]$; but in the present case of constant domains the replacement by $\exp[(t-s)A(s)]$ does not affect the above property, and moreover it makes it possible to drop the differentiability assumption on the semi-group $t \to \exp[\xi A(t)]$, which played a crucial role in [1].

Let us describe now the subjects of the next sections. Section 1 contains a list of our notations, definitions and assumptions. In Section 2 we establish some preliminary results. In Section 3 we derive the basic material which is needed to prove our main theorems. In Section 4 we discuss strict solutions and their maximal time regularity. Section 5 concerns classical solutions. In Section 6 we study strong solutions. Section 7 is devoted to space regularity results. Finally in Section 8 we describe some examples.

1. – Notations and assumptions.

If A is a linear operator on a Banach space E, we denote by $D(A)$ its domain, and by $R(A)$ its range; $\varrho(A)$ is the resolvent set of A, $\sigma(A)$ its spectrum, and the resolvent operator $(\lambda - A)^{-1}$ is denoted by $R(\lambda, A)$. If X, Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators with domain X and range contained in Y, with norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x\in X-\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}.$$

When $X = Y$ we shall write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

Let Y be a Banach space. We shall make use of the following Banach spaces of functions:

a) $L^p(0, T, Y) = \{u\colon]0, T[\to Y\colon u$ is Bochner measurable and $\int_0^T \|u(t)\|_Y^p\, dt < +\infty\}$, $p \in [1, \infty[$, with norm

$$\|u\|_{L^p(0,T,Y)} = \left[\int_0^T \|u(t)\|_Y^p\, dt\right]^{1/p},$$

and $L^\infty(0, T, Y) = \{u\colon]0, T[\to Y\colon u$ is Bochner measurable and essentially bounded$\}$, with norm

$$\|u\|_{L^\infty(0,T,Y)} = \operatorname*{ess\,sup}_{t\in]0,T[} \|u(t)\|_Y;$$

b) $C([0,T],Y)=\{u\colon [0,T]\to Y \text{ continuous}\}$, with norm

$$\|u\|_{C([0,T],Y)}=\sup_{t\in[0,T]}\|u(t)\|_Y;$$

c) $C_\theta(]0,T],Y)=\{u\colon]0,T]\to Y$ continuous such that $t\to t^\theta u(t)$ is bounded in $]0,T]\}$, $\theta\in[0,1[$, with norm

$$\|u\|_{C_\theta(]0,T],Y)}=\|t^\theta u(\cdot)\|_{L^\infty(0,T,Y)},$$

and its subspace $C_\theta([0,T],Y)=\{u\in C_\theta(]0,T],Y)\colon \exists\lim_{t\to0^+}t^\theta u(t)\in Y\}$; note that if $u\in C_\theta([0,T],Y)$ we can and will consider the function $t\to t^\theta u(t)$ as an element of $C([0,T],Y)$: thus in particular we will identify the spaces $C_0([0,T],Y)$ and $C([0,T],Y)$;

d) $C^\beta([0,T],Y)=\{u\in C([0,T],Y)\colon \|u(t)-u(\tau)\|_Y=O(t-\tau)^\beta$ as $t-\tau\to0^+\}$, $\beta\in]0,1[$, with norm

$$\|u\|_{C^\beta([0,T],Y)}=\|u\|_{C([0,T],Y)}+\sup_{t\neq\tau}\frac{\|u(t)-u(\tau)\|_Y}{|t-\tau|^\beta},$$

and its subspace $h^\beta([0,T],Y)=\{u\in C^\beta([0,T],Y)\colon \|u(t)-u(\tau)\|_Y=o(t-\tau)^\beta$ as $t-\tau\to0^+\}$;

e) $\mathrm{Lip}([0,T],Y)=\{u\in C([0,T],Y)\colon \|u(t)-u(\tau)\|_Y=O(t-\tau)$ as $t-\tau\to0^+\}$, with norm

$$\|u\|_{\mathrm{Lip}([0,T],Y)}=\|u\|_{C([0,T],Y)}+\sup_{t\neq\tau}\frac{\|u(t)-u(\tau)\|_Y}{|t-\tau|};$$

f) $C^1([0,T],Y)=\{u\in C([0,T],Y)\colon \exists u'\in C([0,T],Y)\}$, with norm

$$\|u\|_{C^1([0,T],Y)}=\|u\|_{C([0,T],Y)}+\|u'\|_{C([0,T],Y)};$$

g) $C^{1,\beta}([0,T],Y)=\{u\in C^1([0,T],Y)\colon u'\in C^\beta([0,T],Y)\}$, $\beta\in]0,1[$, with norm

$$\|u\|_{C^{1,\beta}([0,T],Y)}=\|u\|_{C([0,T],Y)}+\|u'\|_{C^\beta([0,T],Y)},$$

and its subspace $h^{1,\beta}([0,T],Y)=\{u\in C^1([0,T],Y)\colon u'\in h^\beta([0,T],Y)\}$.

We shall also consider the following function spaces:

$$C(]0,T],Y)=\bigcap_{\varepsilon\in]0,T[}C([\varepsilon,T],Y),$$

and

$$C^\beta(]0,T],Y),\quad h^\beta(]0,T],Y),\quad \mathrm{Lip}(]0,T],Y),\quad C^1(]0,T],Y),$$
$$C^{1,\beta}(]0,T],Y),\quad h^{1,\beta}(]0,T],Y),$$

which are defined similarly. We explicitly note that $C_0(]0, T], Y)$ and $C(]0, T], Y)$ are different spaces.

Let us list now our assumptions, which are the classical ones of TANABE [34] and SOBOLEVSKII [31].

HYPOTHESIS I. – For each $t \in [0, T]$, $A(t)$ is a closed linear operator on the Banach space E, with domain $D(A(t)) \equiv D(A(0))$ (constant with respect to t), which generates an analytic semi-group $\{\exp[\xi A(t)]\}_{\xi \geq 0}$; more precisely:

(i) there exists $\theta_0 \in]\pi/2, \pi[$ such that

$$\varrho(A(t)) \supseteq \Sigma_{\theta_0} = \{z \in \boldsymbol{C}: z = \varrho \exp[i\theta],\ \varrho \in [0, \infty[,\ \theta \in]-\theta_0, \theta_0[\}, \qquad \forall t \in [0, T];$$

(ii) there exists $M > 0$ such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leqslant M/|\lambda|, \quad \forall \lambda \in \Sigma_{\theta_0} - \{0\}; \qquad \|A(t)^{-1}\|_{\mathcal{L}(E)} \leqslant M, \quad \forall t \in [0, T].$$

HYPOTHESIS II. – There exist $\alpha \in]0, 1[$, $B > 0$ such that

$$\|1 - A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)} \leqslant B|t - \tau|^\alpha, \quad \forall t, \tau \in [0, T].$$

REMARK 1.1. – (*a*) We do not assume that $D(A(0))$ is dense in E: this means that the analytic semi-groups $\xi \to \exp[\xi A(t)]$, $t \in [0, T]$, are not strongly continuous at $\xi = 0$. This is in fact the case in several concrete examples (see ACQUISTAPACE-TERRENI [1], SINESTRARI [30]). However if Hypothesis I holds and E is locally sequentially weakly compact (e.g. E is reflexive) then necessarily $D(A(0))$ is dense in E (see KATO [18]).

(*b*) The assumption that $0 \in \varrho(A(t))$ for each $t \in [0, T]$ is not really restrictive. Indeed, suppose that (i) and (ii) of Hypothesis I are satisfied only in $\Sigma_{\theta_0} - \{0\}$, and replace $u(t)$ by $v(t) = \exp[-\varepsilon t]u(t)$: then $v(t)$ solves Problem (P) with $A(t)$ and $f(t)$ replaced by $A(t) - \varepsilon$ and $\exp[-\varepsilon t]f(t)$ respectively. As $\lambda + \varepsilon \in \Sigma_{\theta_0} - \{0\}$ for each $\lambda \in \Sigma_{\theta_0}$, our weakened assumption yields $\Sigma_{\theta_0} \subseteq \varrho(A(t) - \varepsilon)$ and $\|R(\lambda, A(t) - \varepsilon)\|_{\mathcal{L}(E)} \leqslant M/|\lambda + \varepsilon|$ for each $\lambda \in \Sigma_{\theta_0}$ and $t \in [0, T]$. As $|\lambda|/|\lambda + \varepsilon| \leqslant |\lambda|/|\mathrm{Im}\,\lambda| \leqslant 1/\sin\theta$ for each $\lambda \in \Sigma_{\theta_0}$, we deduce that $\{A(t) - \varepsilon\}_{t \in [0, T]}$ satisfies (i) and (ii) of Hypothesis I in the whole Σ_{θ_0}, with M replaced by $M/\varepsilon \sin\theta$.

REMARK 1.2. – We can slightly weaken Hypotheses I and II by supposing that there exists $\omega_0 > 0$ such that:

(i) $\varrho(A(t)) \supseteq \{z \in \boldsymbol{C}: \mathrm{Re}\, z > \omega_0\} \cup \{\omega_0\}$, $\forall t \in [0, T]$;

(ii) $\|R(\omega_0, A(t))\|_{\mathcal{L}(E)} \leqslant M$, $\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leqslant M/|\lambda - \omega_0|$ if $\mathrm{Re}\,\lambda > \omega_0$, $\forall t \in [0, T]$;

iii) $\|1 - (\omega_0 - A(t))R(\omega_0, A(\tau))\|_{\mathcal{L}(E)} \leqslant B|t - \tau|^\alpha$, $\forall t, \tau \in [0, T]$.

Indeed, if this is true, from the estimate of the resolvent one can easily deduce that $\varrho(A(t))$ contains in fact a sector $\Sigma_{\theta_0} + \omega_0 = \{z \in \boldsymbol{C}: z - \omega_0 \in \Sigma_{\theta_0}\}$ for some $\theta_0 \in]\pi/2, \pi[$; thus, as in Remark 1.1 (*b*), we can go back to the initial situation by considering the function $v(t) = \exp[-\omega_0 t]u(t)$.

Let us specify now what we mean as a «solution» of Problem (P).

DEFINITION 1.3. – Let $f \in C([0, T], E)$, $x \in E$. We say that $u: [0, T] \to E$ is a *strict solution* of (P) if $u \in C^1([0, T], E)$, $u(t) \in D(A(0))$ for each $t \in [0, T]$, and

$$u'(t) - A(t)u(t) = f(t) \qquad \forall t \in [0, T],\ u(0) = x\,.$$

DEFINITION 1.4. – Let $f \in C(]0, T], E)$, $x \in E$. We say that $u: [0, T] \to E$ is a *classical solution* of (P) if $u \in C([0, T], E) \cap C^1(]0, T], E)$, $u(t) \in D(A(0))$ for each $t \in]0, T]$ and

$$u'(t) - A(t)u(t) = f(t) \qquad \forall t \in]0, T],\ u(0) = x\,.$$

DEFINITION 1.5. – Let $f \in C([0, T], E)$, $x \in E$. We say that $u: [0, T] \to E$ is a *strong solution* if $u \in C([0, T], E)$ and there exists $\{u_n\}_{n \in \boldsymbol{N}} \subseteq C^1([0, T], E)$ such that:

$$\begin{aligned}
&u_n(t) \in D(A(0)),\ \ \forall t \in [0, T]\,, \quad t \to A(t)u_n(t) \in C([0, T], E),\ \ \forall n \in \boldsymbol{N}\,,\\
&u_n' - A(\cdot)u_n(\cdot) \equiv f_n \to f \quad \text{in } C([0, T], E) \text{ as } n \to \infty\,,\\
&u_n(0) \equiv x_n \to x \quad \text{in } E \text{ as } n \to \infty\,,\\
&u_n \to u \quad \text{in } C([0, T], E) \text{ as } n \to \infty\,.
\end{aligned}$$

REMARK 1.6. – By definition, a strict solution is a classical and a strong one. If $f \in C([0, T], E)$, it is not obvious that a classical solution is a strong one; we shall see later (Theorem 6.7) that this is true under Hypotheses I, II. The situation in the case of variable domains is different (see Remark 6.5 of [1]).

REMARK 1.7. – Problem (P) with initial time $t_0 \neq 0$ can be treated quite similarly to the case $t_0 = 0$, since in all successive estimates the constants in fact depend on the amplitude of the time interval, and not on its endpoints.

2. – Preliminaries.

Let A be a linear operator on a Banach space E, satisfying Hypothesis I. Then the semi-group $\{\exp[\xi A]\}_{\xi \geqslant 0}$ can be represented by a Dunford integral along a curve $\gamma \subseteq \Sigma_{\theta_0}$, joining $+\infty \exp[-i\theta]$ and $+\infty \exp[i\theta]$. For our purposes it is convenient to choose from now on the curve $\gamma = \gamma^0 \cup \gamma^+ \cup \gamma^-$, where

$$\gamma^0 = \{z \in \boldsymbol{C}\colon |z| = 1,\ |\arg z| \leqslant \theta\}$$
$$\gamma^\pm = \{z \in \boldsymbol{C}\colon z = \varrho \exp[\pm i\theta],\ \varrho \geqslant 1\}\,.$$

Then we have

$$\exp[\xi A] = \frac{1}{2\pi i}\int_\gamma \exp[\xi\lambda] R(\lambda, A)\, d\lambda\,, \qquad \xi > 0\,;$$

in particular $\exp[\xi A]$ maps E into $D(A^n)$, $\forall n \in \boldsymbol{N}$ and

$$A^n \exp[\xi A] = \frac{1}{2\pi i}\int_\gamma \lambda^n \exp[\xi\lambda] R(\lambda, A)\, d\lambda\,, \qquad \xi > 0,\ n \in \boldsymbol{N}\,, \tag{2.1}$$

the integrals being absolutely convergent.

If A is a closed linear operator on the Banach space E, then the subspace $D(A)$ of E, equipped with the graph norm, is itself a Banach space continuously imbedded into E. Hence we can define the intermediate spaces $(D(A), E)_{\sigma,\infty}$, $\sigma \in]0, 1[$, as follows (see LIONS [21], LIONS-PEETRE [22]):

DEFINITION 2.1. – If $x \in E$, we say that $x \in (D(A), E)_{\sigma,\infty}$ (resp. $(D(A), E)_\sigma$) if there exists $u\colon]0, 1] \to D(A)$ having first derivative in the sense of distributions $u'\colon]0, 1] \to E$, such that:

(i) $u', Au(\cdot) \in C_\sigma^{\rightarrow}(]0, 1], E)$ (resp. $C_\sigma([0, 1], E)$);

(ii) $u(0) = x$.

Note that condition (ii) of Definition 2.1 is meaningful since it is easily seen that condition (i) implies $u \in C^{1-\sigma}([0, 1], E)$. It is also clear that

$$D(A) \subseteq (D(A), E)_\sigma \subseteq (D(A), E)_{\sigma,\infty} \subseteq \overline{D(A)}\,, \qquad \forall \sigma \in]0, 1[\,.$$

When A is the infinitesimal generator of a bounded analytic semi-group, the spaces $(D(A), E)_{1-\theta,\infty}$ and $(D(A), E)_{1-\theta}$ are denoted by $D_A(\theta, \infty)$ and $D_A(\theta)$ $(\theta \in]0, 1[)$. We can characterize $D_A(\theta, \infty)$ in several ways (see BUTZER-BERENS [5] for the case $\overline{D(A)} = E$, and Proposition 1.3 and Remark 1.4 of [1] for the general case), namely:

$$D_A(\theta, \infty) = \left\{x \in E\colon \sup_{t>0} \left\|\frac{\exp[tA]x - x}{t^\theta}\right\|_E < \infty\right\} =$$
$$= \left\{x \in E\colon \sup_{t>0} \|t^{1-\theta} A \exp[At]x\|_E < \infty\right\} = \left\{x \in E\colon \sup_{\lambda \in \varrho(A)} |\lambda|^\theta \|AR(\lambda, A)x\|_E < \infty\right\}.$$

$D_A(\theta, \infty)$ is a Banach space with the norm

$$\|x\|_{D_A(\theta,\infty)} = \|x\|_E + \sup_{t>0} \left\| \frac{\exp[tA] - 1}{t^\theta} x \right\|_E , \tag{2.2}$$

and moreover it can be verified (see [5]) that the quantities

$$\sup_{t>0} \left\| \frac{\exp[tA] - 1}{t^\theta} x \right\|_E , \qquad \sup_{t>0} \|t^{1-\theta} A \exp[tA] x\|_E , \qquad \sup_{\lambda \in \varrho(A)} |\lambda|^\theta \|AR(\lambda, A)x\|_E$$

define equivalent semi-norms on $D_A(\theta, \infty)$. If, in addition, $0 \in \varrho(A)$, they become equivalent norms on $D_A(\theta, \infty)$, all being equivalent to (2.2).

If $0 < \beta < \theta < 1$ we have

$$\begin{aligned}
&\|x\|_E \leqslant C\|x\|_{D_A(\beta,\infty)} , && \forall x \in D_A(\beta, \infty) \\
&\|x\|_{D_A(\beta,\infty)} \leqslant C\|x\|_{D_A(\theta,\infty)} , && \forall x \in D_A(\theta, \infty) \\
&\|x\|_{D_A(\theta,\infty)} \leqslant C\|Ax\|_E , && \forall x \in D(A) .
\end{aligned}$$

Here and in what follows we will denote by C any constant arising in our estimates.

It is easy to show that for each $\theta \in]0, 1[$, $D_A(\theta)$ is a closed subspace of $D_A(\theta, \infty)$ and that the following characterizations hold:

$$\begin{aligned}
D_A(\theta) = \left\{ x \in D_A(\theta, \infty) \colon \lim_{t \to 0^+} \left\| \frac{\exp[tA] - 1}{t^\theta} x \right\|_E = 0 \right\} = \\
= \left\{ x \in D_A(\theta, \infty) \colon \lim_{t \to 0^+} \|t^{1-\theta} A \exp[tA] x\|_E = 0 \right\} = \\
= \left\{ x \in D_A(\theta, \infty) \colon \lim_{\substack{\lambda \in \varrho(A) \\ |\lambda| \to \infty}} |\lambda|^\theta \|AR(\lambda, A)x\|_E \right\} = 0 .
\end{aligned}$$

Finally it is not difficult to verify that $D_A(\theta)$ is the closure of $D(A)$ in the norm of $D_A(\theta, \infty)$ (see [5], Chapter III, Proposition 3.16, or SINESTRARI [30], Proposition 1.8).

Now we go back to our situation and prove a lemma which will be systematically used in the following.

LEMMA 2.2. – *Under Hypotheses I, II we have:*

$$\|R(\lambda, A(t)) - R(\lambda, A(\tau))\|_{\mathfrak{L}(E)} \leqslant C \frac{(t-\tau)^\alpha}{|\lambda|} , \qquad \forall \lambda \in \Sigma_{\theta_0} , \ \forall t, \tau \in [0, T] .$$

PROOF. – It is enough to observe that

$$R(\lambda, A(t)) - R(\lambda, A(\tau)) = R(\lambda, A(t))\big(A(t)A(\tau)^{-1} - 1\big)A(\tau)R(\lambda, A(\tau)) . \qquad ///$$

As a consequence of the fact that $D(A(t))$ does not depend on t, we have the following

PROPOSITION 2.3. – *Under Hypotheses I, II, for each* $t \in [0, T]$ *and* $\theta \in]0, 1[$ *we have*

$$D_{A(t)}(\theta, \infty) = D_{A(0)}(\theta, \infty) \quad \textit{and} \quad D_{A(t)}(\theta) = D_{A(0)}(\theta)\,.$$

In addition there exist C_1 *and* C_2 *such that if* $x \in D_{A(0)}(\theta, \infty)$

$$\sup_{\xi>0}\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}x\right\|_E \leqslant C_1 \sup_{\xi>0}\left\|\frac{\exp[\xi A(t)]-1}{\xi^\theta}x\right\|_E \leqslant$$
$$\leqslant C_2 \sup_{\xi>0}\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}x\right\|_E, \qquad \forall t \in [0, T]\,.$$

From now on we will write

$$\|x\|_\theta = \sup_{\xi>0}\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}x\right\|_E; \tag{2.3}$$

since $0 \in \varrho(A(0))$, (2.3) is a norm on $D_{A(0)}(\theta, \infty)$.

PROOF. – The equalities follow by definition. To prove the estimates it is sufficient to show that there exists $C > 0$ such that

$$\sup_{\xi>0}\left\|\frac{\exp[\xi A(t)]-\exp[\xi A(0)]}{\xi^\theta}x\right\|_E \leqslant C\left[\|x\|_\theta \wedge \sup_{\xi>0}\left\|\frac{\exp[\xi A(t)]-1}{\xi^\theta}x\right\|_E\right],$$
$$\forall x \in D_{A(0)}(\theta, \infty),\ \forall t \in [0, T]\,.$$

Now if $\xi > 0$ we have:

$$\frac{\exp[\xi A(t)]-\exp[\xi A(0)]}{\xi^\theta} = \frac{1}{2\pi i}\int_\gamma \exp[\xi\lambda]\frac{1}{\xi^\theta}[R(\lambda, A(t)) - R(\lambda, A(0))]\,d\lambda =$$
$$= \frac{1}{2\pi i}\int_\gamma \exp[\xi\lambda]\frac{1}{\xi^\theta}R(\lambda, A(t))\big(A(t)A(0)^{-1}-1\big)A(0)R(\lambda, A(0))\,d\lambda\,.$$

Hence

$$\left\|\frac{\exp[\xi A(t)]-\exp[\xi A(0)]}{\xi^\theta}x\right\|_E \leqslant C\int_\gamma \exp[\xi\cdot\mathrm{Re}\,\lambda]\frac{t^\alpha|\lambda|^\theta}{\xi^\theta|\lambda|^{1+\theta}}\|A(0)R(\lambda, A(0))x\|_E|d\lambda| \leqslant$$
$$\leqslant Ct^\alpha \sup_{\lambda\in\varrho(A(0))}|\lambda|^\theta\|A(0)R(\lambda, A(0))x\|_E \leqslant CT^\alpha\|x\|_\theta\,.$$

By reversing the roles of t and 0 we find also, for each $\xi > 0$:

$$\left\|\frac{\exp[\xi A(t)]-\exp[\xi A(0)]}{\xi^\theta}x\right\|_E \leqslant CT^\alpha \sup_{\xi>0}\left\|\frac{\exp[\xi A(t)]-1}{\xi^\theta}x\right\|_E. \qquad ///$$

LEMMA 2.4. – *Under Hypotheses I, II we have:*

(i) $\|\exp[tA(s)]\|_{\mathcal{L}(E)} \leq C, \quad \forall t, s \in [0, T].$

(ii) $\|A(\tau)\exp[tA(s)]\|_{\mathcal{L}(E)} \leq \dfrac{C}{t}, \quad \forall \tau, s \in [0, T], \forall t \in]0, T].$

(iii) $\|A(\tau)\exp[tA(s)]\|_{\mathcal{L}(D_{A(0)}(\beta,\infty),E)} \leq \dfrac{C}{t^{1-\beta}}, \quad \forall \tau, s \in [0, T], \forall t \in]0, T], \forall \beta \in]0, 1[.$

(iv) *If* $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1[$, *then* $\lim_{t\to 0^+} t^{1-\beta}\|A(\tau)\exp[tA(s)]x\|_E = 0, \quad \forall \tau, s \in [0, T].$

(v) $\|A(\tau)\exp[tA(s)]\|_{\mathcal{L}(D(A(0)),E)} \leq C, \quad \forall \tau, s, t \in [0, T].$

(vi) *If* $x \in \overline{D(A(0))}$ *then* $\lim_{t\leftarrow 0^+} \sup_{s,\tau\in[0,T]} t\|A(\tau)\exp[tA(s)]x\|_E = 0.$

PROOF. – (i) It can be proved exactly as in the classical theory of analytic semigroups (see e.g. MARTIN [24]).

(ii) Similarly (see [24])

$$\|A(\tau)\exp[tA(s)]x\|_E \leq \|A(\tau)A(s)^{-1}\|_{\mathcal{L}(E)} \cdot \|A(s)\exp[tA(s)]x\|_E \leq \frac{C}{t}\|x\|_E .$$

(iii) We have for $x \in D_{A(0)}(\beta, \infty)$

$$\|A(\tau)\exp[tA(s)]x\|_E \leq \|A(\tau)A(s)^{-1}\|_{\mathcal{L}(E)} \cdot \|A(s)\exp[tA(s)]x\|_E \leq \frac{C}{t^{1-\beta}}\|x\|_\beta .$$

(iv) We have for $x \in D_{A(0)}(\beta)$

$$\lim_{t\to 0^+} \|t^{1-\beta}A(\tau)\exp[tA(s)]x\|_E \leq C \lim_{t\to 0^+} \|t^{1-\beta}A(s)\exp[tA(s)]x\|_E = 0 .$$

(v) We have for $x \in D(A(0))$

$$\|A(\tau)\exp[tA(s)]x\|_E \leq \|A(\tau)A(s)^{-1}\|_{\mathcal{L}(E)}\|A(s)\exp[tA(s)]x\|_E \leq C\|A(0)x\|_E:$$

this proves (v).

(vi) If $x \in D(A(0))$ the result follows by (v); the general case is a consequence of (iii). ///

LEMMA 2.5. – *Under Hypotheses I, II we have:*

(i) $$\overline{D(A(0))} = \{x \in E : t \to \exp[tA(s)]x \in C_0([0, T], E)\} = \{x \in E : \lim_{t\to 0^+} \|\exp[tA(s)]x - x\|_E = 0\}, \quad \forall s \in [0, T].$$

(ii) $$D(A(0)) \subseteq \left\{x \in E : t \to \frac{\exp[tA(s)] - 1}{t} x \in C_0(]0, T], E)\right\} = \{x \in E : t \to A(s)\exp[tA(s)]x \in C_0(]0, T], E)\}, \quad \forall s \in [0, T].$$

(iii) $\{x \in D(A(0)) \colon A(0)x \in \overline{D(A(0))}\} =$

$$= \left\{x \in E \colon t \to \frac{\exp[tA(0)] - 1}{t} x \in C_0([0, T], E)\right\} =$$

$$= \left\{x \in E \colon \lim_{t\to 0^+} \left\| \frac{\exp[tA(0)] - 1}{t} x - A(0)x \right\|_E = 0\right\} =$$

$$= \{x \in E \colon t \to A(0) \exp[tA(0)]x \in C_0([0, T], E)\} =$$

$$= \left\{x \in E \colon \lim_{t\to 0^+} \|A(0) \exp[tA(0)]x - A(0)x\|_E = 0\right\}.$$

(iv) $\overline{D(A(0))} = \overline{D(A(0)^2)} =$ closure of $\{x \in D(A(0)) \colon A(0)x \in \overline{D(A(0))}\}$.

PROOF. – (i)-(ii)-(iii) See [30], Proposition 1.2.

(iv) We have $D(A(0)) \supseteq \{x \in D(A(0)) \colon A(0)x \in \overline{D(A(0))}\} \supseteq D(A(0)^2)$ and, by (i),

$$\exp[tA(0)]x \to x, \quad \forall x \in \overline{D(A(0))} \text{ as } t \to 0^+;$$

Since $\exp[tA(0)]x \in D(A(0)^2)$, the proof is complete. ///

Set

$$K(t, s) = (A(t) - A(s)) \exp[(t - s)A(s)], \quad 0 \leqslant s \leqslant t \leqslant T. \tag{2.4}$$

Then clearly $K(t, s) \in \mathcal{L}(E)$ and in particular $K(s, s) = 0$, $\forall s \in [0, T]$, but $t \to K(t, s)$ is not continuous in general at $t = s$. In fact we have:

LEMMA 2.6. – *Under Hypotheses I, II the following estimate holds*:

$$\|K(t, s)\|_{\mathcal{L}(E)} \leqslant \frac{C}{(t - s)^{1-\alpha}}, \quad 0 \leqslant s < t \leqslant T.$$

PROOF. – It follows from the equality

$$K(t, s) = (1 - A(t)A(s)^{-1}) A(s) \exp[(t - s)A(s)]. \quad ///$$

3. – Basic results.

This section contains a series of technical lemmata which examine in detail the operators and functions appearing in the representation formula (F) of the introduction. We follow the same lines of Section 3 in [1], where a similar sequence of statements is given; however in the present case we have simpler and more precise results.

a) The function $t \to \exp[tA(0)]x$.

LEMMA 3.1. – *Let* $s \in [0, T]$. *Under Hypotheses I, II we have*:

(i) *If* $x \in E$, *then* $\exp[tA(s)]x \in D(A(s)^n)$, $\forall n \in \mathbf{N}$, $\forall t \in]0, T]$.

(ii) *If* $x \in E$, *then* $t \to \exp[tA(s)]x \in C^\infty(]0, T], E)$ *and*

$$\left(\frac{d}{dt}\right)^n \exp[tA(s)]x = A(s)^n \exp[tA(s)]x, \qquad \forall n \in \mathbf{N},\ \forall t \in]0, T],$$

(iii) $x \in \overline{D(A(0))}$ *if and only if* $t \to \exp[tA(s)]x \in C([0, T], E)$, *and in this case* $[\exp[tA(s)]x]_{t=0} = x$.

(iv) $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1[$, *if and only if* $t \to \exp[tA(s)]x \in C^\beta([0, T], E)$, *and in this case*

$$\|\exp[tA(s)]x\|_{C^\beta([0,T],E)} \leq C\|x\|_\beta .$$

(v) $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1[$, *if and only if* $t \to \exp[tA(s)]x \in h^\beta([0, T], E)$.

(vi) *If* $x \in D(A(0))$ *then* $t \to \exp[tA(s)]x \in \mathrm{Lip}([0, T], E)$ *and*

$$\|\exp[tA(s)]x\|_{\mathrm{Lip}([0,T],E)} \leq C\|A(0)x\|_E .$$

(vii) $x \in D(A(0))$ *and* $A(0)x \in \overline{D(A(0))}$ *if and only if* $t \to \exp[tA(0)]x \in C^1([0, T], E)$, *and in this case*

$$\left[\frac{d}{dt}\exp[tA(0)]x\right]_{t=0} = A(0)x .$$

(viii) $x \in \overline{D(A(0))}$ *if and only if*

$$t \to \frac{\exp[tA(0)] - 1}{t} A(0)^{-1}x \in C([0, T], E),$$

and in this case

$$\left[\frac{\exp[tA(0)] - 1}{t} A(0)^{-1}x\right]_{t=0} = x .$$

PROOF. – (i)-(ii)-(iii) Easy consequences of (2.1) and Lemma 2.5 (i).

(iv) Let $x \in D_{A(0)}(\beta, \infty)$; then by Proposition 2.3

$$\|\exp[tA(s)]x - \exp[\tau A(s)]x\|_E \leq C\|(\exp[(t-\tau)A(s)] - 1)x\|_E \leq C(t-\tau)^\beta\|x\|_\beta ,$$

where $\|x\|_\beta$ is defined in (2.3).

Conversely, if $t \to \exp[tA(s)]x \in C^\beta([0, T], E)$, then in particular

$$\|(\exp[tA(s)] - 1)x\|_E \leq C(s)t^\beta ,$$

which implies, again by Proposition 2.3, the result.

(v) Quite similar to (iv).

(vi)-(vii) See [30], Theorem 3.1 (*d*)-(*e*).

(viii) It follows by Lemma 2.5 (iii). ///

b) The function $t \to K(t, 0)x = [A(0) - A(t)]\exp[tA(0)]x$.

LEMMA 3.2. – *Under Hypotheses I, II we have:*

(i) $K(\cdot, 0) \in \mathcal{L}\big(E, C_{1-\alpha}(]0, T], E)\big)$.

(ii) *If* $x \in E$ *then* $K(\cdot, 0)x \in C^\alpha(]0, T], E)$.

(iii) $K(\cdot, 0) \in \mathcal{L}\big(D_{A(0)}(\beta, \infty), C_{1-\alpha-\beta}(]0, T], E)\big)$, $\forall \beta \in]0, 1-\alpha]$.

(iv) *If* $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1-\alpha]$, *then* $K(\cdot, 0)x \in C_{1-\alpha-\beta}([0, T], E)$ *and* $[t^{1-\alpha-\beta}K(t, 0)x]_{t=0} = 0$.

(v) $K(\cdot, 0) \in \mathcal{L}\big(D_{A(0)}(\beta, \infty), C^{\alpha+\beta-1}([0, T], E)\big)$, $\forall \beta \in]1-\alpha, 1[$.

(vi) *If* $x \in D_{A(0)}(\beta)$, $\beta \in]1-\alpha, 1[$, *then* $K(\cdot, 0)x \in h^{\alpha+\beta-1}([0, T], E)$.

(vii) $K(\cdot, 0) \in \mathcal{L}\big(D(A(0)), C^\alpha([0, T], E)\big)$.

PROOF. – (i) Evidently Lemma 2.6 yields

$$\|K(t, 0)x\|_E \leq \frac{C}{t^{1-\alpha}}\|x\|_E .$$

(ii) If $t > \tau \geq \varepsilon$ we have

$$\|K(t, 0)x - K(\tau, 0)x\|_E \leq \|[A(\tau)A(t)^{-1} - 1]A(t)\exp[tA(0)]x\|_E + \\ + \left\|[1 - A(\tau)A(0)^{-1}]\int_\tau^t A(0)^2 \exp[\sigma A(0)]x\, d\sigma\right\|_E \leq \frac{C}{\varepsilon}(t-\tau)^\alpha\|x\|_E + \frac{C}{\varepsilon^{2-\alpha}}(t-\tau)\|x\|_E .$$

(iii) If $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1-\alpha]$, from the characterization of $D_{A(0)}(\beta, \infty)$ we get

$$\|K(t, 0)x\|_E \leq Ct^\alpha\|A(0)\exp[tA(0)]x\|_E \leq \frac{C}{t^{1-\alpha-\beta}}\|x\|_\beta .$$

(iv) If $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1-\alpha]$, from the characterization of $D_{A(0)}(\beta)$ we have

$$\lim_{t\to 0^+} \|t^{1-\alpha-\beta}K(t,0)x\|_E \leqslant \lim_{t\to 0^+} t^{-\alpha}\|A(t)A(0)^{-1}-1\|_{\mathfrak{L}(E)}\|t^{1-\beta}A(0)\exp[tA(0)]x\|_E = 0\ .$$

(v) Suppose $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]1-\alpha, 1[$; if $t > \tau \geqslant 0$ as in (ii) we get

$$\|K(t,0)x - K(\tau,0)x\|_E \leqslant C\frac{(t-\tau)^\alpha}{t^{1-\beta}}\|x\|_\beta + C\int_\tau^t \frac{d\xi}{\xi^{2-\beta-\alpha}}\|x\|_\beta \leqslant C(t-\tau)^{\alpha+\beta-1}\|x\|_\beta\,.$$

(vi) Let $x \in D_{A(0)}(\beta)$, $\beta \in]1-\alpha, 1[$. As in (ii), we have as $t-\tau \to 0^+$:

$$\begin{aligned}
&\|K(t,0)x - K(\tau,0)x\|_E \leqslant \\
&\quad \leqslant \|A(\tau)A(t)^{-1}-1\|_{\mathfrak{L}(E)}\|A(t)A(0)^{-1}\|_{\mathfrak{L}(E)}\|\exp[\tau A(0)]\|_{\mathfrak{L}(E)}\|A(0)\exp[(t-\tau)A(0)]x\|_E + \\
&\quad + \|1 - A(\tau)A(0)^{-1}\|_{\mathfrak{L}(E)}\|\exp[\tfrac{1}{2}\tau A(0)]\|_{\mathfrak{L}(E)}\int_\tau^t \|A(0)\exp[\tfrac{1}{2}\xi A(0)]\|_{\mathfrak{L}(E)}\cdot \\
&\quad \cdot\|A(0)\exp[\tfrac{1}{2}(\xi-\tau)A(0)]x\|_E\,d\xi \leqslant C(t-\tau)^\alpha\frac{o(1)}{(t-\tau)^{1-\beta}} + C\int_\tau^t \frac{\tau^\alpha\,d\xi}{\xi(\xi-\tau)^{1-\beta}}o(1) = \\
&\qquad\qquad = o(t-\tau)^{\alpha+\beta-1} + C\int_\tau^t \frac{d\sigma}{(\sigma-\tau)^{2-\beta-\alpha}}o(1) = o(t-\tau)^{\alpha+\beta-1}\,.
\end{aligned}$$

(vii) Let $x \in D(A(0))$; as in (ii) we have for $t > \tau > 0$:

$$\begin{aligned}
&\|K(t,0)x - K(\tau,0)x\|_E \leqslant C(t-\tau)^\alpha\|A(t)\exp[tA(0)]x\|_E + \\
&\qquad + C\tau^\alpha\int_\tau^t \|A(0)\exp[\xi A(0)]A(0)x\|_E\,d\xi \leqslant C(t-\tau)^\alpha\|A(0)x\|_E\,. \qquad ///
\end{aligned}$$

c) The operator $K\varphi(t) = \int_0^t K(t,s)\varphi(s)\,ds = \int_0^t [A(s)-A(t)]\exp[(t-s)A(s)]\varphi(s)\,ds.$

LEMMA 3.3. – *Under Hypotheses I, II, we have*:

(i) $K \in \mathfrak{L}(L^p(0,T,E))$, $\forall p \in [1,\infty]$.

(ii) *If* $\varphi \in C_\theta(]0,T],E)$, $\theta \in [0,1[$, *then* $K\varphi \in C^\delta(]0,T],E)$, $\forall\delta \in]0,\alpha[$.

(iii) *If* $\varphi \in C_\theta(]0,T],E) \cap C^\sigma(]0,T],E)$, $\theta \in [0,1[$, $\sigma \in]0,1[$, *then* $K\varphi \in C^\alpha(]0,T],E)$.

(iv) $K \in \mathfrak{L}\big(C_\theta(]0,T],E),\ C_{\theta-\alpha}(]0,T],E)\big)$, $\forall\theta \in [\alpha,1[$.

(v) $K \in \mathfrak{L}\big(C_\theta(]0,T],E),\ C^\delta([0,T],E)\big)$, $\forall\theta \in [0,\alpha[$, $\forall\delta \in]0,\alpha-\theta[$; *in particular* $K\varphi(0) = 0$, $\forall\varphi \in C_\theta(]0,T],E)$.

(vi) $K \in \mathfrak{L}\big(C^\sigma([0,T],E),\ C^\alpha([0,T],E)\big)$, $\forall\sigma \in]0,1[$.

PROOF. - (i) If $p=\infty$ the result is obvious. If $p<\infty$ we have by Lemma 2.6

$$\|K\varphi\|^p_{L^p(0,T,E)} \leqslant \int_0^T \left(\int_0^t \frac{C}{(t-s)^{1-\alpha}} \|\varphi(s)\|_E \, ds\right)^p dt =$$

$$= \int_0^T \left(\int_0^t \frac{C}{(t-s)^{(1-\alpha)/p}(t-s)^{(1-\alpha)(p-1)/p}} \|\varphi(s)\|_E \, ds\right)^p dt \leqslant \int_0^T \left[\int_0^t \frac{C}{(t-s)^{1-\alpha}} \|\varphi(s)\|^p_E \, ds\right] \cdot$$

$$\cdot \left(\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\right)^{p-1} dt \leqslant C\left(\frac{T^\alpha}{\alpha}\right)^{p-1} \int_0^T \|\varphi(s)\|^p_E \int_s^T \frac{dt}{(t-s)^{1-\alpha}} \, ds \leqslant C\frac{T^\alpha}{\alpha} \|\varphi\|^p_{L^p(0,T,E)},$$

where we have used Hölder's inequality and Fubini-Tonelli's Theorem.

(ii) Let $t>\tau\geqslant\varepsilon$. Then

$$\|K\varphi(t)-K\varphi(\tau)\|_E \leqslant \int_\tau^t \|K(t,s)\|_{\mathcal{L}(E)} \|\varphi(s)\|_E \, ds + \left\|\int_0^\tau (K(t,s)-K(\tau,s))\varphi(s)\, ds\right\|_E \leqslant \tag{3.1}$$

$$\leqslant \int_\tau^t \|K(t,s)\|_{\mathcal{L}(E)} \|\varphi(s)\|_E \, ds + \left\|\int_0^\tau (1-A(t)A(\tau)^{-1})A(\tau)\exp\left[(t-s)A(s)\right]\varphi(s)\, ds\right\|_E +$$

$$+ \left\|\int_0^\tau (1-A(\tau)A(s)^{-1})\int_{\tau-s}^{t-s} A(s)^2 \exp\left[\xi A(s)\right]\varphi(s)\, d\xi\, ds\right\|_E.$$

Hence we can write

$$\|K\varphi(t)-K\varphi(\tau)\|_E \leqslant C\Bigg[\int_\tau^t \frac{\|\varphi(s)\|_E}{(t-s)^{1-\alpha}} ds +$$

$$+ \left[\int_0^{\varepsilon/2} + \int_{\varepsilon/2}^\tau\right] \|1-A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)} \|A(\tau)\exp\left[(t-s)A(s)\right]\varphi(s)\|_E \, ds +$$

$$+ \int_0^\tau \|1-A(\tau)A(s)^{-1}\|_{\mathcal{L}(E)} \int_{\tau-s}^{t-s} \|A(s)^2 \exp\left[\xi A(s)\right]\varphi(s)\|_E \, d\xi\, ds\Bigg] \leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)} \Bigg[\frac{(t-\tau)^\alpha}{\varepsilon^\theta} + \frac{(t-\tau)^\alpha\cdot(\varepsilon/2)^{1-\theta}}{\varepsilon/2} + \frac{(t-\tau)^\alpha}{(\varepsilon/2)^\theta} \log\frac{t-\varepsilon/2}{t-\tau} +$$

$$+ \frac{(t-\tau)\cdot(\varepsilon/2)^\theta}{(\varepsilon/2)^{2-\alpha}} + \int_{\varepsilon/2}^\tau \frac{1}{(\varepsilon/2)^\theta}\left[\frac{1}{(\tau-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\right] ds\Bigg] \leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)} \Bigg[\frac{(t-\tau)^\alpha}{\varepsilon^\theta} + \frac{(t-\tau)^\alpha}{\varepsilon^\theta}\log\left(1+\frac{\tau}{t-\tau}\right) +$$

$$+ \frac{(t-\tau)}{\varepsilon^{1+\theta-\alpha}} + \frac{1}{\varepsilon^\theta}\left((\tau-\varepsilon/2)^\alpha - (t-\varepsilon/2)^\alpha + (t-\tau)^\alpha\right)\Bigg] \leqslant$$

$$\leqslant C(\varepsilon,\delta)\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau)^\delta, \qquad \forall \delta\in\,]0,\alpha[\,.$$

(iii) We can repeat the preceding calculation, with a different estimate for the second term in the last member of (3.1). In fact we have from Lemma 2.2

$$\|A(s)\exp[\xi A(s)]-A(t)\exp[\xi A(t)]\|_{\mathcal{L}(E)}\leqslant C\,\frac{|t-s|^{\alpha}}{\xi}\,,\qquad \forall\xi>0\,,$$

and consequently

$$\begin{aligned}
&\left\|\int_0^\tau (1-A(t)A(\tau)^{-1})\big(A(\tau)\exp[(t-s)A(s)]\varphi(s)\big)\,ds\right\|_E\leqslant\\
&\qquad\leqslant\int_0^\tau \|1-A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)}\|A(\tau)A(s)^{-1}\|_{\mathcal{L}(E)}\cdot\\
&\qquad\cdot\|[A(s)\exp[(t-s)A(s)]-A(t)\exp[(t-s)A(t)]]\varphi(s)\|_E\,ds+\\
&\qquad+\int_0^\tau \|1-A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)}\|A(\tau)A(s)^{-1}-1\|_{\mathcal{L}(E)}\cdot\\
&\qquad\cdot\|A(t)\exp[(t-s)A(t)]\varphi(s)\|_E\,ds+\int_0^\tau \|1-A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)}\cdot\\
&\qquad\cdot\|A(t)\exp[(t-s)A(t)]\|_{\mathcal{L}(E)}\|\varphi(s)-\varphi(t)\|_E\,ds+\\
&\qquad+\|1-A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)}\|(\exp[tA(t)]-\exp[(t-\tau)A(t)])\varphi(t)\|_E\leqslant\\
&\qquad\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\left[(t-\tau)^\alpha\int_0^\tau\frac{1}{(t-s)^{1-\alpha}s^\theta}\,ds+(t-\tau)^\alpha\int_0^\tau\frac{(\tau-s)^\alpha}{(t-s)s^\theta}\,ds\right]+\\
&\qquad+C\|\varphi\|_{C_\theta(]0,T],E)}\left[\frac{(t-\tau)^\alpha}{\varepsilon/2}(\varepsilon/2)^{1-\theta}\right]+C\|\varphi\|_{C^\sigma([\varepsilon/2,T],E)}(t-\tau)^\alpha\int_{\varepsilon/2}^\tau\frac{ds}{(t-s)^{1-\sigma}}+\\
&\qquad+C\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau)^\alpha\leqslant C(\varepsilon)\big[\|\varphi\|_{C_\theta(]0,T],E)}+\|\varphi\|_{C^\sigma([\varepsilon/2,T],E)}\big](t-\tau)^\alpha\,.
\end{aligned}$$

In the second step we have used Lemma 2.2.

(iv) Let $\varphi\in C_\theta(]0,T],E)$, $\theta\in[\alpha,1[$. Then for each $t\in]0,T]$ we have:

$$\|t^{\theta-\alpha}K\varphi(t)\|_E\leqslant C\int_0^t\frac{t^{\theta-\alpha}\,ds}{(t-s)^{1-\alpha}s^\theta}\|\varphi\|_{C_\theta(]0,T],E)}=C\int_0^1\frac{dy}{(1-y)^{1-\alpha}y^\theta}\|\varphi\|_{C_\theta(]0,T],E)}\,,$$

and (iv) follows.

(v) Take $\delta\in]0,\alpha-\theta[$, and choose $p\in](\alpha-\delta)^{-1},\theta^{-1}[$; then $(1-\alpha)p'<(1-\alpha+\delta)p'<1$, where $p'=p/(p-1)$. Hence if $t>\tau\geqslant0$ by (3.1) we check as in (ii):

$$\begin{aligned}
&\|K\varphi(t)-K\varphi(\tau)\|_E\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\cdot\\
&\qquad\cdot\left[\int_\tau^t\frac{ds}{(t-s)^{1-\alpha}s^\theta}+\int_0^\tau\frac{(t-\tau)^\alpha\,ds}{(t-s)s^\theta}+\int_0^\tau(\tau-s)^\alpha\left[\frac{1}{\tau-s}-\frac{1}{t-s}\right]\frac{ds}{s^\theta}\right]\leqslant
\end{aligned}$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\left[\int_\tau^t \frac{ds}{(t-s)^{1-\alpha}s^\theta}+\int_0^\tau \frac{(t-\tau)^\delta\,ds}{(t-s)^{1-\alpha+\delta}s^\theta}+\int_0^\tau \frac{(t-\tau)\,ds}{(\tau-s)^{1-\alpha}(t-s)s^\theta}\right]\leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\left\{\left[\int_\tau^t \frac{ds}{(t-s)^{(1-\alpha)p'}}\right]^{1/p'}\left[\int_\tau^t \frac{ds}{s^{\theta p}}\right]^{1/p}+\right.$$

$$\left.+(t-\tau)^\delta\left[\int_0^\tau \frac{ds}{(t-s)^{(1-\alpha+\delta)p'}}\right]^{1/p'}\left[\int_0^\tau \frac{ds}{s^{\theta p}}\right]^{1/p}+(t-\tau)^\delta\left[\int_0^\tau \frac{ds}{(\tau-s)^{(1-\alpha+\delta)p'}}\right]^{1/p'}\right\}\leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}](t-\tau)^{\alpha-1/p}(t-\tau)^{1/p-\theta}+(t-\tau)^\delta\tau^{\alpha-\delta-1/p}\tau^{1/p-\theta}]\leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau)^\delta\,.$$

(vi) If $\varphi\in C^\sigma([0,T],E)$ we get, as in (iii)

$$\left\|\int_0^\tau (1-A(t)A(\tau)^{-1})A(\tau)\exp[(t-s)A(s)]\varphi(s)\,ds\right\|_E\leqslant$$

$$\leqslant C\|\varphi\|_{C(]0,T],E)}\left[(t-\tau)^\alpha\int_0^\tau \frac{ds}{(t-s)^{1-\alpha}}+(t-\tau)^\alpha\int_0^\tau \frac{(\tau-s)^\alpha\,ds}{(t-s)}\right]+$$

$$+C\|\varphi\|_{C^\sigma([0,T],E)}(t-\tau)^\alpha\int_0^\tau \frac{ds}{(t-s)^{1-\sigma}}+C\|\varphi\|_{C([0,T],E)}(t-\tau)^\alpha\leqslant C\|\varphi\|_{C^\sigma([0,T],E)}(t-\tau)^\alpha;$$

the remaining terms appearing in the last member of (3.1) can be estimated as follows:

$$\int_\tau^t \|K(t,s)\|_{\mathcal{L}(E)}\|\varphi(s)\|_E\,ds\leqslant C(t-\tau)^\alpha\|\varphi\|_{C([0,T],E)};$$

$$\left\|\int_0^\tau (1-A(\tau)A(s)^{-1})\int_{\tau-s}^{t-s} A(s)^2\exp[\xi A(s)]\varphi(s)\,d\xi\,ds\right\|_E\leqslant C\|\varphi\|_{C(]0,T],E)}\int_0^\tau\int_{\tau-s}^{t-s}\frac{d\xi}{\xi^{2-\alpha}}\,ds\leqslant$$

$$\leqslant C\|\varphi\|_{C([0,T],E)}\int_0^\tau\left[\frac{1}{(\tau-s)^{1-\alpha}}-\frac{1}{(t-s)^{1-\alpha}}\right]ds\leqslant C\|\varphi\|_{C(]0,T],E)}(t-\tau)^\alpha\,,$$

and the result follows. ///

d) The operator $(1+K)^{-1}$.

LEMMA 3.4. – *Under Hypotheses I, II we have*:

(i) $(1+K)^{-1}$ *exists and is in* $\mathcal{L}(L^p(0,T,E))$, $\forall p\in[1,\infty]$.

(ii) $(1+K)^{-1}\in\mathcal{L}(C_\theta(]0,T],E))$, $\forall\theta\in[0,1[$.

(iii) $(1+K)^{-1}\in\mathcal{L}(C([0,T],E))$ *and, in addition,* $(1+K)^{-1}\varphi(0)=\varphi(0)$, $\forall\varphi\in C([0,T],E)$.

(iv) *If* $\theta \in [0, 1[$ *and* $\varphi \in C_\theta(]0, T], E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \alpha]$ (*resp.* $C_\theta(]0, T], E) \cap h^\delta(]0, T], E)$, $\delta \in]0, \alpha[$) *then* $(1 + K)^{-1}\varphi \in C^\delta(]0, T], E)$ (*resp.* $h^\delta(]0, T], E)$).

(v) $(1 + K)^{-1} \in \mathcal{L}\big(C^\delta([0, T], E)\big)$, $\forall \delta \in]0, \alpha]$.

(vi) *If* $\varphi \in h^\delta([0, T], E)$, $\delta \in]0, \alpha[$, *then* $(1 + K)^{-1}\varphi \in h^\delta([0, T], E)$.

PROOF. – (i) Define a new norm in $L^p(0, T, E)$, $p \in [1, \infty]$, setting

$$\|\varphi\|_p^* = \|\exp[-\omega t]\varphi(\cdot)\|_{L^p(0,T,E)}, \qquad \forall \varphi \in L^p(0, T, E),$$

where $\omega > 0$ is to be chosen later. Clearly $\|\varphi\|_p^*$ is equivalent to the usual norm in $L^p(0, T, E)$, and moreover it is easily seen that

$$\|K\varphi\|_p^* \leqslant K \int_0^T \frac{\exp[-\omega s]}{s^{1-\alpha}}\, ds \|\varphi\|_p^* \leqslant K\Gamma(\alpha)\omega^{-\alpha}\|\varphi\|_p^*, \qquad \forall \varphi \in L^p(0, T, E).$$

Hence if we choose ω large enough, we get

$$\|K\varphi\|_p^* \leqslant \tfrac{1}{2}\|\varphi\|_p^*, \qquad \forall \varphi \in L^p(0, T, E),$$

so that $(1 + K)$ has bounded inverse in $L^p(0, T, E)$.

(ii) If $\varphi \in C_\theta(]0, T], E)$ then $K\varphi \in C_\theta(]0, T], E)$ by Lemma 3.3 (iv)-(v). As before, define

$$\|\varphi\|_\theta^* = \|\exp[-\omega t]\varphi(\cdot)\|_{C_\theta(]0,T],E)}, \qquad \forall \varphi \in C_\theta(]0, T], E);$$

then it is easy to verify that

$$\|t^\theta \exp[-\omega t]K\varphi(t)\|_E \leqslant K\|\varphi\|_\theta^* t^\alpha \int_0^t \frac{\exp[-\omega t x]}{x^{1-\alpha}(1-x)^\theta}\, dx, \qquad \forall t \in]0, T].$$

Choose $p \in]1, \theta^{-1} \wedge (1-\alpha)^{-1}[$; then

$$\|t^\theta \exp[-\omega t]K\varphi(t)\|_E \leqslant K\|\varphi\|_\theta^* t^\alpha \left[\int_0^1 \exp[-\omega t x p']\, dx\right]^{1/p'} \cdot \left[\int_0^1 \frac{dx}{x^{(1-\alpha)p}(1-x)^{\theta p}}\right]^{1/p} \leqslant C\frac{T^{\alpha-1/p'}}{\omega^{1/p'}}\|\varphi\|_\theta^*.$$

For ω large enough, again we can deduce that $\|K\varphi\|_\theta^* \leqslant \frac{1}{2}\|\varphi\|_\theta^*$, $\forall \varphi \in C_\theta(]0, T], E)$, so that $(1 + K)$ has bounded inverse in $C_\theta(]0, T], E)$.

(iii) $K \in \mathfrak{L}\big(C([0, T], E)\big)$ by Lemma 3.3 (v); in addition by (i) we get that $(1 + K)^{-1} \in \mathfrak{L}(L^\infty(0, T, E))$. Hence for any $g \in C([0, T], E)$ we have

$$(1 + K)^{-1}g = g + \sum_{n=1}^{\infty} (-K)^n g \in C([0, T], E)$$

since $K^n g \in C([0, T], E)$, $\forall n \in N$ and the series converges in $C([0, T], E)$ with respect to the norm $\|\cdot\|_\infty^*$. In particular $((1 + K)^{-1}g)(0) = g(0)$ by Lemma 3.3 (v).

(iv) Let $\theta \in [0, 1[$ and suppose $\varphi \in C_\theta(]0, T], E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \alpha]$ (resp. $C_\theta(]0, T], E) \cap h^\delta(]0, T], E)$, $\delta \in]0, \alpha[$). By (ii) we can set $g = (1 + K)^{-1}\varphi$, so that $g = \varphi - Kg \in C_\theta(]0, T], E)$; hence $Kg \in C^\sigma(]0, T], E)$, $\forall \sigma \in]0, \alpha[$ by Lemma 3.3 (ii). This implies $g \in C_\theta(]0, T], E) \cap C^{\delta \wedge \sigma}(]0, T], E)$, $\forall \sigma \in]0, \alpha[$ (resp. $C_\theta(]0, T], E) \cap h^\delta(]0, T], E)$). The result is proved if $\delta < \alpha$, otherwise this in turn gives $Kg \in C^\alpha(]0, T], E)$ (Lemma 3.3 (iii)), so we conclude that $g \in C^\alpha(]0, T], E)$.

(v) Let $\varphi \in C^\delta([0, T], E)$, $\delta \in]0, \alpha]$. As in (iv), we get $g \in C^\delta([0, T], E)$ since $Kg \in C^\alpha([0, T], E)$ by Lemma 3.3 (vi). Moreover if $\delta \in]0, \alpha[$ for each $t, \tau \in [0, T]$ we have by Lemma 3.3 (v) and by (iii)

$$\frac{\|g(t) - g(\tau)\|_E}{|t - \tau|^\delta} \leqslant \frac{\|\varphi(t) - \varphi(\tau)\|_E}{|t - \tau|^\delta} + \\ + \frac{\|Kg(t) - Kg(\tau)\|_E}{|t - \tau|^\delta} \leqslant \frac{\|\varphi(t) - \varphi(\tau)\|_E}{|t - \tau|^\delta} + C\|g\|_{C([0,T],E)} \leqslant C\|\varphi\|_{C^\delta([0,T],E)};$$

on the other hand if $\delta = \alpha$ we can apply Lemma 3.3 (vi) obtaining

$$\frac{\|Kg(t) - Kg(\tau)\|_E}{|t - \tau|^\alpha} \leqslant C\|g\|_{C^\delta([0,T],E)} \leqslant C\|\varphi\|_{C^\delta([0,T],E)} \leqslant C\|\varphi\|_{C^\alpha([0,T],E)},$$

and the result follows.

(vi) If $\varphi \in h^\delta([0, T], E)$, $\delta \in]0, \alpha[$, then by (v) we have $(1 + K)^{-1}\varphi \in C^\delta([0, T], E)$; moreover by Lemma 3.3 (vi) $K(1 + K)^{-1}\varphi \in C^\alpha([0, T], E) \subseteq h^\delta([0, T], E)$, which implies $(1 + K)^{-1}\varphi = \varphi - K(1 + K)^{-1}\varphi \in h^\delta([0, T], E)$. ///

e) The operator $T\varphi(t) = \int_0^t \exp[(t - s)A(s)]\varphi(s)\, ds$.

LEMMA 3.5. – *Under Hypotheses I, II we have*:

(i) $T \in \mathfrak{L}\big(L^1(0, T, E), C([0, T], E)\big)$ *and* $T\varphi(0) = 0$, $\forall \varphi \in L^1(0, T, E)$.

(ii) $T \in \mathfrak{L}\big(C_\theta(]0, T], E), C^{1-\theta}([0, T], E)\big)$, $\forall \theta \in]0, 1[$.

(iii) *If* $\varphi \in C_\theta(]0, T], E)$, $\theta \in]0, 1[$, *then* $T\varphi \in C^\delta(]0, T], E)$, $\forall \delta \in]0, 1[$.

(iv) $T \in \mathfrak{L}\big(C_0(]0, T], E), C^\delta([0, T], E)\big)$, $\forall \delta \in]0, 1[$.

(v) *If* $\theta \in [0, 1[$ *and* $\varphi \in C_\theta(]0, T], E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \alpha]$ $\big($*resp.* $C^\theta(]0, T], E) \cap h^\delta(]0, T], E)$, $\delta \in]0, \alpha[\big)$, *then* $T\varphi \in C^{1,\delta}(]0, T], E)$ $\big($*resp.* $h^{1,\delta}(]0, T], E)\big)$ *and*

$$(T\varphi)'(t) = \int_0^t A(s) \exp[(t-s)A(s)]\big(\varphi(s) - \varphi(t)\big)\, ds + \\ + \int_0^t \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\varphi(t)\, ds + \exp[tA(t)]\varphi(t), \\ \forall t \in]0, T].$$

(vi) *If* $\theta \in [0, 1[$ *and* $\varphi \in C_\theta(]0, T], E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \alpha]$ $\big($*resp.* $C_\theta(]0, T], E) \cap h^\delta(]0, T], E)$, $\delta \in]0, \alpha[\big)$, *then* $T\varphi(t) \in D(A(0))$, $\forall t \in [0, T]$, $t \to A(t)T\varphi(t) \in C^\delta(]0, T], E)$ $\big($*resp.* $h^\delta(]0, T], E)\big)$ *and*

$$A(t)T\varphi(t) = (T\varphi)'(t) - \varphi(t) - K\varphi(t), \quad \forall t \in]0, T].$$

(vii) *If* $\varphi \in C^\delta([0, T], E)$, $\delta \in]0, \alpha]$, *then*

$$(T\varphi)'(t) - (T\varphi)'(\tau) = O\big((t-\tau)^\delta\big) + \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\varphi(0) \quad \text{as } t - \tau \to 0^+.$$

(viii) *If* $\varphi \in h^\delta([0, T], E)$, $\delta \in]0, \alpha[$, *then* $(T\varphi)'$, $A(\cdot)T\varphi(\cdot) \in h^\delta(]0, T], E)$ *and*

$$(T\varphi)'(t) - (T\varphi)'(\tau) = o\big((t-\tau)^\delta\big) + \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\varphi(0) \quad \text{as } t - \tau \to 0^+.$$

PROOF. - (i) Let $t \in]0, T]$; we have

$$\|T\varphi(t)\|_E \leqslant C\int_0^t \|\varphi(s)\|_E\, ds;$$

it follows that $T\varphi(t) \to 0$ as $t \to 0^+$, and moreover

$$\sup_{t \in]0,T]} \|T\varphi(t)\|_E \leqslant C\|\varphi\|_{L^1(0,T,E)}.$$

Let us show that $T\varphi$ is continuous at each $t \in]0, T]$. Take $\varepsilon \in]0, t[$; then for each $\tau \in]t-\varepsilon, t[$ we have

$$\|T\varphi(t) - T\varphi(\tau)\|_E \leqslant C\int_\tau^t \|\varphi(s)\|_E\, ds + \left\|\int_0^\tau \big(\exp[(t-s)A(s)] - \exp[(\tau-s)A(s)]\big)\varphi(s)\, ds\right\|_E.$$

In the second term on the right-hand side the integrand goes to 0 as $\tau \to t^-$ for each $s \in]0, \tau[$, due to the fact that $\xi \to \exp[\xi A(s)]y$ is continuous for $\xi > 0$ when $y \in E$ is fixed; hence by Lebesgue's Theorem we check $T\varphi(\tau) \to T\varphi(t)$ as $\tau \to t^-$. Interchanging t and τ we get that the same is true as $\tau \to t^+$.

(ii) If $t > \tau \geqslant 0$ we have

$$\|T\varphi(t) - T\varphi(\tau)\|_E \leqslant C\int_\tau^t \|\varphi(s)\|_E\, ds + \int_0^\tau \int_{\tau-s}^{t-s} \|A(s)\exp[\xi A(s)]\|_{\mathfrak{L}(E)}\, d\xi \|\varphi(s)\|_E\, ds \leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\left[(t-\tau)^{1-\theta} + \int_0^\tau \log\left(1 + \frac{t-\tau}{\tau - s}\right)\frac{ds}{s^\theta}\right] \leqslant C(\theta)\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau)^{1-\theta}.$$

(iii) Let $\delta \in]0,1[$. If $t > \tau \geqslant \varepsilon$ we have, as in (ii)

$$\|T\varphi(t) - T\varphi(\tau)\|_E \leqslant C\int_\tau^t \|\varphi(s)\|_E\, ds + \left[\int_0^{\varepsilon/2} + \int_{\varepsilon/2}^\tau\right]\int_{\tau-s}^{t-s} \|A(s)\exp[\xi A(s)]\|_{\mathfrak{L}(E)}\|\varphi(s)\|_E\, d\xi\, ds \leqslant$$

$$\leqslant C\|\varphi\|_{C_\theta(]0,T],E)}\left[\frac{t-\tau}{\varepsilon^\theta} + \frac{t-\tau}{\varepsilon/2}(\varepsilon/2)^{1-\theta} + \int_{\varepsilon/2}^\tau \log\left(1 + \frac{t-\tau}{\tau-s}\right)\frac{ds}{(\varepsilon/2)^\theta}\right] \leqslant$$

$$\leqslant C(\varepsilon, \delta)\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau)^\delta.$$

(iv) As in (ii) we find

$$\|T\varphi(t) - T\varphi(\tau)\|_E \leqslant C\|\varphi\|_{L^\infty(0,T,E)}\left[(t-\tau) + \int_0^\tau \log\left(1 + \frac{t-\tau}{\tau-s}\right)ds\right] \leqslant$$

$$\leqslant C(\delta)\|\varphi\|_{L^\infty(0,T,E)}(t-\tau)^\delta, \qquad \forall \delta \in]0,1[.$$

(v) We consider only the case $\varphi \in C_\theta(]0,T], E) \cap C^\delta(]0,T], E)$, $\delta \in]0,\alpha]$, since the other is analogous. Let $t \in]0,T]$, and choose $\varepsilon \in]0,t[$. If $t > \tau \geqslant \varepsilon$ we can write

$$\frac{T\varphi(t) - T\varphi(\tau)}{t-\tau} = \frac{1}{t-\tau}\int_\tau^t \exp[(t-s)A(s)](\varphi(s) - \varphi(t))\, ds +$$

$$+ \frac{1}{t-\tau}\int_\tau^t (\exp[(t-s)A(s)] - \exp[(t-s)A(t)])\varphi(t)\, ds + \frac{\exp[(t-\tau)A(t)] - 1}{t-\tau}\varphi(t) +$$

$$+ \int_0^\tau \frac{\exp[(t-s)A(s)] - \exp[(\tau-s)A(s)]}{t-\tau}(\varphi(s) - \varphi(\tau))\, ds +$$

$$+ \int_0^\tau \left[\frac{\exp[(t-s)A(s)] - \exp[(\tau-s)A(s)]}{t-\tau} - \frac{\exp[(t-s)A(\tau)] - \exp[(\tau-s)A(\tau)]}{t-\tau}\right]\cdot$$

$$\cdot \varphi(\tau)\, ds + \frac{\exp[(t-\tau)A(\tau)] - 1}{t-\tau}A(\tau)^{-1}(\exp[\tau A(\tau)] - 1)\varphi(\tau) = \sum_{i=1}^6 B_i.$$

As $\tau \to t^-$ we deduce that

$$B_1 = O((t-\tau)^\delta), \quad B_2 = O((t-\tau)^\delta),$$

while

$$B_3 + B_6 = \left[\frac{\exp[(t-\tau)A(t)]-1}{t-\tau}A(t)^{-1} - \frac{\exp[(t-\tau)A(\tau)]-1}{t-\tau}A(\tau)^{-1}\right]\varphi(t) +$$
$$+ \frac{\exp[(t-\tau)A(\tau)]-1}{t-\tau}A(\tau)^{-1}\big(\varphi(t)-\varphi(\tau)\big) + \frac{\exp[(t-\tau)A(\tau)]-1}{t-\tau}\cdot$$
$$\cdot A(\tau)^{-1}\exp[\tau A(\tau)]\varphi(\tau) = O\big((t-\tau)^\alpha\big) + O\big((t-\tau)^\delta\big) + \exp[tA(t)]\varphi(t)\,.$$

Consider B_4: the integrand converges to $A(s)\exp[(t-s)A(s)]\big(\varphi(s)-\varphi(t)\big)$ as $\tau\to t^-$, and is dominated by the functions

$$G_\tau(s) = C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{1}{\varepsilon/2}\|\varphi(s)-\varphi(\tau)\|_E + \chi_{[\varepsilon/2,\tau]}(s)\frac{1}{t-\tau}\big((t-s)^\delta-(\tau-s)^\delta\big)\right\}.$$

Since

$$\lim_{\tau\to t^-} G_\tau(s) = C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{1}{\varepsilon/2}\|\varphi(s)-\varphi(t)\|_E + \chi_{[\varepsilon/2,t]}(s)\frac{\delta}{(t-s)^{1-\delta}}\right\}, \qquad \forall s\in\,]0,t[\,,$$

and

$$\lim_{\tau\to t^-}\int_0^t G_\tau(s)\,ds = \int_0^t C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{1}{\varepsilon/2}\|\varphi(s)-\varphi(t)\|_E + \chi_{[\varepsilon/2,t]}(s)\frac{\delta}{(t-s)^{1-\delta}}\right\}ds\,,$$

we conclude that

$$B_4 = o(1) + \int_0^t A(s)\exp[(t-s)A(s)]\big(\varphi(s)-\varphi(t)\big)\,ds\,, \quad \text{as } \tau\to t^-\,.$$

Finally, the integrand of B_5 converges to $\big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\varphi(t)$, $\forall s\in\,]0,t[$, and is dominated by the functions

$$F_\tau(s) = C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{2}{\varepsilon/2}\|\varphi(\tau)\|_E + \chi_{[\varepsilon/2,\tau]}(s)\frac{(t-s)^\alpha-(\tau-s)^\alpha}{t-\tau}\|\varphi(\tau)\|_E\right\};$$

since

$$\lim_{\tau\to t^-} F_\tau(s) = C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{2}{\varepsilon/2}\|\varphi(t)\|_E + \chi_{[\varepsilon/2,t]}(s)\frac{\alpha}{(t-s)^{1-\alpha}}\|\varphi(t)\|_E\right\}, \qquad \forall s\in\,]0,t[\,,$$
$$\lim_{\tau\to t^-}\int_0^t F_\tau(s)\,ds = \int_0^t C\left\{\chi_{[0,\varepsilon/2]}(s)\frac{2}{\varepsilon/2}\|\varphi(t)\|_E + \chi_{[\varepsilon/2,t]}(s)\frac{\alpha}{(t-s)^{1-\alpha}}\|\varphi(t)\|_E\right\}ds\,,$$

we get

$$B_5 = o(1) + \int_0^t\big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\varphi(t)\,ds\,, \quad \text{as } \tau\to t^-\,.$$

The case $\tau \to t^+$ is treated similarly, interchanging τ and t.

Let us show now that $(T\varphi)' \in C^\delta(]0, T], E)$. Suppose $t > \tau \geqslant \varepsilon$; then

$$\begin{aligned}
&(T\varphi)'(t) - (T\varphi)'(\tau) = \int_\tau^t A(s) \exp[(t-s)A(s)]\big(\varphi(s) - \varphi(t)\big)\, ds + \\
&+ \int_0^t \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)\big)\big(\varphi(\tau) - \varphi(t)\big)\, ds + \\
&+ \big(\exp[tA(t)] - \exp[(t-\tau)A(t)]\big)\big(\varphi(\tau) - \varphi(t)\big) + \int_0^\tau A(s)\big(\exp[(t-s)A(s)] - \\
&- \exp[(\tau - s)A(s)]\big)\big(\varphi(s) - \varphi(\tau)\big)\, ds + \int_\tau^t \big(A(s)\exp[(t-s)A(s)] - \\
&- A(t)\exp[(t-s)A(t)]\big)\varphi(t)\, ds + \big(\exp[tA(\tau)] - \exp[(t-\tau)A(\tau)] - \exp[tA(t)] + \\
&+ \exp[(t-\tau)A(t)]\big)\, \varphi(t) + \int_0^\tau \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\big(\varphi(t) - \\
&- \varphi(\tau)\big)\, ds + \int_0^\tau \big[\big(A(s)\exp[(t-s)A(s)] - A(\tau)\exp[(t-s)A(\tau)]\big) - \\
&- \big(A(s)\exp[(\tau-s)A(s)] - A(\tau)\exp[(\tau-s)A(\tau)]\big)\big]\varphi(\tau)\, ds + \big(\exp[tA(t)] - \\
&- \exp[tA(\tau)]\big)\varphi(t) + \big[\big(\exp[tA(\tau)] - \exp[\tau A(\tau)]\big) - \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\big]\varphi(t) + \\
&\qquad + \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\varphi(t) + \big(\exp[\tau A(\tau)] - 1\big)\big(\varphi(t) - \varphi(\tau)\big)\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\|(T\varphi)'(t) - (T\varphi)'(\tau)\|_E \leqslant C\|\varphi\|_{C^\delta([\varepsilon,T],E)}(t-\tau)^\delta + \\
&\quad + C\|\varphi\|_{C^\delta([\varepsilon,T],E)}(t-\tau)^\delta t^\alpha + C\|\varphi\|_{C^\delta([\varepsilon,T],E)}(t-\tau)^\delta + \\
&\quad + C\|\varphi\|_{C_\theta(]0,T],E)}\frac{t-\tau}{(\varepsilon/2)^2}t^{1-\theta} + C\|\varphi\|_{C^\delta([\varepsilon/2,T],E)}\int_{\varepsilon/2}^\tau \left[\frac{1}{(\tau-s)^{1-\delta}} - \frac{1}{(t-s)^{1-\delta}}\right] ds + \\
&\quad + C\|\varphi\|_{C_\theta(]0,T],E)}\frac{1}{\varepsilon^\theta}(t-\tau)^\alpha + \\
&\quad + C\|\exp[(t-\tau)A(\tau)] - \exp[(t-\tau)A(t)]\|_{\mathcal{L}(E)}\frac{1}{\varepsilon^\theta}\|\varphi\|_{C_\theta(]0,T],E)} + \\
&\quad + C\|\varphi\|_{C^\delta([\varepsilon,T],E)}(t-\tau)^\delta t^\alpha + C\|\varphi\|_{C_\theta(]0,T],E)}\frac{1}{\varepsilon^\theta}\int_0^\tau \left[\frac{1}{(\tau-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\right] ds + \\
&\quad + C\|\varphi\|_{C_\theta(]0,T],E)}\frac{1}{\varepsilon^\theta}\int_\tau^t \frac{d\xi}{\xi^{1-\alpha}} + \frac{C}{\varepsilon^{\theta+1}}\|\varphi\|_{C_\theta(]0,T],E)}(t-\tau) + C\|\varphi\|_{C^\delta([\varepsilon,T],E)}(t-\tau)^\delta \leqslant \\
&\qquad \leqslant C(\varepsilon)\big[\|\varphi\|_{C_\theta(]0,T],E)} + \|\varphi\|_{C^\delta([\varepsilon/2,T],E)}\big](t-\tau)^\delta\,.
\end{aligned}$$

(vi) Again we consider only the case $\varphi \in C_\theta(]0, T], E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \alpha]$.

Fix $t \in]0, T]$; then

$$T\varphi(t) = \int_0^t [\exp[(t-s)A(s)] - \exp[(t-s)A(t)]]\varphi(s)\, ds +$$
$$+ \int_0^t \exp[(t-s)A(t)]\big(\varphi(s)-\varphi(t)\big)\, ds + A(t)^{-1}\big(\exp[tA(t)]-1\big)\varphi(t) =$$
$$= A(t)^{-1}\Big\{\int_0^t \big(A(t)A(s)^{-1}-1\big)A(s)\exp[(t-s)A(s)]\varphi(s)\, ds + \int_0^t \big(A(s)\exp[(t-s)A(s)] -$$
$$- A(t)\exp[(t-s)A(t)]\big)\varphi(s)\, ds + \int_0^t A(t)\exp[(t-s)A(t)]\big(\varphi(s)-\varphi(t)\big)\, ds +$$
$$+ \big(\exp[tA(t)]-1\big)\varphi(t)\Big\},$$

and all integrals are convergent, since

$$\Big\|\int_0^t \big(A(t)A(s)^{-1}-1\big)A(s)\exp[(t-s)A(s)]\varphi(s)\, ds\Big\|_E \leqslant C\int_0^t \frac{ds}{s^\theta(t-s)^{1-\alpha}}\|\varphi\|_{C_\theta(]0,T],E)},$$
$$\Big\|\int_0^t \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\varphi(s)\, ds\Big\|_E \leqslant$$
$$\leqslant C\int_0^t \frac{ds}{s^\theta(t-s)^{1-\alpha}}\|\varphi\|_{C_\theta(]0,T],E)},$$

and

$$\Big\|\int_0^t A(t)\exp[(t-s)A(t)]\big(\varphi(s)-\varphi(t)\big)\, ds\Big\|_E \leqslant C\int_0^{\varepsilon/2} \frac{ds}{(t-s)s^\theta}\|\varphi\|_{C_\theta(]0,T],E)} +$$
$$+ C\int_{\varepsilon/2}^t \frac{ds}{(t-s)^{1-\delta}}\|\varphi\|_{C^\delta([\varepsilon/2,T],E)},$$

where $\varepsilon < \frac{1}{2}t$. It follows that $T\varphi(t) \in D\big(A(0)\big)$ and

$$A(t)T\varphi(t) = \int_0^t \big(A(t)A(s)^{-1}-1\big)A(s)\exp[(t-s)A(s)]\varphi(s)\, ds +$$
$$+ \int_0^t \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)\varphi(s)\, ds +$$
$$+ \int_0^t A(t)\exp[(t-s)A(t)]\big(\varphi(s)-\varphi(t)\big)\, ds + \big(\exp[tA(t)]-1\big)\varphi(t), \qquad \forall t \in]0, T].$$

In particular by (v) we have

$$A(t)T\varphi(t) - (T\varphi)'(t) = -\varphi(t) - \int_0^t \big(A(t)A(s)^{-1}-1\big)A(s)\exp[(t-s)A(s)]\varphi(s)\, ds =$$
$$= -\varphi(t) - K\varphi(t), \qquad \forall t \in]0, T];$$

hence, by (v) and Lemma 3.3 (iii), we conclude that $A(\cdot)T\varphi(\cdot) \in C^\delta(]0, T], E)$.

(vii) We split $(T\varphi)'(t)-(T\varphi)'(\tau)$ as in (v), obtaining for $t>\tau>0$:

$$\begin{aligned}|||(T\varphi)'(t)-(T\varphi)'(\tau)\|_E &\leqslant C\|\varphi\|_{C^\delta([0,T],E)}[(t-\tau)^\delta+(t-\tau)^\alpha]+\\ &+\|(\exp[tA(0)]-\exp[\tau A(0)])\varphi(t)\|_E\leqslant C\|\varphi\|_{C^\delta([0,T],E)}(t-\tau)^\delta+\\ &+\|\exp[tA(0)]-\exp[\tau A(0)]\|_{\mathcal{L}(E)}[\|\varphi(t)-\varphi(\tau)\|_E+\|\varphi(\tau)-\varphi(0)\|_E]+\\ &+\|(\exp[tA(0)]-\exp[\tau A(0)])\varphi(0)\|_E\leqslant C\|\varphi\|_{C^\delta([0,T],E)}(t-\tau)^\delta+\\ &\qquad+\|(\exp[tA(0)]-\exp[\tau A(0)])\varphi(0)\|_E\,.\end{aligned}$$

(viii) As in (vii) we get now as $t-\tau\to 0^+$:

$$\begin{aligned}\|(T\varphi)'(t)-(T\varphi)'(\tau)\|_\delta &\leqslant C\|\varphi\|_{C^\delta([0,T],E)}\cdot o(1)[(t-\tau)^\delta+(t-\tau)^\alpha]+\\ &\qquad+|||(\exp[tA(0)]-\exp[\tau A(0)])\varphi(0)|||_E\,,\end{aligned}$$

which implies $(T\varphi)'\in h^\delta(]0,T],E)$. Moreover since

$$A(t)T\varphi(t)=(T\varphi)'(t)-\varphi(t)-K\varphi(t)\,,\qquad \forall t\in\,]0,T]\,,$$

by Lemma 3.3 (iii) and the inclusion $C^\alpha(]0,T],E)\subseteq h^\delta(]0,T],E)$, we deduce $A(\cdot)T\varphi(\cdot)\in h^\delta(]0,T],E)$. ///

4. – Strict solutions.

In this section we study the properties of strict solutions of Problem (P) (see Definition 1.3). We start with a direct proof of uniqueness and of necessary compatibility conditions for existence. Next, we prove that the strict solution exists and is given by the representation formula (F) of the Introduction, provided f is Hölder continuous and the compatibility conditions hold. From the fact that (F) holds, we will easily deduce an estimate for the strict solution in terms of the data x, f when f is Hölder continuous; a general « a priori » estimate for strict solutions is then obtained for continuous f, by an approximation argument. We note that a direct derivation of this « a priori » estimate is not possible, since if f is merely continuous then the function defined by Formula (F) needs not be a strict solution of (P).

Finally at the end of this section we will prove maximal time regularity for the strict solution.

THEOREM 4.1. – *Under Hypotheses I, II Problem* (P) *with* $x\in E$ *and* $f\in C([0,T],E)$ *has at most one strict solution.*

PROOF. – Let u be a strict solution of Problem (P) with $x = 0$ and $f = 0$; fix $t \in]0, T]$. Define

$$v(s) = \exp\,[(t-s)A(t)]u(s)\,, \qquad s \in [0, t]\,.$$

For each $s \in]0, t[$ we have:

$$v'(s) = -A(t)\exp\,[(t-s)A(t)]u(s) + \exp\,[(t-s)A(t)]u'(s) = \\ = \exp\,[(t-s)A(t)]\big(A(s) - A(t)\big)u(s);$$

As the right-hand side is continuous and bounded in $[0, t[$, by integration we find

$$u(t) = \int_0^t \exp\,[(t-s)A(t)]\big(1 - A(t)A(s)^{-1}\big)A(s)u(s)\,ds\,,$$

which implies

$$\|A(t)u(t)\|_E = \left\|\int_0^t A(t)\exp\,[(t-s)A(t)]\big(1 - A(t)A(s)^{-1}\big)A(s)u(s)\,ds\right\|_E \leqslant \\ \leqslant C\int_0^t \frac{1}{(t-s)^{1-\alpha}}\,\|A(s)u(s)\|_E\,ds\,.$$

By a well-known generalization of Gronwall's Lemma (see e.g. AMANN [3], Corollary 2.4) we deduce that $A(t)u(t) \equiv 0$ in $[0, T]$; this in turn gives $u'(t) \equiv 0$ in $[0, T]$, or $u(t) \equiv u(0) = 0$. ///

About existence of the strict solution, we have an evident necessary condition on the initial data:

THEOREM 4.2. – *Under Hypotheses I, II, if u is a strict solution of Problem* (P) *with $x \in E$ and $f \in C([0, T], E)$, then $x \in D(A(0))$ and $A(0)x + f(0) \in \overline{D(A(0))}$.*

PROOF. – By definition, $x = u(0) \in D(A(0))$. Moreover, $u(t) - x \in D(A(0))$, $\forall t \in [0, T]$, so that

$$A(0)x + f(0) = u'(0) = \lim_{t\to 0^+} \frac{u(t) - x}{t} \in \overline{D(A(0))}\,. \qquad ///$$

The above condition is also sufficient for existence, provided f is a little more regular. In fact we have:

THEOREM 4.3. – *Under Hypotheses I, II suppose $x \in D(A(0))$, $f \in C^\sigma([0, T], E)$, $\sigma \in]0, 1[$, and $A(0)x + f(0) \in \overline{D(A(0))}$. Then the function $u(t)$ given by Formula* (F) *of the Introduction is a strict solution of* (P) *and belongs to $C^{1,\sigma\wedge\alpha}(]0, T], E)$. If in addition $f \in h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$, then $u \in h^{1,\sigma}(]0, T], E)$.*

PROOF. – By Lemma 3.1 (iii) and Lemma 3.5 (i) we have $u \in C([0, T], E)$ and $u(0) = x$. Lemma 3.1 (ii) implies that $t \to \exp[tA(0)]x \in C^\infty(]0, T], E)$; on the other hand by Lemma 3.2 (vii) we have $K(\cdot, 0)x \in C^\alpha([0, T], E)$, so that the function $g = \big((1 + K)^{-1}(f - K(\cdot, 0)x)\big)$ is in $C^{\sigma\wedge\alpha}([0, T], E)$ by Lemma 3.4 (v); therefore $Tg \in C^{1,\sigma\wedge\alpha}(]0, T], E)$ (Lemma 3.5 (v)). Summing up, we find $u \in C^{1,\sigma\wedge\alpha}(]0, T], E)$ and

$$u'(t) = A(0)\exp[tA(0)]x + \int_0^t A(s)\exp[(t-s)A(s)]\big(g(s) - g(t)\big)\,ds +$$
$$+ \int_0^t \big(A(s)\exp[(t-s)A(s)] - A(t)\exp[(t-s)A(t)]\big)g(t)\,ds + \exp[tA(t)]g(t)\,, \qquad \forall t \in]0, T]$$

If in addition $f \in h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$, then $g \in h^\sigma([0, T], E)$ by Lemma 3.4 (vi), hence $Tg \in h^{1,\sigma}(]0, T], E)$ by Lemma 3.5 (viii) and finally $u \in h^{1,\sigma}(]0, T], E)$.

Next, by Lemma 3.1 (i) and Lemma 3.5 (vi) we have $u(t) \in D(A(0))$, $\forall t \in [0, T]$ and

$$A(t)u(t) = \begin{cases} A(0)x & \text{if } t = 0\,, \\ A(t)\exp[tA(0)]x + \dfrac{d}{dt}\displaystyle\int_0^t \exp[(t-s)A(s)]g(s)\,ds - g(t) - \int_0^t K(t, s)g(s)\,ds & \\ & \text{if } t \in]0, T]. \end{cases}$$

Let us show now that $u' \in C([0, T], E)$. As $t \to 0^+$ we get

$$u'(t) = \exp[tA(0)]A(0)x + O(t^{\sigma\wedge\alpha}) + O(t^\alpha) + \exp[tA(0)]g(0) =$$
$$= O(t^{\sigma\wedge\alpha}) + \exp[tA(0)]\big(A(0)x + f(0)\big)\,,$$

and since $A(0)x + f(0) \in \overline{D(A(0))}$, we find that

$$\lim_{t\to 0^+} u'(t) = A(0)x + f(0)\,.$$

Hence

$$\exists u'(0) = A(0)x + f(0) = \lim_{t\to 0^+} u'(t)\,;$$

this means $u' \in C([0, T], E)$ and

$$u'(t) - A(t)u(t) = f(t)\,, \qquad \forall t \in [0, T]\,.$$

so that u is a strict solution of (P). ///

Our next goal is to prove an « a priori » estimate for strict solutions. We first assume f to be Hölder continuous.

THEOREM 4.4. – *Under Hypotheses I, II let u be a strict solution of* (P) *with* $x \in D(A(0))$ *and* $f \in C^\sigma([0, T], E)$, $\sigma \in]0, 1[$; *then*

$$\|u(t)\|_E \leq C\left\{\|x\|_E + \int_0^t \|f(s)\|_E \, ds\right\}, \quad \forall t \in [0, T]; \tag{4.1}$$

$$\|A(t)u(t)\|_E \leq C\{\|A(0)x\|_E + \|f\|_{C^\sigma([0,T],E)}\}, \quad \forall t \in [0, T]. \tag{4.2}$$

PROOF. – The first inequality is an immediate consequence of Formula (F), which holds in this case by Theorems 4.2, 4.3 and 4.1, and the fact that $(1 + K)^{-1} \in \mathcal{L}(L^1(0, t, E))$, $\forall t \in [0, T]$; the latter property can be proved as Lemma 3.4 (i).

To verify the second estimate, we repeat the argument used in the proof of Theorem 4.1, obtaining now:

$$A(t)u(t) = A(t)\exp[tA(t)]x + \int_0^t A(t)\exp[(t-s)A(t)]\big(1 - A(t)A(s)^{-1}\big)A(s)u(s)\, ds +$$
$$+ \int_0^t A(t)\exp[(t-s)A(t)]\big(f(s) - f(t)\big)\, ds + \big(\exp[tA(t)] - 1\big) f(t), \quad \forall t \in [0, T].$$

Hence

$$\|A(t)u(t)\|_E \leq C\|A(0)x\|_E + C\|f\|_{C^\sigma([0,T],E)} + C\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \|A(s)u(s)\|_E \, ds \cdot$$

Again, a Gronwall-type argument (see Corollary 2.4 of [3]) leads to the result. ///

THEOREM 4.5. – *Under Hypotheses I, II let u be a strict solution of* (P), *with* $x \in D(A(0))$ *and* $f \in C([0, T], E)$. *Then*

$$\|u(t)\|_E \leq C\left\{\|x\|_E + \int_0^t \|f(s)\|_E \, ds\right\}, \quad \forall t \in [0, T].$$

PROOF. – We remark that we cannot use Formula (F) directly, as in Theorem 4.4, since f is not Hölder continuous. Let $\varepsilon > 0$. Consider a function $\varphi \in C^\infty(\boldsymbol{R})$ with support contained in $[-1, 1]$, such that $\varphi \geq 0$ and $\int_{-1}^{1} \varphi(s)\, ds = 1$, and define $\varphi_n(t) = n\varphi(nt)$. Since $u'(t) - A(t)u(t) = f(t)$ in $[\varepsilon, T - \varepsilon]$, we have for each $n > \varepsilon^{-1}$:

$$(\varphi_n * u)'(t) - \big(\varphi_n * A(\cdot)u\big)(t) = \varphi_n * \big(u' - A(\cdot)u(\cdot)\big)(t) = (\varphi_n * f)(t), \quad \forall t \in [\varepsilon, T - \varepsilon].$$

where

$$(\varphi_n * v)(t) = \int_{t-1/n}^{t+1/n} \varphi_n(t-s)v(s)\, ds, \quad \forall v \in C([0, T], E).$$

Note that $\varphi_n * u \in C^\infty([\varepsilon, T-\varepsilon], E)$; in addition $(\varphi_n * u)(t) \in D(A(0))$, $\forall t \in [\varepsilon, T-\varepsilon]$, $A(\cdot)\varphi_n * u \in C([\varepsilon, T-\varepsilon], E)$ and

$$A(t)(\varphi_n * u)(t) = \int_{t-1/n}^{t+1/n} \varphi_n(t-s) A(t) u(s)\, ds\,, \qquad \forall t \in [\varepsilon, T-\varepsilon]\,.$$

Hence $\varphi_n * u$ is a strict solution of

$$\begin{cases} (\varphi_n * u)'(t) - A(t)(\varphi_n * u)(t) = (\varphi_n * f)(t) + (\varphi_n * A(\cdot)u)(t) - A(t)(\varphi_n * u)(t)\,, \\ \hfill t \in [\varepsilon, T-\varepsilon]\,, \\ (\varphi_n * u)(\varepsilon) = (\varphi_n * u)(\varepsilon)\,. \end{cases}$$

The right-hand side of this equation is in $C^\alpha([\varepsilon, T-\varepsilon], E)$: indeed, $\varphi_n * f \in C^\infty([\varepsilon, T-\varepsilon], E)$ and moreover, denoting by $\psi_n(t)$ the function $(\varphi_n * A(\cdot)u)(t) - A(t)(\varphi_n * u)(t)$, we have

$$\psi_n(t) = \int_{t-1/n}^{t+1/n} \varphi_n(t-s)\big(1 - A(t)A(s)^{-1}\big) A(s)u(s)\, ds\,,$$

which implies for $t > \tau$

$$\begin{aligned} \|\psi_n(t) - \psi_n(\tau)\|_E \leqslant & \int_{t-1/n}^{t+1/n} n|\varphi(n(t-s)) - \varphi(n(\tau-s))| \|1 - A(t)A(s)^{-1}\|_{\mathfrak{L}(E)} \|A(s)u(s)\|_E\, ds + \\ & + \int_{\tau-1/n}^{\tau+1/n} n\varphi(n(\tau-s)) \|1 - A(t)A(\tau)^{-1}\|_{\mathfrak{L}(E)} \|A(\tau)A(s)^{-1}\|_{\mathfrak{L}(E)} \|A(s)u(s)\|_E\, ds \leqslant \\ & \leqslant n^2 \|\varphi'\|_{C([-1,1])}(t-\tau) K \int_{\tau-1/n}^{t+1/n} |t-s|^\alpha \|A(s)u(s)\|_E\, ds + n\|\varphi\|_{C([-1,1])} K(t-\tau)^\alpha \cdot C \cdot \\ & \cdot \int_{\tau-1/n}^{\tau+1/n} \|A(s)u(s)\|_E\, ds \leqslant C(n)(t-\tau)^\alpha \sup_{t\in[0,T]} \|A(t)u(t)\|_E\,. \end{aligned}$$

Then we can apply Theorem 4.4 (recall also Remark 1.7), checking that

$$\|(\varphi_n * u)(t)\|_E \leqslant C\left\{\|(\varphi_n * u)(\varepsilon)\|_E + \int_\varepsilon^t \|(\varphi_n * f)(s) + \psi_n(s)\|_E\, ds\right\}, \qquad \forall t \in [\varepsilon, T-\varepsilon]\,.$$

Now as $n \to \infty$ we have $\varphi_n * u \to u$, $\varphi_n * f \to f$ and $\psi_n \to 0$ in $C([\varepsilon, T-\varepsilon], E)$, since

$$\|\psi_n(t)\|_E \leqslant n\|\varphi\|_{C([-1,1])} \int_{t-1/n}^{t+1/n} K|t-s|^\alpha \|A(s)u(s)\|_E\, ds \leqslant \frac{C}{n^\alpha} \sup_{t\in[0,T]} \|A(t)u(t)\|_E;$$

hence as $n \to \infty$ we conclude that

$$\|u(t)\|_E \leqslant C\left\{\|u(\varepsilon)\|_E + \int_\varepsilon^t \|f(s)\|_E \, ds\right\}, \qquad \forall t \in [\varepsilon, T - \varepsilon] .$$

and letting $\varepsilon \to 0^+$ we get the result. ///

REMARK 4.6. – In Theorem 4.3 it would be sufficient to suppose that $f \in C([0, T], E)$ and there exists $t^0 \in]0, T]$ such that the oscillation $\omega(\cdot)$ of f satisfies

$$\int_0^{t^0} \frac{\omega(\tau)}{\tau} \, d\tau < \infty . \tag{4.3}$$

This assumption, together with $x \in D(A(0))$ and $A(0)x + f(0) \in \overline{D(A(0))}$, still guarantees that $u(t)$, defined by Formula (F), is the unique strict solution of Problem (P) (but, of course, it is no longer Hölder continuous). We omit the proof, which needs a series of preliminary lemmata similar to those of Section 3, in order to verify that condition (4.3) in fact assures the convergence of all integrals involved. This generalizes a result of CRANDALL-PAZY [10] relative to the autonomous case, and is in perfect analogy with the case of variable domains (see [1], Remarks 4.3 and 5.2). Note that it is not possible to assume f is merely continuous (without condition (4.3)): for example if E is reflexive and $A(t) \equiv A \notin \mathcal{L}(E)$ then there exists a continuous f such that Problem (P) has no continuously differentiable solution (see BAILLON [4], TRAVIS [36], DA PRATO-GRISVARD [12]).

We conclude this section with the study of maximal time regularity of the strict solution of (P).

THEOREM 4.7. – *Under Hypotheses I, II let* $x \in D(A(0))$, $f \in C^\sigma([0, T], E)$, $\sigma \in]0, \alpha]$, *and let u be a strict solution of* (P); *then* $u \in C^{1,\sigma}([0, T], E)$ *if and only if* $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$.

PROOF. – By Theorems 4.2, 4.3 and 4.1 $u(t)$ is given by Formula (F). If in addition $u \in C^{1,\sigma}([0, T], E)$, then in particular

$$u'(t) - u'(0) = O(t^\sigma) \qquad \text{as } t \to 0^+;$$

but from the proof of Theorem 4.3 we know that

$$u'(t) - u'(0) = O(t^\sigma) + (\exp[tA(0)] - 1)(A(0)x + f(0)) \qquad \text{as } t \to 0^+ ,$$

hence $(\exp[tA(0)] - 1)(A(0)x + f(0)) = O(t^\sigma)$ as $t \to 0^+$, i.e. $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$. Suppose conversely that $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$, and let u be a strict solution.

of (P); then $u(t)$ is given by Formula (F) and

$$u'(t) = \exp[tA(0)]A(0)x + (Tg)'(t)\,, \qquad \forall t \in [0, T]\,,$$

where $g = (1+K)^{-1}(f - K(\cdot, 0)x)$. Since $g \in C^\sigma([0, T], E)$ (Lemma 3.2 (vii) and Lemma 3.4 (v)), by Lemma 3.5 (vii) we have as $t - \tau \to 0^+$

$$\begin{aligned} u'(t) - u'(\tau) &= \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\big(A(0)x + g(0)\big) + 0\big((t-\tau)^\sigma\big) = \\ &= \exp[\tau A(0)]\big(\exp[(t-\tau)A(0)] - 1\big)\big(A(0)x + f(0)\big) + O\big((t-\tau)^\sigma\big) = 0\big((t-\tau)^\sigma\big)\,, \end{aligned}$$

hence $u' \in C^\sigma([0, T], E)$. ///

THEOREM 4.8. – *Under Hypotheses I, II let* $x \in D(A(0))$, $f \in h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$, *and let* u *be a strict solution of* (P). *Then* $u \in h^{1,\sigma}([0, T], E)$ *if and only if* $A(0)x + f(0) \in D_{A(0)}(\sigma)$.

PROOF. – If $u \in h^{1,\sigma}([0, T], E)$, by (F) we get

$$u'(t) - u'(0) = o(t^\sigma) \qquad \text{as } t \to 0^+\,,$$

and from the proof of Theorem 4.3 we have

$$u'(t) - u'(\tau) = o(t^\sigma) + \big(\exp[tA(0)] - 1\big)\big(A(0)x + f(0)\big) \qquad \text{as } t \to 0^+;$$

these two facts imply $A(0)x + f(0) \in D_{A(0)}(\sigma)$. If conversely $A(0)x + f(0) \in D_{A(0)}(\sigma)$, again by (F) we check

$$u'(t) = \exp[tA(0)]A(0)x + (Tg)'(t)\,, \qquad \forall t \in [0, T]\,,$$

where $g = (1+K)^{-1}(f - K(\cdot, 0)x) \in h^\sigma([0, T], E)$ by Lemma 3.2 (vii) and Lemma 3.4 (vi); so by Lemma 3.5 (viii)

$$\begin{aligned} u'(t) - u'(\tau) &= \big(\exp[tA(0)] - \exp[\tau A(0)]\big)\big(A(0)x + g(0)\big) + o\big((t-\tau)^\sigma\big) = \\ &= o\big((t-\tau)^\sigma\big) \qquad \text{as } t \to 0^+\,, \end{aligned}$$

hence $u' \in h^\sigma([0, T], E)$. ///

REMARK 4.9. – The necessary and sufficient conditions $A(0)x + f(0) \in \overline{D(A(0))}$ and $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$ of Theorems 4.3 and 4.7 do not differ from those holding in the case of variable domains. In that case under the stronger hypotheses I, II and III of [1] one obtains

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D(A(0))} \tag{4.4}$$

or

$$A(0)x + f(0) - \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)x \in D_{A(0)}(\sigma, \infty) . \tag{4.5}$$

Now, suppose $D(A(t)) \equiv D(A(0))$ and $t \to A(t)^{-1}$ continuously differentiable in $\mathcal{L}(E)$; then it is immediate to verify that the range of the operator $(d/dt)A(t)^{-1}$ lies in $\overline{D(A(0))}$ for each $t \in [0, T]$, so that condition (4.4) reduces to $A(0)x + f(0) \in \overline{D(A(0))}$.

Suppose moreover that $(d/dt)A(t)^{-1} \in C^{\eta}([0, T], \mathcal{L}(E))$, $\eta \in]0, 1[$, and take $\sigma \leqslant \eta \wedge \alpha$ in (4.5) (here α is the number defined in Hypothesis II in Section 1). Then assuming again $D(A(t)) \equiv D(A(0))$ we have for each $y \in E$ and $t \in [0, T]$:

$$\begin{aligned} &\left\| \frac{\exp[\xi A(0)] - 1}{\xi^{\sigma}} \left[\frac{d}{dt} A(t)^{-1} y\right] \right\|_E \leqslant \\ &\qquad \leqslant \left\| \frac{\exp[\xi A(0)] - 1}{\xi^{\sigma}} \left[\frac{d}{dt} A(t)^{-1} - \frac{A(t+\xi)^{-1} - A(t)^{-1}}{\xi}\right] y \right\|_E + \\ &\qquad + \left\| \frac{1}{\xi} \int_0^{\xi} \frac{\exp[\tau A(0)] \left(A(0)A(t+\xi)^{-1} - A(0)A(t)^{-1}\right)}{\xi^{\sigma}} y \, d\tau \right\|_E \leqslant \\ &\qquad \leqslant C \|\exp[\xi A(0)] - 1\|_{\mathcal{L}(E)} \xi^{\eta-\sigma} \|y\|_E + \frac{1}{\xi} \int_0^{\xi} \|\exp[\tau A(0)]\|_{\mathcal{L}(E)} \, d\tau \cdot \\ &\qquad \cdot K \xi^{\alpha-\sigma} \|A(0)A(t)^{-1}\|_{\mathcal{L}(E)} \|y\|_E \leqslant C \|y\|_E \xi^{(\eta\wedge\alpha)-\sigma} \qquad \text{if } \xi \in]0, 1] , \end{aligned}$$

while obviously

$$\left\| \frac{\exp[\xi A(0)] - 1}{\xi^{\sigma}} \frac{d}{dt} A(t)^{-1} y \right\|_E \leqslant C \|y\|_E \quad \text{if } \xi > 1 ;$$

this means that the range of $(d/dt)A(t)^{-1}$ lies in $D_{A(0)}(\eta \wedge \alpha, \infty)$, and therefore condition (4.5) reduces to $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$.

REMARK 4.10. – Replace Hypothesis II in Section 1 by the following stronger one:

$$\|1 - A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)} = o(|t - \tau|^{\alpha}) \quad \text{as } |t - \tau| \to 0^{+} ; \tag{4.6}$$

then we can improve the results of Theorems 4.3 and 4.8 as follows:

(*a*) if $x \in D(A(0))$, $f \in h^{\alpha}([0, T], E)$ and $A(0)x + f(0) \in \overline{D(A(0))}$, then the strict solution u is in $h^{1,\alpha}(]0, T], E)$;

(*b*) if $x \in D(A(0))$, $f \in h^{\alpha}([0, T], E)$ and $A(0)x + f(0) \in \overline{D(A(0))}$, then the strict solution u is in $h^{1,\alpha}([0, T], E)$ if and only if $A(0)x + f(0) \in D_{A(0)}(\alpha)$.

We omit the proof, which does not differ from those of Theorems 4.3 and 4.8; it requires some further statements in Lemmata 3.2, 3.4 and 3.5, which however can be easily proved according to the same lines used in Section 3.

5. – Classical solutions.

This section is devoted to the study of classical solutions of Problem (P) (see Definition 1.4). As remarked in the Introduction, most of the results of this section were first proved by TANABE [34] and SOBOLEVSKII [31], who supposed however $D(A(0))$ to be dense in E. As in Section 4, we start with uniqueness under general assumptions on f (i.e. $f \in C(]0, T], E)$); with a little more regularity of f, namely $f \in C_\theta(]0, T], E)$, we get an «a priori» estimate for classical solutions as a consequence of the similar one relative to strict solutions. Next, we deduce an easy necessary condition for existence; finally we will see that existence (and time regularity) is guaranteed by adding more regularity on f (i.e. $f \in C_\theta(]0, T], E) \cap C^\sigma(]0, T], E)$).

THEOREM 5.1. – *Under Hypotheses I, II Problem* (P) *with $x \in E$ and $f \in C(]0, T], E)$ has at most one classical solution.*

PROOF. – Let u, v be two classical solutions of (P) with $x \in E$ and $f \in C(]0, T], E)$; then for each $\varepsilon \in]0, T[$ the function $u - v$ is a strict solution of (P) in $[\varepsilon, T]$ with data $u(\varepsilon) - v(\varepsilon)$, 0. Hence by Theorem 4.5 we have (recall Remark 1.7)

$$\|u(t) - v(t)\|_E \leqslant C\|u(\varepsilon) - v(\varepsilon)\|_E\,, \qquad \forall t \in [\varepsilon, T]\,;$$

as $\varepsilon \to 0^+$ we get $u \equiv v$. ///

THEOREM 5.2. – *Under Hypotheses I, II let u be a classical solution of* (P) *with $x \in E$ and $f \in C_\theta(]0, T], E)$, $\theta \in [0, 1[$. Then*

$$\|u(t)\|_E \leqslant C\left\{\|x\|_E + \int_0^t \|f(s)\|_E\, ds\right\}, \qquad \forall t \in [0, T]\,.$$

PROOF. – As u is a strict solution in $[\varepsilon, T]$ for each $\varepsilon \in]0, T[$, Theorem 4.5 and Remark 1.7 yield

$$\|u(t)\|_E \leqslant C\left\{\|u(\varepsilon)\|_E + \int_\varepsilon^t \|f(s)\|_E\, ds\right\}, \qquad \forall t \in [\varepsilon, T]\,.$$

Letting $\varepsilon \to 0^+$, the result follows, since f is integrable over $]0, T[$. ///

THEOREM 5.3. – *Under Hypotheses I, II if u is a classical solution of* (P) *with* $x \in E$ *and* $f \in C(]0, T], E)$, *then* $x \in \overline{D(A(0))}$.

PROOF. – Since $u \in C([0, T], E)$ by definition, we have $x = \lim_{t\to 0^+} u(t)$; the proof is complete since $u(t) \in D(A(0))$, $\forall t \in]0, T]$. ///

THEOREM 5.4. – *Under Hypotheses I, II let* $x \in \overline{D(A(0))}$ *and* $f \in C_\theta(]0, T], E) \cap$ $\cap\, C^\sigma(]0, T], E)$, $\theta \in [0, 1[$, $\sigma \in]0, 1[$. *Then the function u given by Formula* (F) *is a classical solution of* (P) *and belongs to* $C^{1,\sigma\wedge\alpha}(]0, T], E)$. *If in addition* $f \in h^\sigma(]0, T], E)$, $\sigma \in]0, \alpha[$, *then* $u \in h^{1,\sigma}(]0, T], E)$.

PROOF. – As in the first part of the proof of Theorem 4.3, we get $u \in C([0, T], E)$ and $t \to \exp[tA(0)]x \in C^\infty(]0, T], E)$; by Lemma 3.2 (ii) we have $K(\cdot, 0)x \in C^\alpha(]0, T], E)$, so that

$$g \equiv (1 + K)^{-1}(f - K(\cdot, 0)x) \in C^{\sigma\wedge\alpha}(]0, T], E) \cap C_\theta(]0, T], E)$$

by Lemma 3.4 (ii)-(iv); therefore $Tg \in C^{1,\sigma\wedge\alpha}(]0, T], E)$ (Lemma 3.5 (v)). If in addition $f \in h^\sigma(]0, T], E)$, $\sigma \in]0, \alpha[$, then $g \in h^\sigma(]0, T], E)$ by Lemma 3.4 (iv), hence $Tg \in h^{1,\sigma}(]0, T], E)$ by Lemma 3.5 (v), and finally $u \in h^{1,\sigma}(]0, T], E)$. ///

REMARK 5.5. – As in the case of strict solutions, Theorem 5.4 still holds (except for the Hölder regularity of u') assuming $f \in C_\theta(]0, T], E)$, $\theta \in [0, 1[$, and the following property (instead of Hölder continuity): for each $\varepsilon \in]0, T[$, the oscillation $\omega_\varepsilon(\cdot)$ of $f|_{[\varepsilon,T]}$ satisfies condition (4.3). This is no longer true if f is merely in $C_\theta(]0, T], E)$ (see Remark 4.6).

REMARK 5.6. – Replacing Hypothesis II with (4.6), we obtain the following improvement of Theorem 5.4: if $x \in \overline{D(A(0))}$ and $f \in C_\theta(]0, T], E) \cap h^\alpha(]0, T], E)$, $\theta \in [0, 1[$, then the classical solution u is in $h^{1,\alpha}(]0, T], E)$.

6. – Strong solutions.

Let us consider now strong solutions of Problem (P). The basic tool is the « a priori » estimate for strict solutions (Theorem 4.5). As a first consequence we derive an « a priori » estimate for strong solutions, and consequently uniqueness. Next, after an easy necessary condition for existence, we prove that Formula (F) indeed gives the strong solution of (P), and study its Hölder regularity. Finally we show that under Hypotheses I, II a classical solution of (P) with $f \in C([0, T], E)$ is in fact a strong one.

THEOREM 6.1. – *Under Hypotheses I, II let u be a strong solution of* (P) *with $x \in E$ and $f \in C([0, T], E)$. Then*

$$\|u(t)\|_E \leqslant C\left\{\|x\|_E + \int_0^t \|f(s)\|_E \, ds\right\}, \qquad \forall t \in [0, T].$$

PROOF. – By definition, there exists $\{u_n\}_{n\in N} \subseteq C^1([0, T], E) \cap C([0, T], D(A(0)))$ such that, defining $x_n = u_n(0)$, $f_n = u_n' - A(\cdot)u_n(\cdot)$, we have

$$\text{(6.1)} \qquad \begin{cases} \text{i) } u_n \to u \text{ in } C([0, T], E),\ f_n \to f \text{ in } C([0, T], E),\ x_n \to x \text{ in } E; \\ \text{ii) } u_n \text{ is a strict solution of} \\ \qquad \begin{cases} u_n'(t) - A(t)u_n(t) = f_n(t), & t \in [0, T], \\ u_n(0) = x_n . \end{cases} \end{cases}$$

By Theorem 4.5 we have

$$\|u_n(t)\|_E \leqslant C\left\{\|x_n\|_E + \int_0^t \|f_n(s)\|_E \, ds\right\}, \qquad \forall t \in [0, T],$$

and as $n \to \infty$ we get the result. ///

COROLLARY 6.2. – *Under Hypotheses I, II Problem* (P) *with $x \in E$ and $f \in C([0, T], E)$ has at most one strong solution.* ///

THEOREM 6.3. – *Under Hypotheses I, II let u be a strong solution of* (P) *with $x \in E$ and $f \in C([0, T], E)$; then $x \in \overline{D(A(0))}$.*

PROOF. – Let $\{u_n\}_{n\in N} \subseteq C^1([0, T], E) \cap C([0, T], D(A(0)))$ satisfy (6.1); then $x = u(0) = \lim_{n\to\infty} u_n(0)$, and $u_n(0) \in D(A(0))$, $\forall n \in N$. ///

THEOREM 6.4. – *Under Hypotheses I, II let $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$. Then the function $u(t)$ defined by Formula* (F) *is a strong solution of* (P) *and belongs to $C^\sigma(]0, T], E)$, $\forall \sigma \in]0, 1[$.*

PROOF. – By Lemmata 3.1 (iii) and 3.5 (i), $u \in C([0, T], E)$ and $u(0) = x$. Since $x \in \overline{D(A(0))}$, we can choose $\{x_n\}_{n\in N} \subseteq D(A(0))$ such that $x_n \to x$ in E and $A(0)x_n + f(0) \in \overline{D(A(0))}$. Indeed, take $\{w_n\}_{n\in N} \subseteq D(A(0)^2)$ such that $w_n \to x + A(0)^{-1}f(0)$ (this is possible by Lemma 2.5 (iv)), and define $x_n = w_n - A(0)^{-1}f(0)$, obtaining $A(0)x_n + f(0) = w_n \in D(A(0)) \subseteq \overline{D(A(0))}$. Now fix $\delta \in]0, \alpha]$; since $C^\delta([0, T], E)$ is dense in $C([0, T], E)$, there exists $\{f_n\}_{n\in N} \subseteq C^\delta([0, T], E)$ such that $f_n(0) = f(0)$ and $f_n \to f$ in

$C([0, T], E)$. Let u_n be the strict solution of

$$\begin{cases} u_n'(t) - A(t)u_n(t) = f_n(t)\,, & t \in [0, T]\,, \\ u_n(0) = x_n\,, \end{cases}$$

which exists by Theorem 4.3 since $A(0)x_n + f_n(0) = A(0)x_n + f(0) \in \overline{D(A(0))}$.

Then $u_n(t)$ is given by Formula (F). It follows that for each $t \in [0, T]$

$$\begin{aligned} \|u_n(t) - u(t)\|_E &\leqslant \|x_n - x\|_E + C\int_0^T \|(1 + K)^{-1}(f_n - f - K(\cdot, 0)(x_n - x))(s)\|_E\, ds \leqslant \\ &\leqslant C\|x_n - x\|_E + C\|(1 + K)^{-1}\|_{\mathcal{L}(L^1(0,T,E))}\left[T \cdot \|f_n - f\|_{C([0,T),E)} + \frac{T^\alpha}{\alpha}\|x_n - x\|_E\right], \end{aligned}$$

hence $u_n \to u$ in $C([0, T], E)$. This shows that u is a strong solution of (P).

Finally by Lemmata 3.1 (ii), 3.2 (i), 3.4 (ii) and 3.5 (iii) we deduce that $u \in C^\sigma(]0, T], E)$, $\forall \sigma \in]0, 1[$. ///

About Hölder regularity of the strong solution up to $t = 0$ we have:

THEOREM 6.5. – *Under Hypotheses I, II let u be a strong solution of* (P) *with $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$. Then $u \in C^\sigma([0, T], E)$, $\sigma \in]0, 1[$, if and only if $x \in D_{A(0)}(\sigma, \infty)$.*

PROOF. – Let $x \in D_{A(0)}(\sigma, \infty)$. By Lemma 3.1 (iv), $t \to \exp[tA(0)]x \in C^\sigma([0, T], E)$. On the other hand, Lemma 3.2 (iii)-(iv) implies that

$$K(\cdot, 0)x \in \begin{cases} C_{1-\alpha-\sigma}(]0, T], E) & \text{if } 0 < \sigma \leqslant 1 - \alpha\,, \\ C^{\alpha+\sigma-1}([0, T], E) & \text{if } 1 - \alpha < \sigma < 1\,. \end{cases}$$

Hence, by Lemma 3.4 (ii)-(iii)

$$(1 + K)^{-1}(f - K(\cdot, 0)x) \in \begin{cases} C_{1-\alpha-\sigma}(]0, T], E) & \text{if } 0 < \sigma \leqslant 1 - \alpha\,, \\ C([0, T], E) & \text{if } 1 - \alpha < \sigma < 1\,, \end{cases}$$

and by Lemma 3.5 (ii)-(iv)

$$T(1 + K)^{-1}(f - K(\cdot, 0)x) \in \begin{cases} C^{\alpha+\sigma}([0, T], E) & \text{if } 0 < \sigma < 1 - \alpha\,, \\ C^\delta([0, T], E),\ \forall \delta \in]0, 1[& \text{if } 1 - \alpha \leqslant \sigma < 1\,. \end{cases}$$

Since u is given by Formula (F) (by Theorem 6.4 and Corollary 6.2), we conclude that $u \in C^\sigma([0, T], E)$.

Conversely, let $x \in \overline{D(A(0))}$, $f \in C([0, T], E)$, and suppose that u is a strong solution of (P) belonging to $C^\sigma([0, T], E)$, $\sigma \in]0, 1[$; then u is given by Formula (F). Moreover by Lemmata 3.2 (i), 3.4 (ii), 3.5 (ii) we have $Tg \in C^\alpha([0, T], E)$, where $g = (1 + K)^{-1}(f - K(\cdot, 0)x)$, and therefore $\exp[tA(0)]x = u - Tg \in C^{\sigma\wedge\alpha}([0, T], E)$, which by Lemma 3.1 (iv) means $x \in D_{A(0)}(\sigma\wedge\alpha, \infty)$. If $\sigma \in]0, \alpha]$ the proof is complete, otherwise we know that $x \in D_{A(0)}(\alpha, \infty)$.

As before, we then deduce $Tg \in C^{(2\alpha)\wedge\delta}([0, T], E)$, $\forall \delta \in]0, 1[$, and hence

$$\exp[tA(0)]x \in C^{\sigma\wedge(2\alpha)}([0, T], E) , \quad \text{or} \quad x \in D_{A(0)}(\sigma\wedge(2\alpha), \infty) .$$

If $\sigma \in]0, 2\alpha]$ the proof is complete, otherwise after a finite number of steps we get $x \in D_{A(0)}(\sigma, \infty)$. ///

THEOREM 6.6. – *Under Hypotheses I, II let u be a strong solution of* (P) *with $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$. Then $u \in h^\sigma([0, T], E)$, $\sigma \in]0, 1[$, if and only if $x \in D_{A(0)}(\sigma)$.*

PROOF. – If $x \in D_{A(0)}(\sigma)$ then $\exp[tA(0)]x \in h^\sigma([0, T], E)$ by Lemma 3.1 (v). On the other hand, as in the proof of Theorem 6.5, we have $Tg \in C^{(\sigma+\alpha)\wedge\delta}([0, T], E)$, $\forall \delta \in]0, 1[$, where $g = (1 + K)^{-1}(f - K(\cdot, 0)x)$, and in particular $Tg \in h^\delta([0, T], E)$, $\forall \delta \in]0, (\sigma + \alpha)\wedge 1[$. Therefore we conclude that $u \in h^\sigma([0, T], E)$.

Conversely, let u be a strong solution belonging to $h^\sigma([0, T], E)$, $\sigma \in]0, 1[$. As in Theorem 6.5, $Tg \in C^\alpha([0, T], E) \subseteq h^\delta([0, T], E)$, $\forall \delta \in]0, \alpha[$. Hence $\exp[tA(0)]x \in h^{\sigma\wedge\delta}([0, T], E)$, $\forall \delta \in]0, \alpha[$, or $x \in D_{A(0)}(\sigma\wedge\delta)$, $\forall \delta \in]0, \alpha[$. By possibly a finite number of steps we get $x \in D_{A(0)}(\sigma)$. ///

THEOREM 6.7. – *Under Hypotheses I, II every classical solution of Problem* (P) *with $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$ is also a strong one.*

PROOF. – Choose $\varepsilon \in]0, T[$: u is a strict solution of

$$\begin{cases} u'(t) - A(t)u(t) = f(t) , & t \in [\varepsilon, T] \\ u(\varepsilon) = u(\varepsilon) . \end{cases}$$

Let v be the strong solution of (P) (it exists by Theorem 6.4). Then by definition there exists $\{v_n\}_{n\in\mathbf{N}}$ such that $v_n \to v$ in $C([0, T], E)$ as $n \to \infty$, and for each $n \in \mathbf{N}$, v_n is the strict solution of

$$\begin{cases} v_n'(t) - A(t)v_n(t) = f_n(t) , & t \in [0, T] \\ v_n(0) = x_n , \end{cases}$$

where $f_n \to f$ in $C([0, T], E)$ and $x_n \to x$ in E as $n \to \infty$.

Hence $w_n = u - v_n$ is the strict solution of

$$\begin{cases} w_n'(t) - A(t)w_n(t) = f(t) - f_n(t)\,, & t \in [\varepsilon, T] \\ w_n(\varepsilon) = u(\varepsilon) - v_n(\varepsilon)\,. \end{cases}$$

By Theorem 4.5, taking into account Remark 1.7, we get

$$\|u(t) - v_n(t)\|_E \leqslant C\Big\{\|u(\varepsilon) - v_n(\varepsilon)\|_E + \int_\varepsilon^t \|f(s) - f_n(s)\|_E\, ds\Big\}\,, \qquad \forall t \in [\varepsilon, T]\,.$$

As $n \to \infty$ we obtain

$$\|u(t) - v(t)\|_E \leqslant C\|u(\varepsilon) - v(\varepsilon)\|_E\,, \qquad \forall t \in [\varepsilon, T]\,,$$

and letting $\varepsilon \to 0^+$ we check $u \equiv v$, i.e. u coincides with the strong solution of (P). (Thus, in particular, u is given by Formula (F).) ///

We recall however that in the definition of strong solution the function f is required to be continuous in $[0, T]$, while one can speak about classical solutions of (P) provided f is merely in $C(]0, T], E)$.

7. - Space regularity.

In this section we look for conditions assuring that the strong solution of Problem (P) is continuous with values in the intermediate spaces $D_{A(0)}(\sigma, \infty)$ and $D_{A(0)}(\sigma)$. Since the solution is given by Formula (F), we begin to study the behaviour in such spaces of the operators appearing in (F).

LEMMA 7.1. - *Under Hypotheses I, II let $\theta, \beta \in]0, 1[$. Then for each $s \in [0, T]$ and $t \in]0, T]$ we have:*

(i) $\|\exp[tA(s)]\|_{\mathcal{L}(E, D_{A(0)}(\theta))} \leqslant \dfrac{C}{t^\theta}$.

(ii) $\|\exp[tA(s)]\|_{\mathcal{L}(D_{A(0)}(\beta,\infty), D_{A(0)}(\theta))} \leqslant \dfrac{C}{t^{(\theta-\beta)\vee 0}}$.

(iii) $\|A(s)\exp[tA(s)]\|_{\mathcal{L}(E, D_{A(0)}(\theta))} \leqslant \dfrac{C}{t^{1+\theta}}$.

(iv) $\|A(s)\exp[tA(s)]\|_{\mathcal{L}(D_{A(0)}(\beta,\infty), D_{A(0)}(\theta))} \leqslant \dfrac{C}{t^{1+\theta-\beta}}$.

PROOF. - (i) It is a consequence of (1.36) of SINESTRARI [30].

(ii) It follows by using (1.37), (1.41) and (1.14) of [30] in the cases $\beta < \theta$, $\beta > \theta$ and $\beta = \theta$ respectively.

(iii) See (1.38) of [30].

(iv) The conclusion follows by (1.39), (1.43) and (1.14) of [30] in the cases $\beta < \theta$, $\beta > \theta$ and $\beta = \theta$ respactively. ///

LEMMA 7.2. - *Under Hypotheses I, II we have:*

(i) $\|\exp[tA(0)] - \exp[\tau A(0)]\|_{\mathcal{L}(E, D_{A(0)}(\theta))} \leqslant C(\varepsilon)|t-\tau|$, $\forall t, \tau \in [\varepsilon, T]$, $\forall \varepsilon \in]0, T[$, $\forall \theta \in]0, 1[$.

(ii) $t \to \exp[tA(0)] \in C^{\beta-\theta}\big([0, T], \mathcal{L}(D_{A(0)}(\beta, \infty), D_{A(0)}(\theta))\big)$, $\forall \beta \in]0, 1[$, $\forall \theta \in]0, \beta[$.

(iii) $t \to \exp[tA(0)] \in C_{\theta-\beta}\big(]0, T], \mathcal{L}(D_{A(0)}(\beta, \infty), D_{A(0)}(\theta))\big)$, $\forall \beta \in]0, 1[$, $\forall \theta \in [\beta, 1[$.

(iv) *If* $x \in D_{A(0)}(\beta), \beta \in]0, 1[$, *then* $t \to \exp[tA(0)]x \in h^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, \beta[$.

(v) *If* $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1[$, *then* $t \to \exp[tA(0)]x \in C_{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in [\beta, 1[$, *and*

$$[t^{\theta-\beta}\exp[tA(0)]x]_{t=0} = \begin{cases} x & \text{if } \theta = \beta\,, \\ 0 & \text{if } \theta > \beta\,. \end{cases}$$

PROOF. - (i) It is a consequence of the fact that $t \to \exp[tA(0)] \in C^1(]0, T]$, $\mathcal{L}\big(E, D(A(0))\big)\big)$.

(ii) Let $t > \tau \geqslant 0$ and take $x \in D_{A(0)}(\beta, \infty)$. For any $\xi > 0$ we have by Lemma 7.1 (iv):

$$\left\|\frac{\exp[\xi A(0)] - 1}{\xi^\theta}(\exp[tA(0)] - \exp[\tau A(0)])x\right\|_E =$$
$$= \left\|\frac{\exp[\xi A(0)] - 1}{\xi^\theta}\int_\tau^t A(0)\exp[\sigma A(0)]x\, d\sigma\right\|_E \leqslant \int_\tau^t \|A(0)\exp[\sigma A(0)]x\|_\theta\, d\sigma \leqslant$$
$$\leqslant C\int_\tau^t \frac{d\sigma}{\sigma^{1+\theta-\beta}}\|x\|_\beta \leqslant C(t-\tau)^{\beta-\theta}\|x\|_\beta\,.$$

(iii) It follows by (i) and Lemma 7.1 (ii).

(iv) It follows by Theorem 3.1 (*c*) of [30].

(v) By (i) we have $t \to t^{\theta-\beta}\exp[tA(0)]x \in C(]0, T], D_{A(0)}(\theta))$; to prove continuity at $t = 0$, suppose first $\theta = \beta$: in this case we have $\lim_{t\to 0^+}\|\exp[tA(0)x - x\|_\beta = 0$ because $t \to \exp[tA(0)]$ is a strongly continuous semi-group in $D_{A(0)}(\beta)$ (see e.g. (1.11) of [30]).

On the other hand, if $\theta > \beta$, as $x \in D_{A(0)}(\beta)$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|t^{1-\beta}A(0)\exp[tA(0)x\|_E < \varepsilon$ if $0 < t < \delta$; therefore

$$t^{\theta-\beta}\|\xi^{1-\theta}A(0)\exp[\xi A(0)]\exp[tA(0)]x\|_E \leqslant$$

$$\leqslant \begin{cases} C\|t^{1-\beta}A(0)\exp[tA(0)]x\|_E \leqslant C\varepsilon & \text{if } \xi \leqslant t, \\ C\|\xi^{1-\beta}A(0)\exp[\xi A\ 0)]x\|_E \leqslant C\varepsilon & \text{if } t < \xi < \delta, \\ Ct^{\theta-\beta}\|\xi A(0)\exp[\xi A(0)]\|_{\mathcal{L}(E)}\dfrac{1}{\delta^\theta}\|\exp[tA(0)]x\|_E \leqslant C(\varepsilon)t^{\theta-\beta} & \text{if } \xi \geqslant \delta, \end{cases}$$

which implies $\lim\limits_{t\to 0^+} t^{\theta-\beta}\|\exp[tA(0)]x\|_\beta = 0$. ///

In our next lemma we prove some results which are similar to those of Lemma 2.12 of FUJITA-KATO [19].

LEMMA 7.3. – *Under Hypotheses I, II we have:*

(i) *If* $\varphi \in C_\sigma(]0, T], E)$, $\sigma \in [0, 1[$, *then* $T\varphi(t) \in D_{A(0)}(\theta)$, $\forall\theta \in]0, 1[$, $\forall t \in [0, T]$.

(ii) $T \in \mathcal{L}\big(C_\sigma(]0, T], E), C^{1-\sigma-\theta}([0, T], D_{A(0)}(\theta))\big)$, $\forall\sigma \in [0, 1[$, $\forall\theta \in]0, 1-\sigma[$.

(iii) *If* $\varphi \in C_\sigma(]0, T], E)$, $\sigma \in]0, 1[$, *then* $T\varphi \in C^{1-\theta}(]0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, 1[$, *and*

$$\|T\varphi(t)\|_\theta \leqslant C(\theta)t^{1-\sigma-\theta}\|\varphi\|_{C_\sigma(]0,T],E)}, \qquad \forall t \in [0, T],\ \forall\theta \in]0, 1[\,.$$

PROOF. – (i) If $\xi \in]0, t[$ we have

$$\left\|\frac{\exp[\xi A(0)] - 1}{\xi^\theta}T\varphi(t)\right\|_E \leqslant \tag{7.1}$$

$$\leqslant \left\|\int_0^{t-\xi}\frac{\exp[\xi A(0)] - 1}{\xi^\theta}A(0)^{-1}\big(A(0)A(s)^{-1}\big)A(s)\exp[(t-s)A(s)]\varphi(s)\,ds\right\|_E +$$

$$+ \left\|\int_{t-\xi}^t\frac{\exp[\xi A(0)] - 1}{\xi^\theta}\exp[(t-s)A(s)]\varphi(s)\,ds\right\|_E \leqslant$$

$$\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left[\int_0^{t-\xi}\frac{\xi^{1-\theta}}{(t-s)s^\sigma}\,s + \int_{t-\xi}^t\frac{ds}{(t-s)^\theta\,s^\sigma}\right].$$

By choosing $\varepsilon < 1-\theta$ we get

$$\left\|\frac{\exp[\xi A(0)] - 1}{\xi^\theta}T\varphi(t)\right\|_E \leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{\xi^{1-\theta-\varepsilon}\int_0^t\frac{ds}{(t-s)^{1-\varepsilon}s^\sigma} + \frac{\xi^{1-\theta}}{(t-\xi)^\sigma}\right\} \leqslant$$

$$\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{\frac{\xi^{1-\sigma-\varepsilon}}{t^{\sigma-\varepsilon}} + \frac{\xi^{1-\theta}}{(t-\xi)^\sigma}\right\},$$

which implies

$$\lim_{\xi\to 0^+}\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}T\varphi(t)\right\|_E = 0\,,\qquad \forall\theta\in\,]0,1[,\ \forall t\in\,]0,T]\,.$$

(ii) Let $\varphi\in C_\sigma(]0,T],E)$ and take $\theta\in\,]0,1-\sigma[$. If $t>\tau\geqslant 0$ and $\xi>0$, by Lemma 7.1 (i)-(iii) we check

$$\begin{aligned}
(7.2)\qquad &\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}\big(T\varphi(t)-T\varphi(\tau)\big)\right\|_E\leqslant\\
&\leqslant\left\|\int_\tau^t\frac{\exp[\xi A(0)]-1}{\xi^\theta}\exp[(t-s)A(s)]\varphi(s)\,ds\right\|_E+\\
&+\left\|\int_0^\tau\int_{\tau-s}^{t-s}\frac{\exp[\xi A(0)]-1}{\xi^\theta}A(s)\exp[\varrho A(s)]\varphi(s)\,d\varrho\,ds\right\|\leqslant\\
&\leqslant C\int_\tau^t\|\exp[(t-s)A(s)]\varphi(s)\|_\theta\,ds+C\int_0^\tau\int_{\tau-s}^{t-s}\|A(s)\exp[\varrho A(s)]\varphi(s)\|_\theta\,d\varrho\,ds\leqslant\\
&\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{\int_\tau^t\frac{ds}{(t-s)^\theta s^\sigma}+\int_0^\tau\int_{\tau-s}^{t-s}\frac{d\varrho}{\varrho^{\theta+1}}\frac{ds}{s^\sigma}\right\}\leqslant\\
&\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{(t-\tau)^{1-\theta-\sigma}+\int_0^\tau\frac{1}{(\tau-s)^{1-\sigma}s^\sigma}\int_{\tau-s}^{t-s}\frac{d\varrho}{\varrho^{\theta+1}}\,ds\right\}\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}(t-\tau)^{1-\theta-\sigma},
\end{aligned}$$

and (ii) is proved.

(iii) If $t>\tau\geqslant\varepsilon$ and $\theta\in\,]0,1[$, we have as in (7.2) for each $\xi>0$:

$$\begin{aligned}
&\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}\big(T\varphi(t)-T\varphi(\tau)\big)\right\|_E\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{\int_\tau^t\frac{ds}{(t-s)^\theta s^\sigma}+\int_0^\tau\int_{\tau-s}^{t-s}\frac{d\varrho}{\varrho^{\theta+1}}\frac{ds}{s^\sigma}\right\}\leqslant\\
&\qquad\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\left\{\frac{(t-\tau)^{1-\theta}}{\varepsilon^\sigma}+\int_0^{\varepsilon/2}\frac{ds}{s^\sigma}\frac{t-\tau}{(\varepsilon/2)^{\theta+1}}+\int_{\varepsilon/2}^\tau\left(\frac{1}{(\tau-s)^\theta}-\frac{1}{(t-s)^\theta}\right)ds\,\frac{1}{(\varepsilon/2)^\theta}\right\}\leqslant\\
&\qquad\leqslant C(\varepsilon)\|\varphi\|_{C_\sigma(]0,T],E)}(t-\tau)^{1-\theta};
\end{aligned}$$

this fact and (i) prove that $T\varphi\in C^{1-\theta}(]0,\ T],\ D_{A(0)}(\theta))$ $\forall\theta\in\,]0,1[$. Finally if $t\in\,]0,T]$, as in (7.1) we have for each $\xi>0$

$$\left\|\frac{\exp[\xi A(0)]-1}{\xi^\theta}T\varphi(t)\right\|_E\leqslant C\|\varphi\|_{C_\sigma(]0,T],E)}\int_0^t\frac{ds}{(t-s)^\theta s^\sigma}\leqslant Ct^{1-\sigma-\theta}\|\varphi\|_{C_\sigma(]0,T],E)}\,,$$

which proves (iii). ///

Now we are able to prove the space regularity theorem for the strong solution of (P).

THEOREM 7.4. – *Under Hypotheses I, II let u be a strong solution of Problem* (P) *with* $x \in \overline{D(A(0))}$ *and* $f \in C([0, T], E)$. *Then we have:*

(i) $u \in C^{1-\theta}(]0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, 1[$.

(ii) $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1[$, *if and only if* $u \in C_0(]0, T], D_{A(0)}(\beta))$.

(iii) *If* $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1[$, *then* $u \in C^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, \beta[$.

(iv) $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1[$, *if and only if* $u \in C([0, T], D_{A(0)}(\beta))$.

(v) *If* $x \in D_{A(0)}(\beta)$, $\beta \in]0, 1[$, *then* $u \in h^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, \beta[$.

PROOF. – (i) Since u is given by Formula (F), for each $t \in [0, T]$ we have $u(t) = \exp[tA(0)]x + Tg(t)$ where $g = (1 + K)^{-1}(f - K(\cdot, 0)x)$. By Lemma 7.2 (ii) we have $t \to \exp[tA(0)]x \in \mathrm{Lip}(]0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, 1[$; moreover, since $K(\cdot, 0)x \in C_{1-\alpha}(]0, T], E)$ (Lemma 3.2 (i)), we get $g \in C_{1-\alpha}(]0, T], E)$ by Lemma 3.4 (ii), and consequently $Tg \in C^{1-\theta}(]0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, 1[$ by Lemma 7.3 (iii). This proves (i).

(ii) Suppose $x \in D_{A(0)}(\beta, \infty)$. By Lemma 7.2 (iii), $t \to \exp[tA(0)]x \in C_0(]0, T], D_{A(0)}(\beta))$; next, Lemma 3.2 (iii)-(v) gives

$$K(\cdot, 0)x \in \begin{cases} C_{1-\alpha-\beta}(]0, T], E) & \text{if } \beta \leqslant 1 - \alpha\,, \\ C^{\alpha+\beta-1}([0, T], E) & \text{if } \beta > 1 - \alpha\,; \end{cases}$$

in any case $K(\cdot, 0)x \in C_{(1-\alpha-\beta)\vee 0}(]0, T], E)$, which implies (Lemma 3.4 (ii)) $g \in C_{(1-\alpha-\beta)\vee 0}(]0, T], E)$. Hence $Tg \in C^{((\alpha+\beta)\wedge 1)-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, (\alpha + \beta)\wedge 1[$ by Lemma 7.3 (ii), and in particular $Tg \in C^{((\alpha+\beta)\wedge 1)-\beta}([0, T], D_{A(0)}(\beta))$. This implies $u \in C_0(]0, T], D_{A(0)}(\beta))$. Suppose conversely $u \in C_0(]0, T], D_{A(0)}(\beta))$; then by definition

$$\left\| \frac{\exp[\xi A(0)] - 1}{\xi^\beta} u(t) \right\|_E \leqslant \|u\|_{C_0(]0,T], D_{A(0)}(\beta))}\,, \qquad \forall t \in]0, T],\ \forall \xi > 0\,;$$

letting $t \to 0^+$, since $u \in C([0, T], E)$ we get

$$\left\| \frac{\exp[\xi A(0)] - 1}{\xi^\beta} x \right\|_E \leqslant \|u\|_{C_0(]0,T], D_{A(0)}(\beta))}\,, \qquad \forall \xi > 0\,,$$

that is $x \in D_{A(0)}(\beta, \infty)$.

(iii) By Lemma 7.2 (ii), $t \to \exp[tA(0)]x \in C^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, \beta[$; as in (ii) we get $Tg \in C^{((\alpha+\beta)\wedge 1)-\theta}([0, T], D_{A(0)}(\theta))$, $\forall \theta \in]0, (\alpha + \beta)\wedge 1[$, and (iii) follows.

(iv) Suppose $x \in D_{A(0)}(\beta)$. Then Lemma 7.2 (v) gives $t \to \exp[tA(0)]x \in C([0, T], D_{A(0)}(\beta))$ while, as above, $Tg \in C^{((\alpha+\beta)\wedge 1)-\theta}([0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, (\alpha+\beta)\wedge 1[$. In particular, $Tg \in C^{((\alpha+\beta)\wedge 1)-\beta}([0, T], D_{A(0)}(\beta))$ and consequently $u \in C([0, T], D_{A(0)}(\beta))$. Conversely if $u \in C([0, T], D_{A(0)}(\beta))$, then obviously $x = u(0) \in D_{A(0)}(\beta)$.

(v) By Lemma 7.2 (iv) we have $t \to \exp[tA(0)]x \in h^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, \beta[$ and, as in (ii), $Tg \in C^{((\alpha+\beta)\wedge 1)-\theta}([0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, (\alpha+\beta)\wedge 1[$; since $\beta < (\alpha+\beta)\wedge 1$, we get in particular $Tg \in h^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, \beta[$ and hence $u \in h^{\beta-\theta}([0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, \beta[$. ///

We finish this section with an «a priori» estimate for the classical solution of Problem (P), which is of interest in the study of the quasi-linear version of (P).

THEOREM 7.5. – *Under Hypotheses I, II let u be a classical solution of* (P) *with* $x \in \overline{D(A(0))}$ *and* $f \in C([0, T], E)$. *Then we have:*

(i) *If* $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1[$, *then*

$$\|u(t)\|_\theta \leqslant C\left[\frac{1}{t^{\theta-\beta}}\|x\|_\beta + t^{1-\theta}\|f\|_{C([0,T],E)}\right], \quad \forall\theta \in [\beta, 1[, \ \forall t \in]0, T].$$

(ii) *If* $x \in D_{A(0)}(\beta, \infty)$ *and* $f \in C^\sigma([0, T], E)$, $\beta, \sigma \in]0, 1[$, *then*

$$\|A(t)u(t)\|_E \leqslant C\left[\frac{1}{t^{1-\beta}}\|x\|_\beta + \|f\|_{C^\sigma([0,T],E)}\right], \quad \forall t \in]0, T].$$

PROOF. – (i) Since, by Theorem 6.7, u is also a strong solution of (P), u is given by Formula (F), i.e.

$$u(t) = \exp[tA(0)]x + Tg_1(t) - Tg_2(t), \quad t \in [0, T],$$

where $g_1 = (1+K)^{-1}f$ and $g_2 = (1+K)^{-1}(K(\cdot, 0)x)$.

By Lemma 7.1 (ii) we have

$$\|\exp[tA(0)]x\|_\theta \leqslant \frac{C}{t^{\theta-\beta}}\|x\|_\beta .$$

Next, by Lemma 3.2 (iii)-(v) it follows that $K(\cdot, 0)x \in C_\lambda(]0, T], E)$, $\lambda = (1 - \alpha - \beta)\vee 0$, and

$$\|K(t, 0)x\|_E \leqslant \frac{C}{t^\lambda}\|x\|_\beta ,$$

hence by Lemma 3.4 (ii) $g_2 \in C_\lambda(]0, T], E)$ and

$$\|g_2\|_{C_\lambda(]0,T],E)} \leqslant C\|K(\cdot, 0)x\|_{C_\lambda(]0,T],E)} \leqslant C\|x\|_\beta ,$$

and finally, by Lemma 7.3 (ii), $Tg_2 \in C^{1-\theta}(]0, T], D_{A(0)}(\theta))$, $\forall\theta \in]0, 1[$ and

$$\|Tg_2(t)\|_\theta \leqslant Ct^{1-\lambda-\theta}\|g_2\|_{C_\lambda(]0,T],E)} \leqslant Ct^{1-\lambda-\theta}\|x\|_\beta .$$

Similarly, since $f \in C([0, T], E)$, we get $\forall\theta \in]0, 1[$

$$\|Tg_1(t)\|_\theta \leqslant Ct^{1-\theta}\|f\|_{C([0,T],E)} .$$

Hence

$$\|u(t)\|_\theta \leqslant \frac{C}{t^{\theta-\beta}}\|x\|_\beta + Ct^{1-\lambda-\theta}\|x\|_\beta + Ct^{1-\theta}\|f\|_{C([0,T],E)} , \qquad \forall\theta \in [\beta, 1[;$$

since $\lambda = (1 - \alpha - \beta)\vee 0$, we have $1 - \lambda = (\alpha + \beta)\wedge 1$ and

$$\|u(t)\|_\theta \leqslant C\left\{\frac{1}{t^{\theta-\beta}}\|x\|_\beta + t^{1-\theta}\|f\|_{C([0,T],E)}\right\}, \qquad \forall\theta \in [\beta, 1[,$$

which proves (i).

(ii) We proceed as in the proof of Theorem 4.4. Fix $\varepsilon \in]0, T[$ and $t \in]\varepsilon, T[$. Setting $v(s) = \exp[(t-s)A(t)]u(s)$, $s \in [\varepsilon, t]$, we easily deduce

$$\begin{aligned} A(t)u(t) &= A(t)\exp[(t-\varepsilon)A(t)]u(\varepsilon) + \\ &+ \int_\varepsilon^t A(t)\exp[(t-s)A(t)]\big(1 - A(t)A(s)^{-1}\big)A(s)u(s)\,ds + \\ &+ \int_\varepsilon^t A(t)\exp[(t-s)A(t)]\big(f(s) - f(t)\big)\,ds + \big(\exp[(t-\varepsilon)A(t)] - 1\big)f(t) , \qquad \forall t \in [\varepsilon, T] . \end{aligned}$$

Hence

$$\|A(t)u(t)\|_E \leqslant \frac{C}{(t-\varepsilon)^{1-\beta}}\|u(\varepsilon)\|_\beta + C\|f\|_{C^\sigma([0,T],E)} + \int_\varepsilon^t \frac{C}{(t-s)^{1-\alpha}}\|A(s)u(s)\|_E\,ds , \qquad \forall t \in [\varepsilon, T] ,$$

which implies (see [3])

$$\|A(t)u(t)\|_E \leqslant C\left\{\frac{1}{(t-\varepsilon)^{1-\beta}}\|u(\varepsilon)\|_\beta + \|f\|_{C^\sigma([0,T],E)}\right\}, \qquad \forall t \in [\varepsilon, T] .$$

By (i) we check

$$\|A(t)u(t)\|_E \leqslant C\left\{\frac{1}{(t-\varepsilon)^{1-\beta}}[\|x\|_\beta + \varepsilon^{1-\beta}\|f\|_{C([0,T],E)}] + \|f\|_{C^\sigma([0,T],E)}\right\}, \qquad \forall t \in [\varepsilon, T] ,$$

and the conclusion follows when $\varepsilon \to 0^+$. ///

8. – Examples.

In the next examples we want to show how Hypotheses I and II of Section 1 can be easily checked in a classical non-autonomous boundary-initial value problem.

In this section, unless otherwise specified, all functions are assumed to be complex-valued.

First example.

Set $E = C([0,1])$, $\|u\|_E = \sup\limits_{x\in[0,1]} |u(x)|$, and define $\forall t \in [0,T]$

$$\begin{cases} D(A(t)) = \{u \in C^2([0,1]) \colon \alpha_0 u(0) - \beta_0 u'(0) = \alpha_1 u(1) + \beta_1 u'(1) = 0\} \\ A(t)u = a(\cdot,t)u'' + b(\cdot,t)u' + c(\cdot,t)u \end{cases} \tag{8.1}$$

where

$$\alpha_i,\ \beta_i \in [0,\infty[,\ \alpha_i + \beta_i > 0\,, \qquad i = 0,1 \tag{8.2}$$

and

$$a,\ b,\ c \in C([0,1]\times[0,T], \boldsymbol{R})\,, \quad \inf_{[0,1]\times[0,T]} a(x,t) > 0\,. \tag{8.3}$$

Obviously $D(A(t))$ does not depend on t; it is not dense in E if and only if β_0 or β_1 is 0. We have the following result:

PROPOSITION 8.1. – *Let* $\{A(t)\}_{t\in[0,T]}$ *be defined by* (8.1) *and suppose* (8.2), (8.3) *hold. Then there exist* ω^0, $M > 0$ *such that*

(i) $\sigma(A(t)) \subseteq\,]-\infty, \omega^0]$, $\forall t \in [0,T]$;

(ii) *If* $\operatorname{Re}\lambda > \omega^0$ *we have*

$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \le \frac{M}{|\lambda - \omega^0|}\,, \qquad \forall t \in [0,T]\,. \tag{8.4}$$

PROOF. – For each $\lambda \in \boldsymbol{C}$ and $t \in [0,T]$ the problem

$$\begin{cases} \lambda u - a(\cdot,t)u'' - b(\cdot,t)u' - c(\cdot,t)u = f \in E \\ \alpha_0 u(0) - \beta_0 u'(0) = 0 \\ \alpha_1 u(1) + \beta_1 u'(1) = 0 \end{cases} \tag{8.5}$$

is equivalent to the following one:

$$(8.6)\qquad \begin{cases} \lambda\cdot\gamma(\cdot,t)u - \big(\psi(\cdot,t)u'\big)' - \varphi(\cdot,t)u = F(\cdot,t)\\ \alpha_0 u(0) - \beta_0 u'(0) = 0\\ \alpha_1 u(1) + \beta_1 u'(1) = 0\end{cases}$$

where

$$\psi(x,t) = \exp\int_0^x b(\xi,t)\big(a(\xi,t)\big)^{-1}\,d\xi\,,\qquad \gamma(x,t) = \big(a(x,t)\big)^{-1}\psi(x,t)\,,$$

$$\varphi(x,t) = c(x,t)\big(a(x,t)\big)^{-1}\psi(x,t)\,,\qquad F(x,t) = f(x)\big(a(x,t)\big)^{-1}\psi(x,t)\,.$$

We note that γ, ψ are strictly positive in $[0,1]\times[0,T]$, and that problem (8.6) is a classical problem of Sturm-Liouville type. So, defining

$$\omega = \max_{[0,1]\times[0,T]} |c(x,t)|$$

it is well-known (see e.g. HARTMAN [15]) that problem (8.6) has countably many eigenvalues, all being real and $\leqslant\omega$; moreover for any other number $\lambda\in\boldsymbol{C}$ the problem is uniquely solvable in $C^2([0,1])$ for each $F(\cdot,t)\in E$, i.e. for each $f\in E$. This proves (i).

To prove (ii), fix $x^0\in[0,1]$, $\varrho>0$ and $\sigma>1$, and take a real function $\theta\in C^\infty(\boldsymbol{R})$, such that $\theta\equiv 1$ on $I_\varrho = [x^0-\varrho, x^0+\varrho]$, $\theta\equiv 0$ outside $I_{\sigma\varrho}$, $0\leqslant\theta\leqslant 1$ and $\|\theta'\|_E\leqslant 2/(\sigma-1)\varrho$. From (8.6), multiplying by $\theta^2\bar u$ and integrating over $]0,1[$, we get

$$(8.7)\qquad \int_0^1(\operatorname{Re}\lambda\cdot\gamma-\varphi)\theta^2|u|^2\,dx + \int_0^1\psi|u'|^2\theta^2\,dx - [\psi u'\bar u\theta^2]_0^1 = \int_0^1\operatorname{Re}(F\bar u)\theta^2\,dx - \int_0^1\operatorname{Re}(u'\bar u)\psi 2\theta\theta'\,dx$$

and

$$(8.8)\qquad \int_0^1\operatorname{Im}\lambda\cdot\gamma\theta^2|u|^2\,dx = \int_0^1\operatorname{Im}(F\bar u)\theta^2\,dx - \int_0^1\operatorname{Im}(u'\bar u)\psi 2\theta\theta'\,dx\,.$$

Set $B_\varrho = I_\varrho\cap\,]0,1[$. As $\operatorname{Re}\lambda\cdot\gamma-\varphi\geqslant C(\operatorname{Re}\lambda-\omega)$ and $-[\psi u'\bar u\theta^2]_0^1\geqslant 0$, by (8.7) we deduce

$$(8.9)\qquad \int_0^1|u'|^2\theta^2\,dx\leqslant C\left\{\int_{B_{\sigma\varrho}}|Fu|\,dx + \frac{1}{(\sigma-1)^2\varrho^2}\int_{B_{\sigma\varrho}}|u|^2\,dx\right\}\qquad \text{if } \operatorname{Re}\lambda>\omega\,.$$

In addition if $|\mathrm{Im}\,\lambda| < \mathrm{Re}\,\lambda - \omega$ we have, again by (8.7),

$$|\lambda-\omega|\int_{B_\varrho}|u|^2\,dx+\int_{B_\varrho}|u'|^2\,dx\leqslant C\left\{\int_{B_{\sigma\varrho}}|Fu|\,dx+\frac{1}{(\sigma-1)^2\varrho^2}\int_{B_{\sigma\varrho}}|u|^2\,dx\right\}$$

while if $0<\mathrm{Re}\,\lambda-\omega\leqslant|\mathrm{Im}\,\lambda|$ by (8.8) and (8.9) we get

$$|\lambda-\omega|\int_{B_\varrho}|u|^2\,dx\leqslant C\int_{B_{\sigma\varrho}}|Fu|\,dx+\frac{C}{(\sigma-1)^2\varrho^2}\int_{B_{\sigma\varrho}}|u|^2\,dx\,; \tag{8.10}$$

thus we conclude that

$$|\lambda-\omega|\int_{B_\varrho}|u|^2\,dx+\int_{B_\varrho}|u'|^2\,dx\leqslant C\left\{\int_{B_{\sigma\varrho}}|Fu|\,dx+\frac{1}{(\sigma-1)^2\varrho^2}\int_{B_{\sigma\varrho}}|u|^2\,dx\right\}\quad \text{if } \mathrm{Re}\,\lambda>\omega\,. \tag{8.11}$$

Now for each $\lambda\in\boldsymbol{C}$ such that $\mathrm{Re}\,\lambda>\omega+1$, set $\varrho=|\lambda-\omega|^{-\frac{1}{2}}$, and choose σ large enough so that $C\sigma/(\sigma-1)^2<1/8$ in (8.10). Then if x^0 is a maximum point for $|u(\cdot,t)|$ in $[0,1]$, we can rewrite (8.11) as follows:

$$\frac{1}{\varrho^2}\int_{B_\varrho}|u|^2\,dx+\int_{B_\varrho}|u'|^2\,dx\leqslant 2C\sigma\varrho\|F(\cdot,t)\|_E\|u(\cdot,t\|_E+\frac{2C\sigma}{(\sigma-1)^2\varrho}\|u(\cdot,t)\|_E^2\,. \tag{8.12}$$

On the other hand it is easy to verify that

$$\|u(\cdot,t)\|_E^2=|u(x^0,t)|^2\leqslant 2\left|\int_y^{x^0}|u'|^2\,ds\right||x^0-y|+2|u(y,t)|^2\,,\qquad \forall y\in B_\varrho\,,$$

which implies, by integrating over B_ϱ, and noting that $\varrho<1$ and $\varrho\leqslant\mathrm{meas}\,(B_\varrho)\leqslant 2\varrho$,

$$\|u(\cdot,t)\|_E^2\leqslant 2\varrho\int_{B_\varrho}|u'|^2\,dx+\frac{2}{\varrho}\int_{B_\varrho}|u|^2\,dx\,.$$

This inequality, together with (8.12), gives (recalling the choice of σ):

$$\|u(\cdot,t)\|_E^2\leqslant 4C\sigma\varrho^2\|F(\cdot,t)\|_E\|u(\cdot,t)\|_E+\tfrac{1}{2}\|u(\cdot,t)\|_E^2\,,$$

or

$$\|u(\cdot,t)\|_E\leqslant 8C\sigma\varrho^2\|F(\cdot,t)\|_E\leqslant\frac{C}{|\lambda-\omega|}\|f\|_E\,,$$

and setting $\omega^0=\omega+1$, (ii) follows. ///

REMARK 8.2. – By (8.5), (8.6) and the well-known inequality

$$\|u'\|_E^2 \leqslant 4\|u''\|_E \cdot \|u\|_E\,, \qquad \forall u \in C^2([0,1])$$

we easily deduce that

$$\|u''\|_E \leqslant C\|f\|_E$$

and consequently if $\operatorname{Re}\lambda > \omega^0$ the solution of problem (8.5) satisfies

$$|\lambda - \omega^0|\|u\|_E + |\lambda - \omega^0|^{\frac{1}{2}}\|u'\|_E + \|u''\|_E \leqslant C\|f\|_E\,. \tag{8.13}$$

Let us suppose further that the functions a, b, c are Hölder continuous in t uniformly in x, i.e. there exist $\alpha \in]0,1[$ and $B > 0$ such that

$$|a(x,t) - a(x,\tau)| + |b(x,t) - b(x,\tau)| + |c(x,t) - c(x,\tau)| \leqslant B|t-\tau|^\alpha\,, \qquad \forall x \in [0,1],\ \forall t, \tau \in [0,T]\,. \tag{8.14}$$

Then we have the following

PROPOSITION 8.3. – *Let* $\{A(t)\}_{t\in[0,T]}$ *be defined by* (8.1) *and suppose* (8.2), (8.3), (8.14) *hold. Then, setting* $\omega' = \omega^0 + 1$, *there exists* $K > 0$ *such that*

$$\|1 - (\omega' - A(t))R(\omega', A(\tau))\|_{\mathcal{L}(E)} \leqslant K|t-\tau|^\alpha\,, \qquad \forall t,\ \tau \in [0,T]\,.$$

PROOF. – Let $f \in E$ and set $u = u(\cdot,\tau) = R(\omega', A(\tau))f$. By Proposition 8.1 and (8.13) we have $u \in D(A(0)) \subseteq C^2([0,1])$ and

$$\|u\|_E + \|u'\|_E + \|u''\|_E \leqslant C\|f\|_E\,.$$

Hence

$$\|\big(1 - (\omega' - A(t))R(\omega', A(\tau))\big)f\|_E = \|(A(t) - A(\tau))R(\omega', A(\tau))f\|_E \leqslant$$
$$\leqslant B|t-\tau|^\alpha[\|u\|_E + \|u'\|_E + \|u''\|_E] \leqslant C\cdot B|t-\tau|^\alpha\|f\|_E\,,$$

and the proof is complete. ///

By Propositions 8.1 and 8.3, taking into account Remark 1.2, we conclude that the operators $\{A(t) - \omega'\}_{t\in[0,T]}$, with $A(t)$ defined by (8.1), satisfy Hypotheses I and II of Section 1. Hence all results of the preceding sections can be applied to the problem

$$\begin{cases} u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u + \lambda u = f(x,t)\,, & (x,t)\in[0,1]\times[0,T] \\ \alpha_0 u(0,t) - \beta_0 u_x(0,t) = 0\,, & t\in[0,T] \\ \alpha_1 u(1,t) + \beta_1 u_x(1,t) = 0\,, & t\in[0,T] \\ u(x,0) = \varphi(x)\,, & x\in[0,1] \end{cases}$$

where $\lambda\in\boldsymbol{C}$, $f\in C([0,1]\times[0,T])$, $\varphi\in C([0,1])$.

We note that in the case of Dirichlet conditions, i.e. $\beta_0=\beta_1=0$, it is known (see DA PRATO-GRISVARD [12], LUNARDI [23]) that

$$\left.\begin{aligned} D_{A(0)}(\theta,\infty) &= \{u\in C^{2\theta}([0,1]): u(0)=u(1)=0\} \\ D_{A(0)}(\theta) &= \{u\in h^{2\theta}([0,1]): u(0)=u(1)=0\} \end{aligned}\right\}\,,\qquad \forall\theta\in\,]0,1[\,-\{\tfrac12\}\,.$$

Second example.

Let Ω be a bounded open set of $\boldsymbol{R}^n$, $n\geqslant 2$, with boundary of class C^2. Consider the differential operator

$$A(x,t,D) = \sum_{i,j=1}^{n} a_{ij}(x,t)D_iD_j + \sum_{i=1}^{n} b_i(x,t)D_i + c(x,t)I\,,\qquad (x,t)\in\bar\Omega\times[0,T]\,,$$

where $D_i=\partial/\partial x_i$, under the following assumptions:

(A.1) (uniform ellipticity). There exists $E>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqslant E|\xi|^2\,,\qquad \forall(x,t)\in\bar\Omega\times[0,T],\ \forall\xi\in\boldsymbol{R}^n;$$

(A.2) For each $t\in[0,T]$ the real-valued (for the sake of simplicity) functions a_{ij}, b_i, c are in $C(\bar\Omega)$ with bounds independent on t.

Let us recall the definition of Sobolev spaces. If $\alpha\in\boldsymbol{N}^n$, set $|\alpha|=\sum_{i=1}^{n}\alpha_i$, and, $D^\alpha u = D_1^{\alpha_1}D_2^{\alpha_2}\dots D_n^{\alpha_n} = \partial^{|\alpha|}u/(\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n})$; if $k\in\boldsymbol{N}$, set $D^k u = \{D^\alpha u\}_{|\alpha|=k}$: thus $D^k u$ is a vector with $\binom{n+k-1}{k}$ components. Let $H^k(\Omega)$ be the Banach space of functions $u\in L^2(\Omega)$ such that their distributional derivatives $D^\alpha u$ are in $L^2(\Omega)$ for each $\alpha\in\boldsymbol{N}^n$ with $|\alpha|\leqslant k$, with norm

$$\|u\|_{H^k(\Omega)} = \left[\|u\|^2_{L^2(\Omega)} + \sum_{|\alpha|\leqslant k}\|D^\alpha u\|^2_{L^2(\Omega)}\right]^{\frac12}.$$

We denote by $H_0^k(\Omega)$ the closure of $C_0^k(\Omega)$ (the functions of $C^k(\Omega)$ with compact support) in the norm of $H^k(\Omega)$.

Now we give the definition of Morrey spaces. For each $x^0 \in \bar{\Omega}$ and $\sigma > 0$ set

$$B(x^0, \sigma) = \{x \in \boldsymbol{R}^n\colon |x - x^0| < \sigma\}\,, \qquad \Omega(x^0, \sigma) = \Omega \cap B(x^0, \sigma)\,, \qquad d = \operatorname{diam}(\Omega)\,.$$

DEFINITION 8.4. – We say that $u \in L^{2,\mu}(\Omega)$, $\mu \in [0, n]$, if the quantity

$$\sup\left\{\varrho^{-\mu}\int\limits_{\Omega(x^0,\varrho)}|u(x)|^2\,dx,\ x^0\in\bar{\Omega},\ \varrho\in\,]0, d]\right\}$$

is finite.

We say that $u \in H^{k,\mu}(\Omega)$, $\mu \in [0, n]$, $k \in \boldsymbol{N}$, if $u \in H^k(\Omega)$ and $D^\alpha u \in L^{2,\mu}(\Omega)$ for each $\alpha \in \boldsymbol{N}^n$ with $|\alpha| = k$.

The spaces $L^{2,\mu}(\Omega)$ and $H^{k,\mu}(\Omega)$ are Banach spaces with norms

$$\|u\|_{L^{2,\mu}(\Omega)} = \left[\sup\left\{\varrho^{-\mu}\int\limits_{\Omega(x^0,\varrho)}|u(x)|^2\,dx,\ x^0\in\bar{\Omega},\ \varrho\in\,]0, d]\right\}\right]^{\frac{1}{2}},$$

$$\|u\|_{H^{k,\mu}(\Omega)} = \left[\|u\|^2_{H^k(\Omega)} + \sum_{|\alpha|=k}\|D^\alpha u\|^2_{L^{2,\mu}(\Omega)}\right]^{\frac{1}{2}}.$$

We shall write simply $\|D^k u\|_{L^2(\Omega)}$ and $\|D^k u\|_{L^{2,\mu}(\Omega)}$ for

$$\left[\sum_{|\alpha|=k}\|D^\alpha u\|^2_{L^2(\Omega)}\right]^{\frac{1}{2}} \qquad \text{and} \qquad \left[\sum_{|\alpha|=k}\|D^\alpha u\|^2_{L^{2,\mu}(\Omega)}\right]^{\frac{1}{2}}$$

respectively.

The $L^{2,\mu}$-spaces have been introduced by MORREY [25]. More properties of these spaces, with several inclusion theorems, are exposed, for example, in CAMPANATO [6]. Here we only need the following result, which however is not optimal at all:

LEMMA 8.5. – *If* $u \in H^{2,\mu}(\Omega)$ *then* $u, D_i u \in L^{2,\mu}(\Omega)$, $i = 1, \ldots, n$, *and*

$$\|u\|_{L^{2,\mu}(\Omega)} + \|Du\|_{L^{2,\mu}(\Omega)} \leqslant C\|u\|_{H^{2,\mu}(\Omega)}\,.$$

PROOF. – See CAMPANATO [6], [7]. ///

Fix now $\mu \in\,]0, n[$, set $E = L^{2,\mu}(\Omega)$ and define for each $t \in [0, T]$:

$$\left\{\begin{array}{l} D\big(A(t)\big) = \{u \in H^2(\Omega) \cap H^1_0(\Omega)\colon A(\cdot, t, D)u \in L^{2,\mu}(\Omega)\}\,, \\ A(t)u = A(\cdot, t, D)u\,. \end{array}\right. \tag{8.15}$$

According with well-known results of CAMPANATO $\big([9], [7]\big)$ we shall verify now that under assumptions (A.1) and (A.2) there exists $\omega^0 > 0$ such that the operators $\{A(t) - \omega^0\}_{t\in[0,T]}$, with $A(t)$ defined by (8.15), satisfy Hypotheses I, II of Section 1.

First of all consider an operator $A(x, D)$ independent on t and satisfying assumptions (A.1) and (A.2). Fix $\lambda \in \boldsymbol{C}$ and consider the stationary problem

$$\begin{cases} \lambda u - A(\cdot, D)u = f \in L^2(\Omega), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{8.16}$$

It is well-known (see AGMON [2]; see also Theorems 3.8.1-3.8.2 and Lemma 5.3.3 of TANABE [35]) that problem (8.16) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ for each $f \in L^2(\Omega)$, provided λ belongs to a suitable sector containing a half-plane $\{z \in \boldsymbol{C}\colon \operatorname{Re} z > \omega'\}$; moreover the following estimate holds

$$|\lambda - \omega'| \|u\|_{L^2(\Omega)} + |\lambda - \omega'|^{\frac{1}{2}} \|Du\|_{L^2(\Omega)} + \|D^2 u\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}, \tag{8.17}$$

and the constant C does not depend on λ.

With the methods of CAMPANATO [9], [6], [7] the following key inequality can be proved:

THEOREM 8.6. – *There exists* $\omega^0 > \omega' \wedge \max\limits_{[0,1]\times[0,T]} |c(x, t)|$ *with the following property: if* $\mu \in]0, n[$ *and* $f \in L^{2,\mu}(\Omega)$, *then the unique solution of problem* (8.16) *with* $\operatorname{Re} \lambda > \omega^0$ *is in* $H^{2,\mu}(\Omega)$ *and satisfies*

$$\|D^2 u\|_{L^{2,\mu}(\Omega)} \leqslant C\{\|u\|_{H^2(\Omega)} + \|f\|_{L^{2,\mu}(\Omega)}\}, \tag{8.18}$$

and the constant C *does not depend on* λ. ///

We omit the proof: we only remark that the same estimate has been obtained by CAMPANATO [7] and MORSELLI [26], by means of a series of lemmata yielding (8.18) with a constant which possibly depends on λ. In order to avoid any dependence on λ, it is necessary to prove again each lemma, showing that in fact there is no dependence at all. This has been done by CAMPANATO [9] in a different (but similar) situation.

As a consequence of Theorem 8.6 we have:

PROPOSITION 8.7. – *For each* $t \in [0, T]$ *let* $A(t)$ *be the operator defined by* (8.15). *Under assumptions* (A.1) *and* (A.2) *we have*:

(i) $D(A(t)) = D(A(0)) = H^{2,\mu}(\Omega) \cap H_0^1(\Omega)$, $\forall t \in [0, T]$.

(ii) $\varrho(A(t)) \supseteq \{\lambda \in \boldsymbol{C}\colon \operatorname{Re} \lambda > \omega^0\}$, $\forall t \in [0, T]$.

(iii) *If* $\operatorname{Re} \lambda > \omega^0$ *then*

$$\|R(\lambda, A(t))f\|_{L^{2,\mu}(\Omega)} \leqslant \frac{C}{|\lambda - \omega^0|} \|f\|_{L^{2,\mu}(\Omega)}, \quad \forall f \in L^{2,\mu}(\Omega). \tag{8.19}$$

and the constant C *does not depend on* λ.

PROOF. – Fix λ such that $\operatorname{Re}\lambda > \omega^0$ (ω^0 is defined in Theorem 8.6). Let $f \in L^{2,\mu}(\Omega)$ and let $u = u(\cdot, t)$ be the unique solution of the problem (analogous to (8.16))

$$\begin{cases} \lambda u - A(\cdot, t, D)u = f\,, & x \in \Omega \\ u = 0\,, & x \in \partial\Omega\,. \end{cases}$$

By Theorem 8.6, $u \in H^{2,\mu}(\Omega) \cap H^1_0(\Omega)$; this proves (i) and (ii).

To prove (iii) we observe that since $\omega^0 > \max\limits_{[0,1]\times[0,T]} |c(x,t)|$, from the equation

$$\big(\lambda - c(x,t)\big)u = \sum_{i,j=1}^{n} a_{ij}(x,t) D_i D_j u + \sum_{i=1}^{n} b_i(x,t)\, D_i u + f$$

it can be easily deduced that

$$|\lambda - \omega^0| \|u\|_{L^{2,\mu}(\Omega)} \leqslant C\{\|D^2 u\|_{L^{2,\mu}(\Omega)} + \|Du\|_{L^{2,\mu}(\Omega)} + \|f\|_{L^{2,\mu}(\Omega)}\}\,;$$

thus, by Lemma 8.5 and Theorem 8.6 we get

$$|\lambda - \omega^0| \|u\|_{L^{2,\mu}(\Omega)} \leqslant C\{\|u\|_{H^2(\Omega)} + \|f\|_{L^{2,\mu}(\Omega)}\}\,,$$

and finally by (8.17) we obtain

$$|\lambda - \omega^0| \|u\|_{L^{2,\mu}(\Omega)} \leqslant C\{\|f\|_{L^2(\Omega)} + \|f\|_{L^{2,\mu}(\Omega)}\} \leqslant C\|f\|_{L^{2,\mu}(\Omega)}\,,$$

which proves (iii). ///

Let us assume further the following:

(A.3) The functions a_{ij}, b_i, c are Hölder continuous in t uniformly in x, i.e. there exist $\alpha \in]0,1[$ and $L > 0$ such that

$$\sum_{i,j=1}^{n} |a_{ij}(x,t) - a_{ij}(x,\tau)| + \sum_{i=1}^{n} |b_i(x,t) - b_i(x,\tau)| + |c(x,t) - c(x,\tau)| \leqslant L|t-\tau|^\alpha\,, \quad \forall x \in \bar{\Omega},\ \forall t, \tau \in [0,T]\,.$$

PROPOSITION 8.8. – *Let* $A(t)$, $t \in [0,T]$, *be defined by* (8.15), *and set* $\bar\omega = \omega^0 + 1$. *Under assumptions* (A.1), (A.2) *and* (A.3) *we have*

$$\|[1 - (\bar\omega - A(t))R(\bar\omega, A(\tau))]f\|_{L^{2,\mu}(\Omega)} \leqslant C|t-\tau|^\alpha \|f\|_{L^{2,\mu}(\Omega)}\,, \qquad \forall f \in L^{2,\mu}(\Omega)\,.$$

PROOF. – Set $u = u(\cdot,\tau) = R(\bar\omega, A(\tau))f$. Then by (8.17) we have

$$\|u\|_{L^{2,\mu}(\Omega)} + \|Du\|_{L^{2,\mu}(\Omega)} + \|D^2 u\|_{L^{2,\mu}(\Omega)} \leqslant C\|f\|_{L^{2,\mu}(\Omega)}\,.$$

Hence

$$\|[1-(\bar{\omega}-A(t))R(\bar{\omega},A(\tau))]f\|_{L^{2,\mu}(\Omega)}=\|(A(\tau)-A(t))u\|_{L^{2,\mu}(\Omega)}\leqslant$$
$$\leqslant L|t-\tau|^{\alpha}[\|u\|_{L^{2,\mu}(\Omega)}+\|Du\|_{L^{2,\mu}(\Omega)}+\|D^2u\|_{L^{2,\mu}(\Omega)}]\leqslant L|t-\tau|^{\alpha}\|f\|_{L^{2,\mu}(\Omega)} \qquad ///$$

and the result follows. ///

By Propositions 8.7 and 8.8, taking into account Remark 1.2, we see that the operators $\{A(t)-\bar{\omega}\}_{t\in[0,T]}$, with $A(t)$ defined by (8.15), satisfy Hypotheses I and II of Section 1. Hence all results of the preceding sections are applicable to the problem

$$\begin{cases} u_t(x,t)-A(x,t,D)u(x,t)+\lambda u(x,t)=f(x,t)\,, & (x,t)\in\bar{\Omega}\times[0,T] \\ u(x,t)=0\,, & (x,t)\in\partial\Omega\times[0,T] \\ u(x,0)=\varphi(x)\,, & x\in\Omega\,, \end{cases}$$

where $\lambda\in\boldsymbol{C}$, $f\in C([0,T],L^{2,\mu}(\Omega))$, $\varphi\in L^{2,\mu}(\Omega)$.

Acknowledgements. – We wish to thank E. SINESTRARI (Università di Roma I) and A. LUNARDI (Università di Pisa) for their useful suggestions and helpful advice.

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