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Hölder Classes with Boundary Conditions as Interpolation Spaces

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§ 0. Introduction

This paper is concerned with the characterization of certain real interpolation spaces between the domain of an elliptic differential operator A, with general boundary conditions, and the Banach space E of continuous functions in which the domain is imbedded.

The interpolation spaces considered here are the classes $(D_A, E)_{\alpha,\infty}$ introduced by Lions (see Lions-Peetre [14]) and the "continuous interpolation spaces" $(D_A, E)_\alpha$ defined by Da Prato-Grisvard [9]; however, following Grisvard [11], we denote such spaces respectively by $D_A(\theta, \infty)$ and $D_A(\theta)$ (where $\theta = 1-\alpha$), and introduce them by means of an abstract characterization (see Definition 2.1 below) which is valid under suitable hypotheses concerning the behaviour of the resolvent operator $(\lambda - A)^{-1}$.

Such assumptions are satisfied when, in particular, A is the infinitesimal generator of an analytic semigroup. In this situation, the spaces $D_A(\theta, \infty)$ and $D_A(\theta)$ are of great importance in the theory of abstract evolution equations, because of their "maximal regularity" property. Maximal regularity means the following: if f is continuous with values in a Banach space Y, then the evolution problem

$$u'(t) - Au(t) = f(t), \quad t \in [0, T]; \quad u(0) = 0$$

has a unique C^1-solution u such that u' and Au are continuous with values in Y. This property is not true in a general Banach space Y (see Baillon [8]), but it holds when $Y = D_A(\theta)$, where A is the infinitesimal generator of an analytic semigroup in some other Banach space E. Note that we cannot replace $D_A(\theta)$ by $D_A(\theta, \infty)$ (see Da Prato-Grisvard [9]); however a similar property holds for $D_A(\theta, \infty)$ (with A as before), i.e. if f is continuous with values in E and bounded with values in $D_A(\theta, \infty)$, then the same is true for u' and Au. For a proof of these facts see Sinestrari [18].

Thus when A generates an analytic semigroup the spaces $D_A(\theta, \infty)$ and $D_A(\theta)$ have been estensively used in the theory of abstract parabolic equations, in order to obtain existence and sharp regularity results (see, among others [1, 2, 4, 9, 11, 13, 16, 17, 18]. On the other hand in concrete situations the abstract regularity results have to be interpreted, and this in turn requires the characterization of these spaces in such concrete cases. Now, when $E=L^p(\Omega)$ and $A=A(\cdot, D)$ is an elliptic operator of order $2m$, whose domain is determined by a set of m general boundary differential operators $\{B_j(\cdot, D)\}_{1\leqq j\leqq m}$ satisfying the usual assumptions (Agmon [6]), the spaces $D_A(\theta, \infty)$ and $D_A(\theta)$ are known to be the functions f belonging to the Besov-Nikolskii spaces $B^{2m\theta}_{p,\infty}(\Omega)$ and $h^{2m\theta}_{p,\infty}(\Omega)$ which satisfy $B_j(\cdot, D)f=0$ on $\partial\Omega$ whenever it makes sense ([11, 9]). Here we treat instead the case $E=C(\bar{\Omega})$, and we obtain as $D_A(\theta, \infty)$ and $D_A(\theta)$ the functions of the Hölder and "little Hölder" classes $C^{2m\theta}(\bar{\Omega})$ and $h^{2m\theta}(\bar{\Omega})$ which satisfy, as before, the boundary conditions whenever they are meaningful.

Let us conclude with the description of the subject of the next sections. Section 1 is devoted to preliminaries; in Sect. 2 we state our main result, which is proved in Sects. 3 and 4; finally Sect. 5 contains some remarks and generalizations.

§ 1. Notations, Assumptions and Preliminary Results

If $\beta, \gamma\in\mathbb{N}^n$ and $z\in\mathbb{C}^n$, $n\geqq 1$ we set as usual

$$|\beta|:=\sum_{i=1}^n \beta_i, \quad \beta!:=\prod_{i=1}^n \beta_i!, \quad \binom{\beta}{\gamma}:=\prod_{i=1}^n \binom{\beta_i}{\gamma_i}, \quad z^\beta:=\prod_{i=1}^n z_i^{\beta_i}$$

whereas D_β stands for $\dfrac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}$.

Let Ω be an open set of $\mathbb{R}^n$; we list now some Banach spaces which will be used throughout. If $k\in\mathbb{N}$ and $\alpha>0$, $\alpha\notin\mathbb{N}$, we set:

$$C^k(\bar{\Omega}):=\{f\colon \bar{\Omega}\to\mathbb{C}\colon D^\beta f \text{ is uniformly continuous and bounded } \forall\beta\in\mathbb{N}^n \text{ with } |\beta|\leqq k\}$$

$$C^\alpha(\bar{\Omega}):=\{f\in C^{[\alpha]}(\bar{\Omega})\colon D^\beta f \text{ is } (\alpha-[\alpha])\text{-Hölder continuous and bounded } \forall\beta\in\mathbb{N}^n \text{ with } |\beta|=[\alpha]\},$$

where $[\alpha]$ is the greatest integer less than α. The spaces $C^k(\bar{\Omega})$, $C^\alpha(\bar{\Omega})$ are endowed with the norms

$$\|f\|_{C^k(\bar{\Omega})}:=\sum_{|\beta|\leqq k}\|D^\beta f\|_{C^0(\bar{\Omega})}, \qquad \|f\|_{C^\alpha(\bar{\Omega})}:=\|f\|_{C^{[\alpha]}(\bar{\Omega})}+\sum_{|\beta|=[\alpha]}[D^\beta f]_{C^{\alpha-[\alpha]}(\bar{\Omega})},$$

where $\|\cdot\|_{C^0(\bar{\Omega})}$ and, for $\eta\in]0,1[$, $[\cdot]_{C^\eta(\bar{\Omega})}$ are the usual sup-norm and Hölder-seminorm:

$$\|g\|_{C^0(\bar{\Omega})}:=\sup\{|g(x)|\colon x\in\bar{\Omega}\}, \qquad [g]_{C^\eta(\bar{\Omega})}:=\sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\eta}\colon x, y\in\bar{\Omega}, x\neq y\right\}.$$

If $k=0$, we write simply $C(\bar{\Omega})$ instead of $C^0(\bar{\Omega})$.

The spaces $C^k(\partial\Omega)$, $k\in\mathbb{N}$, are defined similarly, clearly involving only tangential derivatives.

If $x_0 \in \bar{\Omega}$, the open ball of center x_0 and radius r is denoted by $B(x_0, r)$. We set

(1.1) $$\Omega(x_0, r) := \Omega \cap B(x_0, r), \qquad x_0 \in \bar{\Omega},\ r > 0.$$

If $\alpha > 0$ and $\alpha \notin \mathbb{N}$ we also set

(1.2) $$h^\alpha(\bar{\Omega}) := \{ f \in C^\alpha(\bar{\Omega}) : \lim_{r \to 0^+} \sup_{x_0 \in \bar{\Omega}} [D^\beta f]_{C^{\alpha - [\alpha]}(\overline{\Omega(x_0, r)})} = 0 \ \forall \beta \in \mathbb{N}^n \text{ with } |\beta| = [\alpha] \};$$

thus if $\alpha \in]0, 1[$ we have $g \in h^\alpha(\bar{\Omega})$ if and only if

$$\lim_{r \to 0^+} \sup \left\{ \frac{|g(x) - g(y)|}{|x - y|^\alpha} : x, y \in \bar{\Omega},\ 0 < |x - y| < r \right\} = 0.$$

The space $h^\alpha(\bar{\Omega})$ is a closed subspace of $C^\alpha(\bar{\Omega})$, and hence it is a Banach space with the norm of $C^\alpha(\bar{\Omega})$. We also need the usual Sobolev spaces: if $\beta \in [1, \infty[$, $k \in \mathbb{N}^+$, we set

$$L^p(\Omega) := \{ f : \Omega \to \mathbb{C} : f \text{ is measurable and } p\text{-integrable} \},$$

$$W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : D^\beta f \in L^p(\Omega)\ \forall \beta \in \mathbb{N}^n \text{ with } |\beta| \leqq k \}$$

(here the derivatives are in the sense of distributions), with the obvious norms

$$\|f\|_{L^p(\Omega)} := \{ \int_\Omega |f(x)|^p \, dx \}^{1/p}, \qquad \|f\|_{W^{k,p}(\Omega)} := \{ \sum_{|\alpha| \leqq k} \|D^\beta f\|^p_{L^p(\Omega)} \}^{1/p}.$$

Let now Ω be a bounded open set of $\mathbb{R}^n$, $n \geqq 1$, with boundary $\partial\Omega$ of class C^{2m}, $m \geqq 1$. We introduce the differential operators

(1.3) $$A(x, D) := \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha, \qquad x \in \bar{\Omega},$$

(1.4) $$B_j(x, D) := \sum_{|\beta| \leqq m_j} b_{j\beta}(x) D^\beta, \qquad x \in \partial\Omega,\quad j = 1, \dots, m$$

under the following assumptions:

(1.5) $a_\alpha \in C(\bar{\Omega})$, $|\alpha| \leqq 2m$; $\quad b_{j\beta} \in C^{2m - m_j}(\partial\Omega)$, $|\beta| \leqq m_j$, $\quad j = 1, \dots, m$

(uniform ellipticity). There exist $\eta \in [0, 2\pi[$, $\nu > 0$ such that

(1.6) $$\nu(|\xi|^{2m} + t^{2m}) \leqq |\sum_{|\alpha| = 2m} a_\alpha(x)\, \xi^\alpha - (-1)^m e^{i\eta}\, t^{2m}| \qquad \forall x \in \bar{\Omega},\ \forall \xi \in \mathbb{R}^n,\ \forall t \in \mathbb{R}.$$

(root condition). If $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $(\xi, t) \neq (0, 0)$, $(\xi | \nu(x)) = 0$ the polynomial

(1.7) $$\zeta \to \sum_{|\alpha| = 2m} a_\alpha(x)(\xi + \zeta \nu(x))^\alpha - (-1)^m e^{i\eta}\, t^{2m}$$

has exactly m roots $\zeta_j^+(x, \xi, t)$ with positive imaginary part (here $v(x)$ is the unit outward normal vector at x and $(\cdot|\cdot)$ is the scalar product in $\mathbb{R}^n$).

(complementing condition). If $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $(\xi, t) \neq (0, 0)$, $(\xi | v(x)) = 0$ the m polynomials

$$\zeta \to \sum_{|\beta| = m_j} b_{j\beta}(x)(\xi + \zeta\, v(x))^\beta \tag{1.8}$$

are linearly independent modulo the polynomial (see (1.7))

$$\zeta \to \prod_{j=1}^{m} (\zeta - \zeta_j^+(x, \xi, t)).$$

(normality) $m_j \in \mathbb{N}$, $j = 1, \ldots, m$, $0 \leqq m_j < m_i \leqq 2m-1$ if $j < i$, and

$$\sum_{|\beta| = m_j} b_{j\beta}\, v(x)^\beta \neq 0 \qquad \forall x \in \partial\Omega,\ j = 1, \ldots, m. \tag{1.9}$$

Let $A(\cdot, D)$ and $B_j(\cdot, D)$ be defined by (1.3) and (1.4). Then we consider the non-homogeneous problem

$$\begin{aligned} &\lambda u - A(\cdot, D)\, u = f \quad \text{in } \Omega, \\ &B_j(\cdot, D)\, u = g_j \quad \text{on } \partial\Omega, \qquad j = 1, \ldots, m \end{aligned} \tag{1.10}$$

with prescribed data $f, g_1, \ldots, g_m$.

The following result is well known (Agmon [6]):

Theorem 1.1. *Suppose that* (1.5), ..., (1.9) *hold. Then there exists* $\lambda_0 \geqq 0$ *such that if* $|\lambda| > \lambda_0$ *and* $\arg\lambda = \eta$ (η *is defined in* (1.6)) *then for each* $f \in L^p(\Omega)$ *and* $g = (g_1, \ldots, g_m) \in \prod_{j=1}^{m} W^{2m - m_j - 1/p, p}(\partial\Omega)$, $p \in]1, \infty[$, *problem* (1.10) *has a unique solution* $u \in W^{2m,p}(\Omega)$; *moreover there exists* $M_p > 0$ *such that*

$$\begin{aligned} &\sum_{k=0}^{2m} |\lambda - \lambda_0|^{1 - \frac{k}{2m}} \|D^k u\|_{L^p(\Omega)} \\ &\leqq M_p \left\{ \|f\|_{L^p(\Omega)} + \sum_{j=1}^{m} \sum_{k=0}^{2m - m_j} |\lambda - \lambda_0|^{1 - \frac{m_j + k}{2m}} \|D^k \tilde{g}_j\|_{L^p(\Omega)} \right\}, \end{aligned} \tag{1.11}$$

where $\tilde{g}_j$ *is any function in* $W^{2m - m_j, p}(\Omega)$ *satisfying* $\tilde{g}_j|_{\partial\Omega} = g_j$.

Proof. For the estimate see e.g. Tanabe [21, Lemma 3.8.1]; a proof of existence is in Triebel [22, Theorems 5.5.2–4.9.1]. □

Theorem 1.1 is basic in order to get an estimate similar to (1.11) in $C(\bar{\Omega})$. Namely we have (Stewart [20]):

Theorem 1.2. *Suppose that* (1.5), ..., (1.9) *hold. Then there exists* $\lambda_1 \geqq 0$ *such that if* $|\lambda| > \lambda_1$ *and* $\arg\lambda = \eta$, *then for each* $f \in C(\bar{\Omega})$ *and* $g = (g_1, \ldots, g_m) \in \prod_{j=1}^{m} C^{2m - m_j}(\partial\Omega)$

problem (1.10) *has a unique solution* $u \in \bigcap_{p \in]1,\infty[} W^{2m,p}(\Omega)$; *moreover for each* $p>n$ *there exists* $N_p>0$ *such that*

$$(1.12)\quad \sum_{k=0}^{2m-1} |\lambda-\lambda_1|^{1-\frac{k}{2m}} \|D^k u\|_{C(\bar\Omega)} + |-\lambda_1|^{\frac{n}{2mp}} \sup_{x_0\in\bar\Omega} \|D^{2m}u\|_{L^p(\Omega(x_0,|\lambda-\lambda_1|^{-1/2m}))}$$
$$\leqq N_p \left\{ \|f\|_{C(\bar\Omega)} + \sum_{j=1}^{m} \sum_{k=0}^{2m-m_j} |\lambda-\lambda_1|^{1-\frac{m_j+k}{2m}} \|D^k \tilde g_j\|_{C(\partial\Omega)} \right\},$$

where $\tilde g_j$ *is any function in* $C^{2m-m_j}(\bar\Omega)$ *satisfying* $\tilde g_j|_{\partial\Omega}=g_j$.

Proof. See the Appendix below. □

We need two further basic results. The first is the well-known Sobolev's imbedding theorem, the second yields a method for extending functions defined on subsets of $\mathbb{R}^n$.

Proposition 1.3. *Suppose that* Ω *is bounded and has Lipschitz boundary* $\partial\Omega$; *let* $q>n$ *and* $\alpha=1-n/q$. *Then* $W^{1,q}(\Omega)\hookrightarrow h^\alpha(\bar\Omega)$; *moreover there exist* $K_1, K_2>0$ *such that for each* $x_0\in\bar\Omega$, $r>0$ *and* $u\in W^{1,q}(\Omega)$ *we have*:

(i) $\|u\|_{C(\overline{\Omega(x_0,r)})} \leqq K_1 r^{-n/q}\{\|u\|_{L^q(\Omega(x_0,r))} + r\|Du\|_{L^q(\Omega(x_0,r))}\}$,

(ii) $[u]_{C^\alpha(\Omega(x_0,r))} \leqq K_2 \|Du\|_{L^q(\Omega(x_0,r))}$.

Proof. See e.g. Adams [5, Lemmata 5.15 and 5.17]. □

Proposition 1.4. (i) *Let* F *be a closed set of* $\mathbb{R}^n$, *let* $k\in\mathbb{N}$. *There exists a mapping* $E_k: C(F)\to C(\mathbb{R}^n)$ *such that*

(a) $E_k(f)|_F \equiv f$,

(b) $\|E_k(f)\|_{C^\alpha(\mathbb{R}^n)} \leqq M_k \|f\|_{C^\alpha(F)}$ $\forall f\in C^\alpha(F)$, $\forall \alpha\in[0,k]$,

where M_k *is independent of the closed set* F *and of* $\alpha\in[0,k]$.

(ii) *Let* Ω *be a bounded open set with Lipschitz boundary* $\partial\Omega$. *There exists a mapping* $E: L^1(\Omega)\to L^1(\mathbb{R}^n)$ *such that*

(a) $E(f)|_\Omega\equiv f$,

(b) $\|E(f)\|_{W^{k,p}(\mathbb{R}^n)} \leqq M_{k,\Omega}\|f\|_{W^{k,p}(\Omega)}$ $\forall f\in W^{k,p}(\Omega)$, $\forall k\in\mathbb{N}$, $\forall p\in[1,\infty[$,

where $M_{k,\Omega}$ *is independent of* $p\in[1,\infty[$.

Proof. Part (i) is due to Whitney; for a proof see Stein [19, Chap. VI, Sect. 2].

The result of (ii) goes back to Calderon, and is also proved in [19, Chap. VI, Sect. 3]. □

We finish this section with the following

Definition 1.5. *Let* $\{B_j(\cdot,D)\}$ *be defined by* (1.4). *If* $p\in[1,\infty[$, $k=0,1,\dots,2m$ *and* $\alpha\in]0,2m]$ *we set*:

$$W_B^{k,p}(\Omega) := \{u\in W^{k,p}(\Omega): B_j(\cdot,D)\,u=0 \text{ on } \partial\Omega \text{ for } m_j<k-1/p\}$$

$$C_B^k(\bar\Omega) := \{u\in C^k(\bar\Omega): B_j(\cdot,D)\,u=0 \text{ on } \partial\Omega \text{ for } m_j\leqq k\}$$

$$C_B^\alpha(\bar\Omega) := C^\alpha(\bar\Omega)\cap C_B^{[\alpha]}(\bar\Omega),$$

$$h_B^\alpha(\bar\Omega) := h^\alpha(\bar\Omega)\cap C_B^{[\alpha]}(\bar\Omega).$$

Remark 1.6. Let $f \in C_B^\alpha(\bar{\Omega})$ and let $m_j < \alpha$. Then, if we extend, via Proposition 1.4, the coefficients of $B_j(\cdot, D)$ to the whole $\bar{\Omega}$, we have $B_j(\cdot, D) f \in C^{\alpha - m_j}(\bar{\Omega})$. Hence the condition $B_j(\cdot, D) f = 0$ on $\partial\Omega$ means in particular that

$$\|B_j(\cdot, D) f\|_{C^r(\partial\Omega)} = 0, \qquad r = 0, 1, \ldots, [\alpha - m_j].$$

§ 2. The Main Result

Let E be a Banach space and let $A\colon D_A \hookrightarrow E \to E$ be a closed linear operator whose domain D_A is possibly not dense in E. We assume that the resolvent $\rho(A)$ of A contains a fixed half-line $R_{\eta,\omega} := \{z \in \mathbb{C} \colon \arg z = \eta, |z| > \omega\}$; more precisely, we suppose that there exist $\omega \geqq 0$, $\eta \in [0, 2\pi[$ and $M > 0$ such that:

$$\rho(A) \supseteq R_{\eta,\omega}, \qquad \|R(z, A)\|_{\mathscr{L}(E)} \leqq \frac{M}{|z - \omega|} \quad \forall z \in R_{\eta,\omega}; \tag{2.1}$$

here $R(z, A) := (z - A)^{-1}$. By replacing possibly A with $e^{i\eta}(A - \omega)$, it is not restrictive to assume, instead of (2.1), that:

$$\rho(A) \supseteq R_{0,0} =]0, \infty[, \qquad \|R(s, A)\|_{\mathscr{L}(E)} \leqq \frac{M}{s} \quad \forall s > 0. \tag{2.2}$$

Then in particular for $s \in [1, \infty[$ we have

$$\|AR(s, A)x\|_E \leqq M \|x\|_E \qquad \forall x \in E,$$

$$s\|AR(s, A)x\|_E \leqq M \|x\|_{D_A} \qquad \forall x \in D_A,$$

where $\|\cdot\|_{D_A}$ is the graph norm. Thus, following Grisvard [11], we are led to define the intermediate spaces $D_A(\theta, \infty)$ and $D_A(\theta)$, $\theta \in]0, 1[$, by:

Definition 2.1. *We set:*

$$D_A(\theta, \infty) := \{x \in E \colon \sup_{s \geqq 1} s^\theta \|AR(s, A)x\|_E < \infty\},$$

$$D_A(\theta) := \{x \in D_A(\theta, \infty) \colon \lim_{s \to \infty} s^\theta \|AR(s, A)x\|_E = 0\}.$$

A norm in $D_A(\theta, \infty)$ is the following:

$$\|x\|_{D_A(\theta,\infty)} := \|x\|_E + \sup_{s \geqq 1} s^\theta \|AR(s, A)x\|_E. \tag{2.3}$$

Clearly $D_A \hookrightarrow D_A(\theta) \hookrightarrow D_A(\theta, \infty) \hookrightarrow D_A(\sigma) \hookrightarrow \overline{D_A}$ if $0 < \sigma < \theta < 1$. Moreover $D_A(\sigma)$ is a closed subspace of $D_A(\theta, \infty)$: indeed, it coincides with the closure of D_A with respect to the norm (2.3) (a proof is readily obtained by adapting that of [11, Lemme 2.5]).

Proposition 2.2. *$D_A(\theta, \infty)$ and $D_A(\theta)$ are real interpolation spaces between D_A and E, namely:*

$$D_A(\theta, \infty) = (D_A, E)_{1-\theta,\infty}, \qquad D_A(\theta) = (D_A, E)_{1-\theta}.$$

(For the precise definition and more properties of the spaces $(D_A, E)_{\alpha,\infty}$ see Lions-Peetre [14] or Triebel [22]; for the spaces $(D_A, E)_\alpha$ see Da Prato-Grisvard [9].)

Proof. See [11, Prop. 5.5] and [9, Théorème 2.5]. □

After these preparations, we are ready to state our main result. Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geqq 1$, with boundary $\partial\Omega$ of class C^{2m}, $m \geqq 1$; let $A(\cdot, D)$, $\{B_j(\cdot, D)\}_{1 \leqq j \leqq m}$ be the differential operators defined by (1.3), (1.4) and suppose that (1.5), ..., (1.9) hold. If we set $E = C(\bar{\Omega})$, by Theorem 1.2 the operator A, defined by

$$D_A := \{u \in \bigcap_{p \geqq 1} W^{2m,p}(\Omega) : A(\cdot, D)\, u \in C(\bar{\Omega}),\ B_j(\cdot, D)\, u = 0 \text{ on } \partial\Omega,\ j = 1, \ldots, m\}$$

$$Au := A(\cdot, D)\, u \tag{2.4}$$

fulfills (2.1) for some $\omega \geqq 0$, $\eta \in [0, 2\pi[$ and $M > 0$.

We will prove the following result:

Theorem 2.3. *Let A be defined by* (2.4) *and suppose that* (2.1) *holds. If $\theta \in]0, 1[$ and $2m\theta$ is not an integer, then*

$$D_A(\theta, \infty) = C_B^{2m\theta}(\bar{\Omega}), \qquad D_A(\theta) = h_B^{2m\theta}(\bar{\Omega}),$$

with equivalence of norms.

(The spaces $C_B^\alpha(\bar{\Omega})$ and $h_B^\alpha(\bar{\Omega})$ were introduced in Definition 1.5.)

The proof of the first equality is contained in Sects. 3 and 4 below; the proof of the second one is quite similar and will be sketched in Sect. 5.

§ 3. The First Inclusion

Let A be defined by (2.4) and suppose that (2.1) holds. Then, considering $e^{i\eta}(A - \omega)$ in place of A, we can assume that (2.2) is true. Then we prove the following:

Theorem 3.1. *If $\theta \in]0, 1[$ and $2m\theta$ is not an integer, then*

$$C_B^{2m\theta}(\bar{\Omega}) \hookrightarrow D_A(\theta, \infty).$$

Proof. It suffices to show that

$$\sup_{s \geqq 1} s^\theta \|AR(s, A) f\|_{C(\bar{\Omega})} \leqq C \|f\|_{C^{2m\theta}(\bar{\Omega})}\ \forall f \in C_B^{2m\theta}(\bar{\Omega}). \tag{3.1}$$

This will be done by constructing, for each fixed $f \in C_B^{2m\theta}(\bar{\Omega})$, a function w: $[1, \infty[\to C(\bar{\Omega})$ such that:

$$\|w(s) - f\|_{C(\bar{\Omega})} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar{\Omega})} \qquad \forall s \geqq 1 \tag{3.2}$$

$$\|AR(s, A)\, w(s)\|_{C(\bar{\Omega})} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar{\Omega})} \qquad \forall s \geqq 1: \tag{3.3}$$

this will imply (3.1) since

$$\|AR(s,A)f\|_{C(\bar\Omega)} \leqq \|AR(s,A)\|_{\mathscr{L}(C(\bar\Omega))} \cdot \|f-w(s)\|_{C(\bar\Omega)} + \|AR(s,A)\,w(s)\|_{C(\bar\Omega)}$$
$$\leqq c\,s^{-\theta}\|f\|_{C^{2m\theta}(\bar\Omega)}.$$

Let $f \in C_B^{2m\theta}(\bar\Omega)$, and consider an extension $F \in C^{2m\theta}(\mathbb{R}^n)$ of f (Prop. 1.4(i)), satisfying

$$\|F\|_{C^{2m\theta}(\mathbb{R}^n)} \leqq c\,\|f\|_{C^{2m\theta}(\mathbb{R}^n)}. \tag{3.4}$$

Define an auxiliary function $v_0\colon]0,1] \to C(\mathbb{R}^n)$ by

$$v_0(t)(x) \equiv v_0(t,x) := \int_{\mathbb{R}^n} \phi(z)\,F(x-tz)\,dz = t^{-n}\int_{\mathbb{R}^n} \phi\left(\frac{x-y}{t}\right) F(y)\,dy, \tag{3.5}$$

where $\phi \in C^\infty(\mathbb{R}^n)$ is a real-valued function such that $0 \leqq \phi \leqq 1$, $\phi \equiv 0$ outside $B(0,1)$, $\int_{\mathbb{R}^n} \phi(z)\,dz = 1$, and ϕ is even in each variable.

We have the following lemma, whose proof is straightforward:

Lemma 3.2. (i) $\lim_{t\to 0^+} \|v_0(t)-F\|_{C(\mathbb{R}^n)} = 0$, i.e. $v_0(0)=f$,

(ii) $v_0 \in C^\infty(]0,1] \times \mathbb{R}^n)$ and

$$\sup_{t\in]0,1]} \left\|\frac{\partial^h v_0(t)}{\partial t^h}\right\|_{C(\mathbb{R}^n)} \leqq c\,\|F\|_{C^h(\mathbb{R}^n)}, \qquad h=0,1,\ldots,[2m\theta]. \quad \square$$

Let us define now

$$v(t)(x) \equiv v(t,x) := \sum_{h=0}^{[2m\theta]} (-1)^h \frac{\partial^h v_0}{\partial t^h}(t,x)\frac{t^h}{h!}, \qquad t\in]0,1],\ x\in\mathbb{R}^n. \tag{3.6}$$

Then clearly $v \in C^\infty(]0,1] \times \mathbb{R}^n)$ and we have the following result:

Lemma 3.3. *For each* $t\in]0,1]$ *we have:*

(i) $\|v(t)-F\|_{C(\mathbb{R}^n)} \leqq c\,t^{2m\theta}\|F\|_{C^{2m\theta}(\mathbb{R}^n)}$

(ii) $\|D^\gamma v(t) - D^\gamma F\|_{C(\mathbb{R}^n)} \leqq c\,t^{2m\theta-|\gamma|}\|F\|_{C^{2m\theta}(\mathbb{R}^n)}$ $\forall \gamma\in\mathbb{N}^n$ *with* $|\gamma| \leqq [2m\theta]$

(iii) $\|D^\gamma v(t)\|_{C(\mathbb{R}^n)} \leqq c\,t^{-(|\gamma|-2m\theta)}\|F\|_{C^{2m\theta}(\mathbb{R}^n)}$ $\forall\gamma\in\mathbb{N}^n$ *with* $|\gamma| > 2m\theta$.

Proof. (i) Let us compute $\frac{\partial^h v_0}{\partial t^h}(t)$ for $h \leqq [2m\theta]$: it is easily seen that

$$\frac{\partial^h v_0}{\partial t^h}(t,x) = \int_{\mathbb{R}^n} \phi(z) \sum_{|\beta|=h} D^\beta F(x-tz)(-z)^\beta \frac{h!}{\beta!}\,dz \tag{3.7}$$

and consequently

$$v(t,x) = \int_{\mathbb{R}^n} \phi(z) \sum_{|\beta|\leqq[2m\theta]} \frac{D^\beta F(x-tz)}{\beta!} z^\beta t^{|\beta|}\,dz.$$

On the other hand for each $z\in\mathbb{R}^n$ and $t\in]0,1]$ we have by Taylor's formula

$$F(x) = \sum_{|\beta|\leqq[2m\theta]} \frac{D^\beta F(x-tz)}{\beta!} z^\beta t^{|\beta|} + \sum_{|\beta|=[2m\theta]} \frac{D^\beta F(\xi)}{\beta!} z^\beta t^{[2m\theta]}$$

where $\xi=\xi(t,z,x)$ is a suitable point in the segment joining x and $x-tz$; hence we get

$$v(t,x)-F(x)=\int_{\mathbb{R}^n}\phi(z)\left\{\sum_{|\beta|\leqq[2m\theta]}\frac{D^\beta F(x-tz)}{\beta!}z^\beta t^{|\beta|}-F(x)\right\}dz$$

$$=\int_{\mathbb{R}^n}\phi(z)\sum_{|\beta|=[2m\theta]}[D^\beta F(x-tz)-D^\beta F(\xi)]\frac{z^\beta t^{[2m\theta]}}{\beta!}dz$$

and finally

$$\|v(t)-F\|_{C(\mathbb{R}^n)}\leqq c\int_{\mathbb{R}^n}\phi(z)|z|^{2m\theta}t^{2m\theta}dz\sum_{|\beta|=[2m\theta]}[D^\beta F]_{C^{2m\theta-[2m\theta]}(\mathbb{R}^n)}$$

$$\leqq c\,t^{2m\theta}\|F\|_{C^{2m\theta}(\mathbb{R}^n)}.$$

(ii) Fix $\gamma\in\mathbb{N}^n$ with $|\gamma|\leqq[2m\theta]$ and compute $D^\gamma\dfrac{\partial^h v_0(t)}{\partial t^h}$ for $h\leqq[2m\theta]$. If $|\gamma|+h\leqq[2m\theta]$, by (3.7) we get:

$$D^\gamma\frac{\partial^h v_0(t,x)}{\partial t^h}=\int_{\mathbb{R}^n}\phi(z)\sum_{|\beta|=h}D^{\beta+\gamma}F(x-tz)(-z)^\beta\frac{h!}{\beta!}dz,\qquad\text{if }|\gamma|+h\leqq[2m\theta].\tag{3.8}$$

On the other hand if $|\gamma|+h>2m\theta$ we choose $\gamma_1,\gamma_2\in\mathbb{N}^n$ such that

$$|\gamma_1|=[2m\theta]-h,\qquad|\gamma_2|=|\gamma|-[2m\theta]+h,\qquad\gamma_1+\gamma_2=\gamma;$$

note that $|\gamma_2|\geqq 1$. Hence using (3.8) we can write

$$D^\gamma\frac{\partial^h v_0}{\partial t^h}(t,x)=D^{\gamma_2}\left(D^{\gamma_1}\frac{\partial^h v_0}{\partial t^h}(t,x)\right)$$

$$=D^{\gamma_2}\left(\int_{\mathbb{R}^n}\phi(z)\sum_{|\beta|=h}D^{\beta+\gamma_1}F(x-tz)(-z)^\beta\frac{h!}{\beta!}dz\right)$$

$$=D^{\gamma_2}\left(t^{-n}\int_{\mathbb{R}^n}\sum_{|\beta|=h}D^{\beta+\gamma_1}F(y)\frac{h!}{\beta!}\phi\left(\frac{x-y}{t}\right)\left(\frac{y-x}{t}\right)^\beta dy\right)$$

$$=t^{-n}\int_{\mathbb{R}^n}\sum_{|\beta|=h}D^{\beta+\gamma_1}F(y)\frac{h!}{\beta!}[D^{\gamma_2}(\phi(\xi)(-\xi)^\beta)]_{\xi=\frac{x-y}{t}}\cdot t^{-|\gamma_2|}dy$$

$$=t^{-|\gamma_2|}\int_{\mathbb{R}^n}\sum_{|\beta|=h}D^{\beta+\gamma_1}F(x-tz)\frac{h!}{\beta!}D^{\gamma_2}(\phi(z)(-z)^\beta)\,dz,$$

and, since $\int_{\mathbb{R}^n}D^{\gamma_2}(\phi(z)(-z)^\beta)\,dz=0$, we obtain

$$D^\gamma\frac{\partial^h v_0}{\partial t^h}(t,x)=t^{-(|\gamma|-[2m\theta]+h)}\int_{\mathbb{R}^n}\sum_{|\beta|=h}[D^{\beta+\gamma_1}F(x-tz)-D^{\beta+\gamma_1}F(x)]$$

$$\cdot\frac{h!}{\beta!}D^{\gamma_2}(\phi(z)(-z)^\beta)\,dz;$$

this implies

$$(3.9)\qquad \left\|D^\gamma \frac{\partial^h v_0}{\partial t^h}(t)\right\|_{C(\mathbb{R}^n)}$$

$$\leqq c\, t^{-(|\gamma|-[2m\theta]+h)} \sum_{|\beta|=[2m\theta]} [D^\beta F]_{C^{2m\theta-[2m\theta]}(\mathbb{R}^n)}\, t^{2m\theta-[2m\theta]}$$

$$\leqq c\, t^{-(|\gamma|+h-2m\theta)} \|F\|_{C^{2m\theta}(\mathbb{R}^n)}, \qquad \text{if } |\gamma|+h>2m\theta.$$

Now by (3.6) we have:

$$\|D^\gamma v(t)-D^\gamma F\|_{C(\mathbb{R}^n)} \leqq \left\| \sum_{h=0}^{[2m\theta]-|\gamma|} (-1)^h D^\gamma \frac{\partial^h v_0}{\partial t^h}(t)\frac{t^h}{h!} - D^\gamma F\right\|_{C(\mathbb{R}^n)}$$

$$+\left\| \sum_{h=[2m\theta]-|\gamma|+1}^{[2m\theta]} (-1)^h D^\gamma \frac{\partial^h v_0}{\partial t^h}(t)\frac{t^h}{h!}\right\|_{C(\mathbb{R}^n)} = I_1+I_2.$$

We estimate I_1 as in (i), by using (3.8) and Taylor's formula for $D^\gamma F$ of order $[2m\theta]-|\gamma|-1$, centered at $x-tz$:

$$I_1 = \left\| \int_{\mathbb{R}^n} \phi(z) \left[\sum_{|\beta|\leqq[2m\theta]-|\gamma|} \frac{D^{\gamma+\beta} F(x-tz)}{\gamma!} z^\beta t^{|\beta|} - D^\gamma F(x)\right] dz \right\|_{C(\mathbb{R}^n)}$$

$$= \left\| \int_{\mathbb{R}^n} \phi(z) \sum_{|\beta|=[2m\theta]-|\gamma|} [D^{\gamma+\beta} F(x-tz) - D^{\gamma+\beta} F(\xi)] \frac{z^\gamma}{\gamma!} t^{[2m\theta]-|\gamma|}\, dz \right\|_{C(\mathbb{R}^n)}$$

$$\leqq c \sum_{|\beta|=[2m\theta]} [D^\beta F]_{C^{2m\theta-[2m\theta]}(\mathbb{R}^n)}\, t^{2m\theta-|\gamma|} \leqq c\, t^{2m\theta-|\gamma|} \|F\|_{C^{2m\theta}(\mathbb{R}^n)}.$$

In order to estimate I_2 we just use (3.9):

$$I_2 \leqq \sum_{h=[2m\theta]-|\gamma|+1}^{[2m\theta]} \left\|D^\gamma \frac{\partial^h v_0}{\partial t^h}(t)\right\|_{C(\mathbb{R}^n)} \frac{t^h}{h!} \leqq c\, t^{2m\theta-|\gamma|} \|F\|_{C^{2m\theta}(\mathbb{R}^n)}.$$

This we obtain

$$\|D^\gamma v(t)-D^\gamma F\|_{C(\mathbb{R}^n)} \leqq I_1+I_2 \leqq c\, t^{2m\theta-|\gamma|} \|F\|_{C^{2m\theta}(\mathbb{R}^n)},$$

and (ii) follows.

(iii) Let $\gamma\in\mathbb{N}^n$ be such that $|\gamma|\geqq 2m\theta$. By using again (3.9) we have:

$$\|D^\gamma v(t)\|_{C(\mathbb{R}^n)} \leqq \sum_{h=0}^{[2m\theta]} \left\|D^\gamma \frac{\partial^h v_0}{\partial t^h}(t)\right\|_{C(\mathbb{R}^n)} \frac{t^h}{h!} \leqq c\, t^{-(|\gamma|-2m\theta)} \|F\|_{C^{2m\theta}(\mathbb{R}^n)},$$

and the proof is complete. □

The desired function w: $[1, \infty[\to C(\bar{\Omega})$ satisfying (3.2) and (3.3) is now

$$(3.10)\qquad w(s)(x) \equiv w(s,x) := v(s^{-1/2m}, x), \qquad s\geqq 1,\ x\in\bar{\Omega};$$

its main properties are summarized as follows:

Corollary 3.4. *We have:*

(i) $w \in C^\infty([1, \infty[\times \bar{\Omega})$,

(ii) $\|w(s)-f\|_{C(\bar{\Omega})} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar{\Omega})}$,

(iii) $\|D^\gamma w(s)-D^\gamma f\|_{C(\bar{\Omega})} \leqq c\, s^{-(\theta-|\gamma|/2m)} \|f\|_{C^{2m\theta}(\bar{\Omega})}$ $\forall \gamma \in \mathbb{N}^n$ *with* $|\gamma| \leqq [2m\theta]$,

(iv) $\|D^\gamma w(s)\|_{C(\bar{\Omega})} \leqq c\, s^{|\gamma|/2m-\theta} \|f\|_{C^{2m\theta}(\bar{\Omega})}$ $\forall \gamma \in \mathbb{N}^n$ *with* $|\gamma| > 2m\theta$.

Proof. It is an immediate consequence of (3.4) and Lemma 3.3. □

By Corollary 3.4(ii) we have shown (3.2). Concerning (3.3) we set:

$$u(s) := s R(s, A)\, w(s), \qquad s \geqq 1, \tag{3.11}$$

and observe that

$$A R(s, A)\, w(s) = u(s) - w(s), \qquad s \geqq 1. \tag{3.12}$$

Now $u(s)$ satisfies:

$$u(s) \in \bigcap_{p \geqq 1} W^{2m,p}(\Omega)$$
$$s u(s, x) + [\lambda_0 - A(x, D)]\, u(s, x) = s\, w(s, x) \quad \text{in } \Omega,$$
$$B_j(x, D)\, u(s, x) = 0, \qquad j = 1, \ldots, m, \quad \text{on } \partial\Omega.$$

Hence $u(s) - w(s)$ is the unique solution of:

$$u(s) - w(s) \in \bigcap_{p \geqq 1} W^{2m,p}(\Omega),$$
$$s[u(s, x) - w(s, x)] + [\lambda_0 - A(x, D)][u(s, x) - w(s, x)]$$
$$= [A(x, D) - \lambda_0]\, w(s, x) \quad \text{in } \Omega$$
$$B_j(x, D)[u(s, x) - w(s, x)] = -B_j(x, D)\, w(s, x), \qquad j = 1, \ldots, m, \quad \text{on } \partial\Omega.$$

Thus by Theorem 1.2 we obtain:

$$\begin{aligned} \|u(s) - w(s)\|_{C(\bar{\Omega})} &\leqq c\, s^{-1} \|[A(\cdot, D) - \lambda_0]\, w(s)\|_{C(\bar{\Omega})} \\ &\quad + c \sum_{j=1}^{m} \sum_{k=0}^{2m-m_j} s^{-\frac{m_j+k}{2m}} \|B_j(\cdot, D)\, w(s)\|_{C^k(\partial\Omega)} \\ &= J_1 + J_2. \end{aligned} \tag{3.13}$$

We estimate J_1 by using Corollary 3.4(iii)–(iv) and recalling that $s \geqq 1$:

$$\begin{aligned} J_1 &\leqq c\, s^{-1} \sum_{|\beta| \leqq 2m} \|D^\beta w(s)\|_{C(\bar{\Omega})} \\ &\leqq c\, s^{-1} \left\{1 + \sum_{h=[2m\theta]+1}^{2m} s^{(h/2m)-\theta}\right\} \|f\|_{C^{2m\theta}(\bar{\Omega})} \\ &\leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar{\Omega})}. \end{aligned} \tag{3.14}$$

To estimate J_2 we split it into three terms. Set

$$j_0 := \max\{j \leqq m: m_j < 2m\theta\};$$

then

$$(3.15)\qquad J_2 = \left(\sum_{j=1}^{j_0} \sum_{k=0}^{[2m\theta]-m_j} + \sum_{j=1}^{j_0} \sum_{k=[2m\theta]-m_j+1}^{2m-m_j} + \sum_{j=j_0+1}^{m} \sum_{k=0}^{2m-m_j}\right)$$

$$\cdot \|B_j(\cdot, D)\, w(s)\|_{C^k(\partial\Omega)} = J_{21} + J_{22} + J_{23};$$

now by Corollary 3.4(iii)–(iv) we get

$$(3.16)\quad J_{22} \leqq c \sum_{j=1}^{j_0} \sum_{k=[2m\theta]-m_j+1}^{2m-m_j} s^{-(m_j+k)/2m} \|w(s)\|_{C^{k+m_j}(\partial\Omega)} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar\Omega)},$$

and similarly

$$(3.17)\qquad J_{23} \leqq c \sum_{j=j_0+1}^{m} \sum_{k=0}^{2m-m_j} s^{-\frac{m_j+k}{2m}} \|w(s)\|_{C^{k+m_j(\partial\Omega)}} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar\Omega)}.$$

Finally, concerning J_{21} we use Remark 1.6 and Corollary 3.4(iii), obtaining

$$(3.18)\qquad J_{21} = \sum_{j=1}^{j_0} \sum_{k=0}^{[2m\theta]-m_j} \|B_j(\cdot, D)[w(s)-f]\|_{C^k(\partial\Omega)}$$

$$\leqq c \sum_{j=1}^{j_0} \sum_{k=0}^{[2m\theta]-m_j} \|w(s)-f\|_{C^{k+m_j(\partial\Omega)}} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar\Omega)}.$$

Collecting (3.13), ..., (3.18) we get

$$\|u(s) - w(s)\|_{C(\bar\Omega)} \leqq c\, s^{-\theta} \|f\|_{C^{2m\theta}(\bar\Omega)}$$

and recalling (3.12) we have proved (3.3).

As (3.1) follows by (3.2) and (3.3), the proof of Theorem 3.1 is complete. □

§ 4. The Second Inclusion

Again, let A be defined by (2.4) and, after the usual modifications, assume that (2.2) holds. We have to prove:

Theorem 4.1. *If* $\theta \in]0, 1[$ *and* $2m\theta$ *is not an integer, then*

$$D_A(\theta, \infty) \hookrightarrow C_B^{2m\theta}(\bar\Omega).$$

Proof. We will construct a function u: $]0, 1] \to D_A$ such that $u(t) \to f$ in $C_B^{[2m\theta]}(\bar\Omega)$ as $t \to 0^+$: this will imply that $f \in C_B^{[2m\theta]}(\bar\Omega)$; next, we will show that $f \in C^{2m\theta}(\bar\Omega)$ by using the approximating function $u(t)$, evaluated at suitable points t.

We start with defining

$$(4.1)\qquad u(t)(x) \equiv u(t, x) := t^{-1}[R(t^{-1}, A) f](x), \qquad t \in]0, 1], \ x \in \bar\Omega.$$

Remark 4.2. Clearly $u \in C^1(]0, 1], C(\bar{\Omega}))$, and it is readily seen that

$$(4.2) \qquad u'(t) = t^{-2} R(t^{-1}, A)\, AR(t^{-1}, A) f, \qquad t \in]0, 1].$$

Thus in particular $u, u' \in C(]0, 1], D_A)$, which implies

$$\left.\begin{array}{l} u(t), u'(t) \in \bigcap_{p \geqq 1} W^{2mp}(\Omega) \hookrightarrow \bigcap_{\alpha \in]0,1[} C^{2m\alpha}(\bar{\Omega}) \\ B_j(\cdot, D)\, u(t) = B_j(\cdot, D)\, u'(t) = 0 \quad \text{on } \partial\Omega \end{array}\right\} \quad \forall t \in]0, 1].$$

As a consequence we have for $|\beta| \leqq 2m-1$

$$\frac{\partial}{\partial t} D^\beta u(t, x) = D^\beta \frac{\partial}{\partial t} u(t, x) \text{ in the sense of } C(]0, 1] \times \bar{\Omega}),$$

and for $|\beta| = 2m$

$$\frac{\partial}{\partial t} D^\beta u(t, x) = D^\beta \frac{\partial}{\partial t} u(t, x) \text{ in the sense of } \bigcap_{p \geqq 1} L^p(]0, 1] \times \Omega).$$

We have the following key lemma:

Lemma 4.3. *For each $p > n$ there exists $C_p > 0$ such that:*

(i) $\displaystyle\sum_{h=1}^{2m-1} t^{-\left(1-\frac{h}{2m}\right)} \sum_{|\beta|=h} \|D^\beta u(t)\|_{C(\bar{\Omega})}$
$\displaystyle + t^{-\frac{n}{2mp}} \sup_{x_0 \in \bar{\Omega}} \Big\{ \sum_{|\beta|=2m} \|D^\beta u(t)\|_{L^p(\Omega(x_0, t^{1/2m}))} \Big\} \leqq C_p\, t^{-1} \|f\|_{C(\bar{\Omega})},$

(ii) $\displaystyle\sum_{h=1}^{2m-1} t^{-\left(1-\frac{h}{2m}\right)} \sum_{|\beta|=h} \|D^\beta u'(t)\|_{C(\bar{\Omega})}$
$\displaystyle + t^{-\frac{n}{2mp}} \sup_{x_0 \in \bar{\Omega}} \Big\{ \sum_{|\beta|=2m} \|D^\beta u'(t)\|_{L^p(\Omega(x_0, t^{1/2m}))} \Big\}$
$\leqq C_p\, t^{-(2-\theta)} \|f\|_{D_A(\theta, \infty)}.$

Proof. (i) It follows readily by (4.1) and Theorem 1.2 with $\lambda = \lambda_0 + t^{-1}$.

(ii) It follows by (4.2), Theorem 1.2 with $\lambda = \lambda_0 + t^{-1}$ and the fact that

$$\|AR(t^{-1}, A) f\|_{C(\bar{\Omega})} \leqq c\, t^\theta \|f\|_{D_A(\theta, \infty)}. \quad \square$$

The next Lemma is a consequence of Lemma 4.3.

Lemma 4.4. *We have:*

(i) $\lim_{t \to 0^+} \|u(t) - f\|_{C(\bar{\Omega})} = 0.$

(ii) $\|D^\beta u(r) - D^\beta u(s)\|_{C(\bar{\Omega})} \leqq c\, |r-s|^{\theta - \frac{|\beta|}{2m}} \|f\|_{D_A(\theta, \infty)}$
$\forall r, s \in]0, 1]$, $\forall \beta \in \mathbb{N}^n$ *with* $|\beta| \leqq [2m\theta]$;

(iii) $\|D^\beta u(t)\|_{C(\bar{\Omega})} \leqq c\, t^{-\left(\frac{|\beta|}{2m} - \theta\right)} \|f\|_{D_A(\theta, \infty)}$
$\forall t \in]0, 1]$. $\forall \beta \in \mathbb{N}^n$ *with* $2m\theta < |\beta| \leqq 2m-1$;

(iv) *for each* $p>\dfrac{n}{2m(1-\theta)}$ *there exists* $C_p>0$ *such that*

$$\sup_{x_0\in\bar{\Omega}} \|D^\beta u(t)\|_{L^p(\Omega(x_0,t^{1/2m}))}$$
$$\leqq C_p\, t^{-\left(1-\theta-\frac{n}{2mp}\right)} \|f\|_{D_A(\theta,\infty)}\ \forall t\in]0,1],\ \forall \beta\in\mathbb{N}^n \quad \textit{with} \quad |\beta|=2m;$$

(v) *for each* $\alpha\in]0,1[$ *there exists* $c_\alpha>0$ *such that*

$$\sup_{x_0\in\bar{\Omega}} [D^\beta u(t)]_{C^\alpha(\overline{\Omega(x_0,t^{1/2m})})}$$
$$\leqq c_\alpha\, t^{\frac{1-\alpha}{2m}+\theta-1} \|f\|_{D_A(\theta,\infty)}\ \forall t\in]0,1],\ \forall \beta\in\mathbb{N}^n \quad \textit{with} \quad |\beta|=2m-1.$$

Proof. (i) We have

$$\|u(t)-f\|_{C(\bar{\Omega})}=\|AR(t^{-1},A)f\|_{C(\bar{\Omega})}\leqq c\, t^\theta \|f\|_{D_A(\theta,\infty)}.$$

(ii) If $|\beta|\leqq[2m\theta]$ and $0<s<r\leqq 1$ we have by Lemma 4.3(ii) (with any fixed $p>n$):

$$\begin{aligned}\|D^\beta u(r)-D^\beta u(s)\|_{C(\bar{\Omega})} &\leqq \int_s^r \|D^\beta u'(\sigma)\|_{C(\bar{\Omega})}\, d\sigma\\ &\leqq c\int_s^r \sigma^{-1+\theta-(|\beta|/2m)}\, d\sigma\, \|f\|_{D_A(\theta,\infty)}\\ &\leqq c(r-s)^{\theta-|\beta|/2m}\|f\|_{D_A(\theta,\infty)}.\end{aligned}$$

(iii) If $2m\theta<|\beta|\leqq 2m-1$ and $t\in]0,1]$ we write:

$$\|D^\beta u(t)\|_{C(\bar{\Omega})}\leqq \int_t^1 \|D^\beta u'(\sigma)\|_{C(\bar{\Omega})}\, d\sigma+\|D^\beta u(1)\|_{C(\bar{\Omega})}. \tag{4.3}$$

Now by Lemma 4.3(i) (again with any fixed $p>n$)

$$\|D^\beta u(1)\|_{C(\bar{\Omega})}\leqq c\,\|f\|_{C(\bar{\Omega})}, \tag{4.4}$$

whereas by Lemma 4.3(ii)

$$\int_t^1 \|D^\beta u'(\sigma)\|_{C(\bar{\Omega})}\, d\sigma\leqq c\int_t^1 \sigma^{-1+\theta-\frac{|\beta|}{2m}}\, d\sigma\, \|f\|_{D_A(\theta,\infty)}\leqq c\, t^{\theta-\frac{|\beta|}{2m}}\|f\|_{D_A(\theta,\infty)}. \tag{4.5}$$

As $t\leqq 1$, by (4.3), (4.4) and (4.5) we get

$$\|D^\beta u(t)\|_{C(\bar{\Omega})}\leqq c\, t^{\theta-\frac{|\beta|}{2m}}\|f\|_{D_A(\theta,\infty)}.$$

(iv) If $p>\dfrac{n}{2m(1-\theta)}$, $|\beta|=2m$ and $x_0\in\bar{\Omega}$, we write:

$$\begin{aligned}\|D^\beta u(t)\|_{L^p(\Omega(x_0,t^{1/2m}))} &\leqq \int_t^1 \|D^\beta u'(\sigma)\|_{L^p(\Omega(x_0,t^{1/2m}))}\, d\sigma\\ &\quad+\|D^\beta u(1)\|_{L^p(\Omega(x_0,t^{1/2m}))};\end{aligned} \tag{4.6}$$

now Lemma 4.3 (i) yields

$$\sup_{x_0\in\bar\Omega} \|D^\beta u(1)\|_{L^p(\Omega(x_0,t^{1/2m}))} \leqq c_p \|f\|_{C(\bar\Omega)}, \tag{4.7}$$

whereas by Lemma 4.3 (ii)

$$\begin{aligned} \sup_{x_0\in\bar\Omega} \int_t^1 \|D^\beta u'(\sigma)\|_{L^p(\Omega(x_0,t^{1/2m}))}\, d\sigma &\leqq c_p \int_t^1 \sigma^{-2+\theta+\frac{n}{2mp}}\, d\sigma\, \|f\|_{D_A(\theta,\infty)} \\ &\leqq c_p\, t^{-1+\theta+\frac{n}{2mp}} \|f\|_{D_A(\theta,\infty)}. \end{aligned} \tag{4.8}$$

As $1-\theta-\dfrac{n}{2mp}>0$, by (4.6), (4.7) and (4.8) we conclude that

$$\sup_{x_0\in\bar\Omega} \|D^\beta u(t)\|_{L^p(\Omega(x_0,t^{1/2m}))} \leqq C_p\, t^{-1+\theta+\frac{n}{2mp}} \|f\|_{D_A(\theta,\infty)}.$$

(v) Let $\alpha\in]0,1[$, $|\beta|=2m-1$, and set $q:=\dfrac{n}{1-\alpha}$. By Proposition 1.3 (ii)

$$[D^\beta u(t)]_{C^\alpha(\overline{\Omega(x_0,t^{1/2m})})} \leqq C_\alpha \sum_{|\gamma|=2m} \|D^\gamma u(t)\|_{L^q(\Omega(x_0,t^{1/2m}))}. \tag{4.9}$$

Now pick $p>\max\left\{q, \dfrac{n}{2m(1-\theta)}\right\}$: by (4.9), Hölder's inequality and part (iv) we get:

$$\begin{aligned} [D^\beta u(t)]_{C^\alpha(\overline{\Omega(x_0,t^{1/2m})})} &\leqq C_\alpha \sum_{|\gamma|=2m} \|D^\gamma u(t)\|_{L^p(\Omega(x_0,t^{1/2m}))}\, t^{\frac{n}{2m}\left(\frac{1}{q}-\frac{1}{p}\right)} \\ &\leqq C_{\alpha,p}\, t^{\frac{n}{2mp}+\theta-1+\frac{n}{2m}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{D_A(\theta,\infty)} \\ &= C_\alpha t^{\frac{1-\alpha}{2m}+\theta-1} \|f\|_{D_A(\theta,\infty)}. \quad \square \end{aligned}$$

By Lemma 4.4 (i)–(ii) we deduce that $u(t)\to f$ in $C^{[2m\theta]}(\bar\Omega)$ as $t\to 0^+$; as $B_j(\cdot,D)\,u(t)=0$ on $\partial\Omega$ for $j=1,\dots,m$, when $t\to 0^+$ we get $B_j(\cdot,D)f=0$ on $\partial\Omega$ if $m_j\leqq[2m\theta]$, i.e. $f\in C_B^{[2m\theta]}(\bar\Omega)$. In addition we get

$$\begin{aligned} \|f\|_{C^{[2m\theta]}(\bar\Omega)} &\leqq \|f-u(1)\|_{C^{[2m\theta]}(\bar\Omega)} + \|u(1)\|_{C^{[2m\theta]}(\bar\Omega)} \\ &\leqq c\, \|f\|_{D_A(\theta,\infty)}. \end{aligned} \tag{4.10}$$

Thus it remains to show that $D^\beta f\in C^{2m\theta-[2m\theta]}(\bar\Omega)$ if $|\beta|=[2m\theta]$. We distinguish two cases: (a) $[2m\theta]<2m-1$, (b) $[2m\theta]=2m-1$. In case (a), let $|\beta|=[2m\theta]<2m-1$, and choose $t:=|x-y|^{2m}$ where $x,y\in\bar\Omega$ and $|x-y|\leqq 1$. Then

$$\begin{aligned} &|D^\beta f(x)-D^\beta f(y)| \\ &\quad\leqq |D^\beta f(x)-D^\beta u(t,x)|+|D^\beta u(t,x)-D^\beta u(t,y)|+|D^\beta u(t,y)-D^\beta f(y)| \\ &\quad\leqq 2\,\|D^\beta f-D^\beta u(t)\|_{C(\bar\Omega)} + C \sum_{|\gamma|=[2m\theta]+1} \|D^\gamma u(t)\|_{C(\bar\Omega)}\, |x-y|, \end{aligned}$$

and by Lemma 4.4 (ii)–(iii)

(4.11)
$$\begin{aligned}
&|D^\beta f(x)-D^\beta f(y)| \\
&\leqq c\, t^{\theta-\frac{[2m\theta]}{2m}} \|f\|_{D_A(\theta,\infty)} + c\, t^{\theta-\frac{[2m\theta]+1}{2m}} |x-y|\, \|f\|_{D_A(\theta,\infty)} \\
&\leqq c\, |x-y|^{2m\theta-[2m\theta]} \|f\|_{D_A(\theta,\infty)}.
\end{aligned}$$

In case (b), let $|\beta|=[2m\theta]=2m-1$ and choose, as before, $t:=|x-y|^{2m}$ where $x, y\in\bar{\Omega}$ and $|x-y|\leqq 1$. Then

$$\begin{aligned}
&|D^\beta f(x)-D^\beta(y)| \\
&\quad\leqq 2\,\|D^\beta f-D^\beta u(t)\|_{C(\bar{\Omega})}+[D^\beta u(t)]_{C^{2m\theta-[2m\theta]}(\overline{\Omega(x,t^{1/2m})})}\, |x-y|^{2m\theta-[2m\theta]},
\end{aligned}$$

and by Lemma 4.4(ii)–(v)

(4.12)
$$\begin{aligned}
&|D^\beta f(x)-D^\beta f(y)| \\
&\leqq c\, t^{\theta-\frac{2m-1}{2m}} \|f\|_{D_A(\theta,\infty)} \\
&\quad + c\, t^{\frac{1-2m\theta+[2m\theta]}{2m}+\theta-1} |x-y|^{2m\theta-[2m\theta]} \|f\|_{D_A(\theta,\infty)} \\
&\leqq c\, |x-y|^{2m\theta-[2m\theta]} \|f\|_{D_A(\theta,\infty)}.
\end{aligned}$$

By (4.11) and (4.12) we conclude that if $|\beta|=[2m\theta]$ then $D^\beta f\in C^{2m\theta-[2m\theta]}(\bar{\Omega})$; moreover recalling (4.10) we also obtain

$$\|f\|_{C^{2m\theta}(\bar{\Omega})}\leqq c\, \|f\|_{D_A(\theta,\infty)},$$

and the proof of Theorem 4.1 is complete. □

§ 5. Improvements and Remarks

By Theorems 3.1 and 4.1 the first equality of Theorem 2.3 is established. In order to check the second one, just a few remarks are needed.

Concerning the first inclusion, we proceed as in Sect. 3. There is only a difference in the basic Lemma 3.3: namely, it turns out that the right-hand sides of the inequalities in (i)–(ii)–(iii) have to be multiplied by $o(1)$ (as $t\to 0^+$), due to the fact that $F\in h^{2m\theta}(\mathbb{R}^n)$. Consequently, the right-hand sides of the inequalities of Corollary 3.4 should also be multiplied by $o(1)$ (as $s\to\infty$). As a result one obtains, instead of (3.2),

(5.1) $$\lim_{s\to\infty} s^\theta \|w(s)-f\|_{C(\bar{\Omega})}=0.$$

Continuing as in Sect. 3, one then arrives to

(5.2) $$\lim_{s\to\infty} s^\theta \|AR(s, A)\, w(s)\|_{C(\bar{\Omega})}=0$$

which replaces (3.3). Finally, recalling (3.12), by (5.1) and (5.2) it follows that

$$\lim_{s\to\infty} s^\theta \|AR(s, A) f\|_{C(\bar{\Omega})}=0,$$

i.e. $f\in D_A(\theta)$.

The second inclusion is easier: we already know that $D_A(\theta) \hookrightarrow D_A(\theta, \infty) = C_B^{2m\theta}(\bar{\Omega})$; hence if $f \in D_A(\theta)$ we have only to show that $f \in h^{2m\theta}(\bar{\Omega})$. Now, recalling that $D_A(\theta)$ is the closure of D_A in $D_A(\theta, \infty)$, we take a sequence $\{u_n\} \subseteq D_A$ such that $u_n \to f$ in $D_A(\theta, \infty)$, i.e. in $C^{2m\theta}(\bar{\Omega})$, as $n \to \infty$. But $D_A \hookrightarrow h^{2m\theta}(\bar{\Omega})$ by Prop. 1.3, and consequently we get $\{u_n\} \subseteq h^{2m\theta}(\bar{\Omega})$. Thus $f \in h^{2m\theta}(\bar{\Omega})$ since $h^{2m\theta}(\bar{\Omega})$ is a closed subspace of $C^{2m\theta}(\bar{\Omega})$. The proof of Theorem 2.3 is now complete. □

Remark 5.1. Theorem 2.3 can be generalized in several directions. Following Amann [7], one can consider elliptic systems of differential operators as in [7, Sects. 12–13], in a possibly unbounded open set Ω which is supposed to be uniformly regular of class C^{2m} ([7, Sect. 11]). The analogue of Theorem 1.1 is proved by Geymonat-Grisvard [10, Sect. 5] and Amann [7, Theorem 12.2], whereas the analogue of Theorem 1.2 can be proved by the same method used in the Appendix below; the arguments of Sects. 3 and 4 then still work.

Remark 5.2. The critical cases $2m\theta \in \mathbb{N}$ are not covered by our theorem: they will be the object of a further paper. However in the case $m=1$ the "critical" spaces $D_A(\frac{1}{2}, \infty)$ and $D_A(\frac{1}{2}, \infty)$ are known. The (single) boundary operator $B(\cdot, D)$ has then one of the following forms:

(a) $B(x, D) = I$ (Dirichlet problem), or

(b) $B(x, D) = \alpha(x)\, I + \sum_{i=1}^{n} \beta_i(x)\, D_i$ (oblique derivative problem), where $(\beta(x) | \nu(x)) > 0\ \forall x \in \partial\Omega$.

Denote by $C^{*,1}(\bar{\Omega})$ and $h^{*,1}(\bar{\Omega})$ the Zygmund spaces defined by:

$$C^{*,1}(\bar{\Omega}) := \left\{ u \in C(\bar{\Omega}) : \sup \left\{ \frac{\left| u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \right|}{|x-y|} : x, y, \frac{x+y}{2} \in \bar{\Omega}, x \neq y \right\} < \infty \right\}$$

$$h^{*,1}(\bar{\Omega}) := \left\{ u \in C(\bar{\Omega}) : \lim_{r \to 0^+} \sup_{x_0 \in \bar{\Omega}} \right.$$

$$\left. \cdot \sup \left\{ \frac{\left| u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \right|}{|x-y|} : x, y, \frac{x+y}{2} \in \overline{\Omega(x_0, r)}, x \neq y \right\} = 0 \right\};$$

then in case (a) (Lunardi [15]) we have

$$D_A(\tfrac{1}{2}, \infty) = \{u \in C^{*,1}(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}, \qquad D_A(\tfrac{1}{2}) = \{u \in h^{*,1}(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\},$$

whereas in case (b) (Acquistapace-Terreni [3]) we obtain

$$D_A(\tfrac{1}{2}, \infty) = \left\{ u \in C^{*,1}(\bar{\Omega}) : \sup \right.$$

$$\left. \cdot \left\{ \frac{|u(x - \sigma\beta(x)) - u(x)|}{\sigma} : x \in \partial\Omega, \sigma > 0, x - \sigma\beta(x) \in \bar{\Omega} \right\} < \infty \right\},$$

$$D_A(\tfrac{1}{2})=\left\{u\in h^{*,1}(\bar\Omega):\ \lim_{\sigma\to 0^+}\frac{u(x-\sigma\beta(x))-u(x)}{\sigma}=\alpha(x)f(x)\ \forall x\in\partial\Omega\right\}.$$

Remark 5.3. The method employed in the proof of Theorem 2.3 still works in different situations. For instance if we choose $E=L^p(\Omega)$, $1<p<\infty$, then we find again Grisvard's characterizations of $D_A(\theta,\infty)$ and $D_A(\theta)$ in this case ([11, 9]), needing on the other hand much less regularity on the coefficients of the differential operator. Even more, we can study by the same method the spaces $D_A(\theta,q)$, $1\leqq q<\infty$, where

$$D_A(\theta,q)=\left\{x\in E:\ \int_0^\infty \|s^\theta AR(s,A)x\|_E^q\frac{ds}{s}<\infty\right\};$$

also in this case we find again old results by Grisvard (see [12] or [22, Theorem 4.3.3(a)]) as well as new results. More details will be published elsewhere.

Appendix: Proof of Theorem 1.2

Let $f\in C(\bar\Omega)$, $g=(g_1,\dots,g_m)\in\prod_{j=1}^m C^{2m-m_j}(\partial\Omega)$. As, clearly, $f\in\bigcap_{p>1}L^p(\Omega)$ and, for $j=1,\dots,m$, $g_j\in\bigcap_{p>1}W^{2m-m_j-\frac{1}{p},p}(\partial\Omega)$, by Theorem 1.1 for each $p\in]1,\infty[$ problem (1.10) has a unique solution $u_p\in W^{2m,p}(\Omega)$; hence if $q>p$ we have $u_p=u_q$ and consequently $u_p\in\bigcap_{q>1}W^{2m,q}(\Omega)$ and is independent of p. Thus a unique solution $u\in\bigcap_{p>1}W^{2m,p}(\Omega)$ of problem (1.10) does exist.

We have to prove (1.12). Fix $p>n$, choose $\lambda_1=\lambda_0+1$ (λ_0 is given in Theorem 1.1) and fix $\lambda\in C$ with $|\lambda|>\lambda_1$ and $\arg\lambda=\eta$; fix also $x_0\in\bar\Omega$ and let $\mu>2$ to be chosen later. Select a function $\phi(x)\equiv\phi(x_0,\lambda,\mu,x)$ with the following properties:

$$\text{(A.1)}\qquad \begin{aligned}&\phi\in C^\infty(\mathbb{R}^n),\quad \phi\equiv 1\ \text{ on } B(x_0,\rho),\quad \phi\equiv 0\ \text{ outside } B(x_0,\mu\rho),\\ &\|D^h\phi\|_{C(\mathbb{R}^n)}\leqq c_h\,\rho^{-h}(\mu-1)^{-h},\quad h=1,\dots,2m,\end{aligned}$$

where we have set

$$\text{(A.2)}\qquad \rho:=|\lambda-\lambda_0|^{-1/2m}.$$

(Note that $\rho<1$.) The function $v(x):=u(x)\cdot\phi(x)$ solves

$$\text{(A.3)}\qquad \begin{aligned}&\lambda v(x)-\sum_{|\alpha|\leqq 2m}a_\alpha(x)D^\alpha v(x)=\phi(x)f(x)+F(x),\qquad x\in\bar\Omega,\\ &\sum_{|\beta|\leqq m_j}b_{j\beta}(x)D^\beta v(x)=\phi(x)g_j(x)+G_j(x),\qquad x\in\partial\Omega,\ j=1,\dots,m,\end{aligned}$$

where

$$\text{(A.4)}\qquad F(x)=\sum_{|\alpha|\leqq 2m}a_\alpha(x)\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}D^\beta u(x)D^{\alpha-\gamma}\phi(x),$$

(A.5) $$G_j(x)=\sum_{|\beta|\leqq m_j} b_{j\beta}(x)\sum_{\delta<\beta}\binom{\beta}{\delta} D^\delta u(x)\, D^{\beta-\delta}\phi(x), \qquad j=1,\dots,m.$$

By Theorem 1.1 we have (denoting again by g_j any $W^{2m-m_j,p}$-extension of g_j to the whole Ω):

$$\sum_{k=0}^{2m}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k v\|_{L^p(\Omega)}$$
$$\leqq M_p\left\{\|\phi f+F\|_{L^p(\Omega)}+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda-\lambda_0|^{1-\frac{m_j+k}{2m}}\|D^k(\phi g_j+G_j)\|_{L^p(\Omega)}\right\},$$

and hence

(A.6) $$\sum_{k=0}^{2m}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k v\|_{L^p(\Omega)}$$
$$\leqq M_p\Bigg\{\|f\|_{L^p(\Omega(x_0,\mu\rho))}+\|F\|_{L^p(\Omega(x_0,\mu\rho))}$$
$$+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda-\lambda_0|^{1-\frac{m_j+k}{2m}}$$
$$\cdot\,[\|D^k(\phi g_j)\|_{L^p(\Omega(x_0,\mu\rho))}+\|D^k G_j\|_{L^p(\Omega(x_0,\mu\rho))}]\Bigg\}.$$

Now by (A.4) and (A.1) we get:

(A.7) $$\|F\|_{L^p(\Omega(x_0,\mu\rho))}\leqq c\sum_{k=0}^{2m-1}\|D^k u\|_{C(\bar\Omega)}\cdot\rho^{-2m+k+n/p}\,\mu^{n/p}(\mu-1)^{-1};$$

moreover if $k=0,1,\dots,2m-m_j$ it is easily seen that

$$|D^k G_j|\leqq c\sum_{h=0}^{k+m_j-1}|D^h u|\cdot\sum_{r=1}^{k+m_j-h}|D^r\phi|,$$

and therefore (A.1) yields

(A.8) $$\|D^k G_j\|_{L^p(\Omega(x_0,\mu\rho))}$$
$$\leqq c\sum_{h=0}^{k+m_j-1}\|D^h u\|_{C(\bar\Omega)}\cdot\rho^{h-k-m_j+n/p}\,\mu^{n/p}(\mu-1)^{-1},$$
$$k=0,1,\dots,2m-m_j.$$

Finally, again by (A.1) it follows that

(A.9) $$\|D^k(\phi g_j)\|_{L^p(\Omega(x_0,\mu\rho))}$$
$$\leqq c\sum_{h=0}^{k}\|D^h g_j\|_{L^p(\Omega(x_0,\mu\rho))}\cdot\rho^{h-k}(\mu-1)^{h-k},$$
$$k=0,1,\dots,2m-m_j.$$

By (A.6), (A.7), (A.8) and (A.9), recalling (A.2) we easily get:

$$\text{(A.10)}\qquad \begin{aligned}&\sum_{k=0}^{2m}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k v\|_{L^p(\Omega)}\\ &\leqq c_p\left\{\|f\|_{L^p(\Omega(x_0,\mu\rho))}+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda-\lambda_0|^{1-\frac{m_j+k}{2m}}\|D^k g_j\|_{L^p(\Omega(x_0,\mu\rho))}\right.\\ &\left.\quad+\sum_{k=0}^{2m-1}|\lambda-\lambda_0|^{1-\frac{k}{2m}-\frac{n}{2mp}}\mu^{n/p}(\mu-1)^{-1}\|D^k u\|_{C(\bar\Omega)}\right\}.\end{aligned}$$

On the other hand, by Proposition 1.3(i) and (A.2),

$$\text{(A.11)}\qquad \begin{aligned}&\sum_{k=0}^{2m-1}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k u\|_{C(\overline{\Omega(x_0,\rho)})}+|\lambda-\lambda_0|^{\frac{n}{2mp}}\|D^{2m}u\|_{L^p(\Omega(x_0,\rho))}\\ &\leqq c\,|\lambda-\lambda_0|^{\frac{n}{2mp}}\sum_{k=0}^{2m}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k v\|_{L^p(\Omega)}.\end{aligned}$$

Now choose as x_0 a point of maximum for the (real) function $\Lambda\in C(\bar\Omega)$ defined by

$$\Lambda(x)=\sum_{k=0}^{2m-1}\rho^k|D^k u(x)|+\rho^{2m-n/p}\|D^{2m}u\|_{L^p(\Omega(x,\rho))},\qquad x\in\bar\Omega;$$

then we have clearly

$$\text{(A.12)}\quad \begin{aligned}|\lambda-\lambda_0|\,\|\Lambda\|_{C(\bar\Omega)}&\leqq\sum_{k=0}^{2m-1}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k u\|_{C(\bar\Omega)}\\ &\quad+|\lambda-\lambda_0|^{\frac{n}{2mp}}\sup_{x\in\bar\Omega}\|D^{2m}u\|_{L^p(\Omega(x,r))}\\ &\leqq(2m+1)\,|\lambda-\lambda_0|\,\|\Lambda\|_{C(\bar\Omega)}.\end{aligned}$$

Choose now μ so large that

$$c_p\,\mu^{n/p}(\mu-1)^{-1}\leqq(4m+2)^{-1};$$

then by (A.10), (A.11) and (A.12) we conclude that

$$\begin{aligned}&(2m+1)^{-1}\left\{\sum_{k=0}^{2m-1}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k u\|_{C(\bar\Omega)}+|\lambda-\lambda_0|^{\frac{n}{2mp}}\sup_{x\in\bar\Omega}\|D^{2m}u\|_{L^p(\Omega(x,\rho))}\right\}\\ &\leqq|\lambda-\lambda_0|\,\Lambda(x_0)\leqq C_p\,|\lambda-\lambda_0|^{\frac{n}{2mp}}\left\{\|f\|_{L^p(\Omega(x_0,\mu\rho))}\right.\\ &\quad\left.+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda-\lambda_0|^{1-\frac{m_j+k}{2m}}\|D^k g_j\|_{L^p(\Omega(x_0,\mu\rho))}\right\}\\ &\quad+(4m+2)^{-1}\sum_{k=0}^{2m-1}|\lambda-\lambda_0|^{1-\frac{k}{2m}}\|D^k u\|_{C(\bar\Omega)},\end{aligned}$$

which clearly implies (1.12). The proof of Theorem 1.2 is complete. □

References

1. Acquistapace, P.: Existence and maximal time regularity for linear parabolic integrodifferential equations. J. Integral Equations **10**, 5–43 (1985) (Supplement)
2. Acquistapace, P., Terreni, B.: Some existence and regularity results for abstract non-autonomous parabolic equations. J. Math. Anal. Appl. **99**, 9–66 (1984)
3. Acquistapace, P., Terreni, B.: Characterization of Hölder and Zygmund classes as interpolation spaces. Pubbl. Dip. di Mat. Univ. Pisa, n. 61 (1984)
4. Acquistapace, P., Terreni, B.: A unified approach to abstract linear non-autonomous parabolic equations. Rend. Sem. Mat. Univ. Padova (to appear).
5. Adams, R.A.: Sobolev spaces. New York: Academic Press 1985
6. Agmon, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Commun. Pure Appl. Math. **15**, 119–147 (1962)
7. Amann, H.: Existence and regularity for semilinear parabolic evolution equations. Ann. Scuola Norm. Sup. Pisa (4) **11**, 593–676 (1984)
8. Baillon, J.B.: Caractére borné de certains générateurs de semigroups linéaires dans les espaces de Banach. C.R. Acad. Sci. Paris **290**, 757–760 (1980)
9. Da Prato, G., Grisvard, P.: Equations d'évolution abstraites non linéaires de type parabolique. Ann. Mat. Pura Appl. (4) **120**, 329–396 (1979)
10. Geymonat, G., Grisvard, P.: Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici. Rend. Sem. Mat. Univ. Padova **38**, 121–173 (1967)
11. Grisvard, P.: Equations différentielles abstraites. Ann. Sci. Ec. Norm. Super. (4) **2**, 311–395 (1969)
12. Grisvard, P.: Equations opérationnelles abstraites et problèmes aux limites dans les domaines non réguliers, Actes Congrés Inter. Math. (Nice 1970), **2**, 731–736. Paris: Gauthier-Villars 1971
13. Labbas, R., Terreni, B.: Somme d'opérateurs linéaires de type parabolique. Boll. Un. Mat. Ital. (to appear)
14. Lions, J.L., Peetre, J.: Sur une classe d'espaces d'interpolation. Inst. Hautes Études Sc. Publ. Math. **19**, 5–68 (1964)
15. Lunardi, A.: Interpolation spaces between domains of elliptic operators and spaces of continuous functions with applications to nonlinear parabolic equations. Math. Nachr. **121**, 295–318 (1985)
16. Lunardi, A.: Abstract quasilinear parabolic equations. Math. Ann. **267**, 395–415 (1984)
17. Lunardi, A., Sinestrari, E.: Fully nonlinear integrodifferential equations in general Banach space. Math. Z. **190**, 225–248 (1985)
18. Sinestrari, E.: On the abstract Cauchy problem of parabolic type in spaces of continuous functions. J. Math. Anal. Appl. **107**, 16–66 (1985)
19. Stein, E.M.: Singular integrals and differentiability properties of functions. Princeton: Princeton University Press 1970
20. Stewart, H.B.: Generation of analytic semigroups by strongly elliptic operators under general boundary conditions. Trans. Am. Math. Soc. **259**, 299–310 (1980)
21. Tanabe, H.: Equations of evolution. London/S. Francisco/Melbourne: Pitman 1979
22. Triebel, H.: Interpolation theory, function spaces, differential operators. Amsterdam/New York/Oxford: North-Holland 1978

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