# Chapter 1.3
# Iteration theory

In this chapter we begin to deal with the main argument of this book: iteration theory. As anticipated in the introduction, we shall mainly discuss hyperbolic Riemann surfaces, where the whole strength of Montel's theorem is available. The idea is that if $X$ is a hyperbolic Riemann surface and $f \in \mathrm{Hol}(X, X)$, then the sequence of iterates of $f$ is a normal family, and so its behavior cannot be chaotic. Indeed, it will turn out that if $f$ is not an automorphism then $\{f^k\}$ is either compactly divergent or converges, uniformly on compact sets, to a constant.

This is clearly the best result of this kind for a generic hyperbolic Riemann surface. But if $D \subset \widehat{X}$ is a hyperbolic domain, then we can say something more. In this case, in fact, $\mathrm{Hol}(D, D)$ is contained in $\mathrm{Hol}(D, \widehat{X})$, a space without compactly divergent sequences; therefore, the sequence of iterates of a function $f \in \mathrm{Hol}(D, D)$ is relatively compact in $\mathrm{Hol}(D, \widehat{X})$, and so it always has converging subsequences, converging possibly to a point of $\partial D$.

This observation (already anticipated in Proposition 1.1.46) leads to the core of this chapter: Heins' theorem, showing that if $D \subset \widehat{X}$ is a hyperbolic domain of regular type and $f \in \mathrm{Hol}(D, D)$ is not an automorphism, then the sequence of iterates of $f$ converges, uniformly on compact sets, to a constant $\tau \in \overline{D}$. The proof of Heins' theorem is divided in three parts. In the first section, we shall study in detail the sequence of iterates of a holomorphic function with a fixed point, using still another generalization of Schwarz's lemma (Theorem 1.3.4). In the second section we shall prove the Wolff-Denjoy theorem (that is Heins' theorem in $\Delta$), which has been the model along which the whole theory has developed (for instance, this book is the conclusion of our efforts to extend the Wolff-Denjoy theorem to several variables). Finally, in the third section we shall show how to generalize the tools used to prove the Wolff-Denjoy theorem — namely, the horocycles — to get Heins' theorem in its full strength.

Iteration theory is the most important but not the only argument discussed in this chapter. We shall complete the study of the relationship between commuting functions and fixed points, showing that two commuting functions in $\mathrm{Hol}(\Delta, \Delta)$ have a common fixed point (even if they have no fixed points… see Theorem 1.3.24), and in the meantime we shall also present Lindelöf's theorem, which will become quite important in the second part of this book.

## 1.3.1 *The fixed point case*

The aim of this first section is the study of the sequence of iterates of a function with a fixed point; our main tool will be another generalization of Schwarz's lemma. If $f \in \mathrm{Hol}(\Delta, \Delta)$ is such that $f(z_0) = z_0$ for some $z_0 \in \Delta$, then Corollary 1.1.4 yields $|f'(z_0)| \le 1$, with equality iff $f \in \mathrm{Aut}(\Delta)$. Well: this is true for any hyperbolic Riemann surface, and it will be of the greatest importance for us.

Our investigation begins with a characterization of the limit points of $\{f^k\}$. We need:

**Lemma 1.3.1:** *Let $X$ be a Riemann surface, and $f \in \mathrm{Hol}(X,X)$. Then $\mathrm{id}_X$ can be a limit point of $\{f^k\}$ only if $f \in \mathrm{Aut}(X)$.*

*Proof:* Obviously, $f$ is one-to-one. Take $z_0 \in X$: by Corollary 1.1.36, $z_0 \in f^{k_\nu}(X) \subset f(X)$ for all $\nu$ large enough, and so $f$ is onto, **q.e.d.**

A non-periodic automorphism $f$ of a Riemann surface $X$ such that $\mathrm{id}_X$ is a limit point of $\{f^k\}$ will be called *pseudoperiodic*. Now we may prove:

**Theorem 1.3.2:** *Let $X$ be a hyperbolic Riemann surface, and $f \in \mathrm{Hol}(X,X)$. Let $h \in \mathrm{Hol}(X,X)$ be a limit point of the sequence $\{f^k\}$. Then either*
(i) *$h$ is a constant $z_0 \in X$, or*
(ii) *$h$ is an automorphism of $X$. In this case, $f \in \mathrm{Aut}(X)$ too.*

*Proof:* Write $h = \lim\limits_{\nu\to\infty} f^{k_\nu}$, and set $m_\nu = k_{\nu+1} - k_\nu$. By Montel's Theorem 1.1.43, up to a subsequence we can assume that $\{f^{m_\nu}\}$ either converges to a holomorphic map $g$ or is compactly divergent. Suppose $h$ is not constant; then $h(X)$ is open in $X$. Now, for any $z \in X$ we have

$$\lim_{\nu\to\infty} f^{m_\nu}\bigl(f^{k_\nu}(z)\bigr) = \lim_{\nu\to\infty} f^{k_{\nu+1}}(z) = h(z);$$

therefore $\{f^{m_\nu}\}$ cannot be compactly divergent, and $g$ is the identity on the open subset $h(X)$ of $X$. Hence $g = \mathrm{id}_X$; by Lemma 1.3.1, $f$ is an automorphism. It remains to show that $h$ itself is an automorphism. But this follows immediately from Corollary 1.1.47, **q.e.d.**

For hyperbolic domains we can be slightly more precise:

**Corollary 1.3.3:** *Let $D \subset \widehat{X}$ be a hyperbolic domain, and $f \in \mathrm{Hol}(D,D)$. Let $h\colon D \to \widehat{X}$ be a limit point in $\mathrm{Hol}(D,\widehat{X})$ of the sequence $\{f^k\}$. Then either*
(i) *$h$ is a constant $z_0 \in \overline{D}$, or*
(ii) *$h$ is an automorphism of $D$. In this case, $f \in \mathrm{Aut}(D)$ too.*

*Proof:* It suffices to notice that if $h$ is not constant then it is contained in $\mathrm{Hol}(D,D)$, and then apply Theorem 1.3.2, **q.e.d.**

To state the generalization of Schwarz's lemma we mentioned before, we ought to define what we mean by derivative at a fixed point of a function defined on a Riemann surface. Let $X$ be a Riemann surface, and let $f \in \mathrm{Hol}(X,X)$ admit a fixed point $z_0 \in X$. Then the differential $df_{z_0}$ sends the complex tangent space $T_{z_0}X$ at $X$ in $z_0$ into itself; hence $df_{z_0}$ acts on $T_{z_0}X$ by multiplication by a complex number, that we shall denote by $f'(z_0)$. $f'(z_0)$ is sometimes called the *multiplier* of $f$ at the fixed point $z_0$. Clearly, if $X$ actually is a plane domain, $f'(z_0)$ is the usual derivative of $f$ at $z_0$.

And now we can prove:

**Theorem 1.3.4:** *Let $X$ be a hyperbolic Riemann surface, $f \in \mathrm{Hol}(X,X)$ and $z_0 \in X$ a fixed point of $f$. Then*
(i) $|f'(z_0)| \le 1$;
(ii) $f'(z_0) = 1$ *iff* $f = \mathrm{id}_X$;
(iii) $|f'(z_0)| = 1$ *iff* $f \in \mathrm{Aut}(X)$.

*Proof:* Let $\pi: \Delta \to X$ be the universal covering map of $X$. Since $\pi$ is a local isometry for the Poincaré distances, there is an open ball $B$ for $\omega_X$ centered about $z_0$ which is biholomorphic to $\Delta$. Since, by Theorem 1.1.40, $f(B) \subset B$, the Schwarz-Pick lemma immediately yields (i) and (ii). In particular, if $f \in \mathrm{Aut}(X)$ then (i) applied to $f$ and $f^{-1}$ yields $|f'(z_0)| = 1$.

Finally, assume $|f'(z_0)| = 1$. By Theorem 1.1.43, the sequence $\{f^k\}$ has a subsequence $\{f^{k_\nu}\}$ converging to a holomorphic function $h$. Obviously, $|h'(z_0)| = 1$; hence $h$ is not constant and, by Theorem 1.3.2, $f$ is an automorphism, **q.e.d.**

On the unit disk $\Delta$ we can find functions $f \in \mathrm{Hol}(\Delta,\Delta)$ with $f(0) = 0$ which are not automorphisms and with $|f'(0)|$ arbitrarily close to 1. Surprisingly, this is not true in multiply connected hyperbolic Riemann surfaces, as shown in the *Aumann-Carathéodory Starrheitssatz*:

**Corollary 1.3.5:** *Let $X$ be a multiply connected hyperbolic Riemann surface. Then for every $z_0 \in X$ we have*

$$\sup\bigl\{|f'(z_0)| \bigm| f \in \mathrm{Hol}(X,X),\ f \notin \mathrm{Aut}(X) \text{ and } f(z_0) = z_0\bigr\} < 1. \tag{1.3.1}$$

*Proof:* If $X = \Delta^*$, then every $f \in \mathrm{Hol}(\Delta^*,\Delta^*)$ is the restriction of a holomorphic function $\tilde f \in \mathrm{Hol}(\Delta,\Delta)$ such that $\tilde f(\Delta^*) \subset \Delta^*$. In particular, $\tilde f^{-1}(0)$ is either empty or $\{0\}$. Assume there exist a sequence $\{f_\nu\} \subset \mathrm{Hol}(\Delta^*,\Delta^*) \setminus \mathrm{Aut}(\Delta^*)$ and a point $z_0 \in \Delta^*$ such that $f_\nu(z_0) = z_0$ and $|f_\nu'(z_0)| \to 1$. We may assume that $f_\nu \to g \in \mathrm{Hol}(\Delta^*,\Delta^*)$. Obviously, $g(z_0) = z_0$ and $|g'(z_0)| = 1$. By Theorem 1.3.4, $g \in \mathrm{Aut}(\Delta^*)$; then $\tilde g \in \mathrm{Aut}(\Delta)$ fixes 0 and $z_0$, and thus $\tilde g = \mathrm{id}_\Delta$. So we have constructed a sequence $\tilde f_\nu \to \mathrm{id}_\Delta$ such that $\tilde f_\nu(z_0) = z_0$ and $\tilde f_\nu(\Delta^*) \subset \Delta^*$ for all $\nu \in \mathbf{N}$. By Corollary 1.1.36, $\tilde f_\nu(0) = 0$ for all sufficiently large $\nu$, and this implies $\tilde f_\nu = \mathrm{id}_\Delta$ eventually, contradiction.

Finally, assume $X$ not biholomorphic to $\Delta^*$. Suppose, by contradiction, there is a sequence $\{f_\nu\} \subset \mathrm{Hol}(X,X)\setminus\mathrm{Aut}(X)$ such that $f_\nu(z_0) = z_0$ for all $\nu \in \mathbf{N}$ and $|f_\nu'(z_0)| \to 1$. By Theorem 1.1.43, up to a subsequence we can assume that $\{f_\nu\}$ tends toward a holomorphic function $g: X \to X$. Obviously, $g(z_0) = z_0$ and $|g'(z_0)| = 1$; therefore, by Theorem 1.3.4, $g \in \mathrm{Aut}(X)$. Hence we have constructed a sequence of non-automorphisms converging toward an automorphism; by Corollary 1.2.24 this is impossible, **q.e.d.**

Using Theorem 1.3.4 we can completely describe the structure of the isotropy group of a point in a hyperbolic Riemann surface:

**Corollary 1.3.6:** *Let $X$ be a hyperbolic Riemann surface. Then either* $\mathrm{Aut}_{z_0}(X)$ *is finite cyclic for all $z_0 \in X$ (and $X$ is multiply connected) or* $\mathrm{Aut}_{z_0}(X)$ *is isomorphic to* $\mathbf{S}^1$ *for all $z_0 \in X$ (and $X$ is simply connected).*

*Proof:* Fix $z_0 \in X$, and define $D\colon \mathrm{Aut}_{z_0}(X) \to \mathbf{S}^1$ by $D(\gamma) = \gamma'(z_0)$. By Theorem 1.3.4, $D$ is a continuous injective homomorphism of $\mathrm{Aut}_{z_0}(X)$ into $\mathbf{S}^1$; since (Corollary 1.1.47) $\mathrm{Aut}_{z_0}(X)$ is compact, $D$ is an isomorphism of topological groups between $\mathrm{Aut}_{z_0}(X)$ and a closed subgroup of $\mathbf{S}^1$.

If $X$ is simply connected, we already know that $\mathrm{Aut}_{z_0}(X)$ is isomorphic to $\mathbf{S}^1$ for all $z_0 \in X$. If $X$ is multiply connected with non-abelian fundamental group, $\mathrm{Aut}_{z_0}(X)$ is discrete for all $z_0 \in X$ (Theorem 1.2.19), and hence finite cyclic. Finally, if $X$ is doubly connected Proposition 1.1.32 shows that $\mathrm{Aut}_{z_0}(X)$ is either cyclic of order 2 or trivial for all $z_0 \in X$, **q.e.d.**

Now we are ready to describe the behavior of the sequence of iterates of a function with a fixed point:

**Theorem 1.3.7:** *Let $X$ be a hyperbolic Riemann surface. Let $f \in \mathrm{Hol}(X,X)$ admit a fixed point $z_0 \in X$. Then either*
(i) $|f'(z_0)| < 1$ *and the sequence $\{f^k\}$ converges to $z_0$, or*
(ii) *$f$ is a periodic automorphism, or*
(iii) *$f$ is a pseudoperiodic automorphism. This latter possibility can occur only if $X$ is simply connected.*

*Proof:* By Theorem 1.3.4, $|f'(z_0)| \le 1$. If $|f'(z_0)| = 1$, $f$ is an automorphism, and the assertion follows from Corollary 1.3.6. So assume $|f'(z_0)| < 1$. Since $f$ has a fixed point, $\{f^k\}$ cannot have compactly divergent subsequences. Let $h_1$, $h_2$ be two limit points of $\{f^k\}$. Since $f \notin \mathrm{Aut}(X)$, by Theorem 1.3.2 both $h_1$ and $h_2$ are constant; but $z_0$ should be a fixed point for both $h_1$ and $h_2$, and so $h_1 \equiv h_2 \equiv z_0$. In other words, $z_0$ is the unique limit point of $\{f^k\}$, and $f^k \to z_0$, **q.e.d.**

For obvious reasons, a point $z_0 \in X$ such that $f(z_0) = z_0$ and $|f'(z_0)| < 1$ is called an *attractive fixed point* of $f$.

We end this section with a corollary we shall need later on:

**Corollary 1.3.8:** *Let $X$ be a hyperbolic Riemann surface, and assume there exists a function $f \in \mathrm{Hol}(X,X)$, $f$ not the identity, with two fixed points. Then $X$ is multiply connected and $f$ is a periodic automorphism of $X$.*

*Proof:* By Schwarz's lemma, $X$ must be multiply connected. If $f$ were not an automorphism, the sequence $\{f^k\}$ would have to converge to each of the distinct fixed points (by Theorem 1.3.7), impossible.

Now let $\pi\colon \Delta \to X$ be the universal covering map of $X$, and denote by $\Gamma \subset \mathrm{Aut}(\Delta)$ the automorphism group of the covering. Let $z_0$, $z_1 \in X$ be the fixed points of $f$, and choose $\tilde z_0$, $\tilde z_1 \in \Delta$ so that $\pi(\tilde z_j) = z_j$ for $j = 0, 1$. We can lift $f$ to an automorphism $\tilde f$ of $\Delta$ such that $\tilde f(\tilde z_0) = \tilde z_0$. Then there exists $\gamma_1 \in \Gamma$ such that

$$\tilde f(\tilde z_1) = \gamma_1(\tilde z_1). \tag{1.3.2}$$

We claim that for every $k \in \mathbf{N}$ there is $\gamma_k \in \Gamma$ such that $\tilde{f}^k(\tilde{z}_1) = \gamma_k(\tilde{z}_1)$. By induction on $k$. For $k = 1$ it is (1.3.2); so assume there is $\gamma_{k-1} \in \Gamma$ such that $\tilde{f}^{k-1}(\tilde{z}_1) = \gamma_{k-1}(\tilde{z}_1)$. Since $\tilde{f}$ is the lifting of a function in $\mathrm{Hol}(X, X)$, there is $\tilde{\gamma} \in \Gamma$ such that $\tilde{f} \circ \gamma_{k-1} = \tilde{\gamma} \circ \tilde{f}$. Then

$$\tilde{f}^k(\tilde{z}_1) = \tilde{f}\big(\gamma_{k-1}(\tilde{z}_1)\big) = \tilde{\gamma}\big(\tilde{f}(\tilde{z}_1)\big) = (\tilde{\gamma} \circ \gamma_1)(\tilde{z}_1),$$

and $\gamma_k = \tilde{\gamma} \circ \gamma_1$ is as claimed.

Now, $\{\tilde{f}^k(\tilde{z}_1)\} = \{\gamma_k(\tilde{z}_1)\}$ is contained in a compact subset of $\Delta$, for $\tilde{f}$ has a fixed point; hence it should be a finite set, for $\Gamma$ acts properly discontinuously on $\Delta$. It follows that $\tilde{f}^{k_0}(\tilde{z}_1) = \tilde{z}_1$ for some $k_0 \in \mathbf{N}$, and so $\tilde{f}$ (and hence $f$) is periodic, **q.e.d.**

Functions satisfying the hypotheses of Corollary 1.3.8 do exists; take for instance the doubly connected domain $D = \{z \in \mathbf{C} \mid 1/2 < |z| < 2\}$ and the function $f(z) = 1/z$.

Now we shall move on to fixed point free functions, the realm of Wolff's lemma.

## 1.3.2 *The Wolff-Denjoy theorem*

In this short (but important) section we shall study iteration theory in $\Delta$, proving the fundamental *Wolff-Denjoy theorem:*

**Theorem 1.3.9:** *Let $f \in \mathrm{Hol}(\Delta, \Delta)$, and assume $f$ is neither an elliptic automorphism nor the identity. Then the sequence of iterates $\{f^k\}$ converges, uniformly on compact sets, to the Wolff point $\tau \in \overline{\Delta}$ of $f$.*

*Proof:* If $f$ has a fixed point, the assertion follows from Theorem 1.3.7; so assume $f$ has no fixed points.

If $f$ is a parabolic automorphism, then transfering everything on $H^+$ it becomes clear that $f^k \to \tau(f)$, the unique fixed point of $f$. If $f$ is a hyperbolic automorphism, then, moving again to $H^+$, we can assume $f(w) = \lambda w$ for some $\lambda \in \mathbf{R}^+$, $\lambda \neq 1$. Then the Wolff point of $f$ is 0 or $\infty$ according to $\lambda < 1$ or $\lambda > 1$, and thus $f^k \to \tau(f)$.

So assume now that $f \notin \mathrm{Aut}(\Delta)$. If $h = \lim\limits_{\nu\to\infty} f^{k_\nu}$ is a limit point of $\{f^k\}$ in $\mathrm{Hol}(\Delta, \mathbf{C})$, by Corollary 1.3.3 $h$ is a constant $\tau \in \overline{\Delta}$. If $\tau$ were an interior point of $\Delta$, we would have

$$f(\tau) = \lim_{\nu\to\infty} f\big(f^{k_\nu}(\tau)\big) = \lim_{\nu\to\infty} f^{k_\nu}\big(f(\tau)\big) = \tau,$$

impossible; therefore $\tau \in \partial\Delta$.

We claim that $\tau = \tau(f)$. In fact, by Wolff's lemma for any $R > 0$ (and $\nu \in \mathbf{N}$) we have

$$f^{k_\nu}\big(E\big(\tau(f), R\big)\big) \subset E\big(\tau(f), R\big);$$

hence

$$\{\tau\} = h\big(E\big(\tau(f), R\big)\big) \subset \overline{E\big(\tau(f), R\big)} \cap \partial\Delta = \{\tau(f)\},$$

that is our claim. But thus $\tau(f)$ is the unique limit point of $\{f^k\}$, and we are done, **q.e.d.**

Therefore, beside the trivial case of elliptic automorphisms, a sequence of iterates always converges, and the limit is always a constant function. If $f \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point $z_0 \in \Delta$, then the result could be expected, for by Schwarz's lemma $f$ contracts the Poincaré disks centered at $z_0$. If $f$ has no fixed points, then again the result could be expected when $f$ has angular derivative strictly less than 1 at its Wolff point $\tau$, for in this case $f$ contracts the horocycles centered at $\tau$, by Wolff's lemma. The remarkable fact is that the theorem holds even if $f$ has angular derivative 1 at $\tau(f)$, when the horocycles are just turned around. Even more remarkable will be the generalization to hyperbolic domains of regular type, but we defer it to the next section.

If $f$ is an automorphism, we can say a bit more. If $f$ is elliptic, we know that $\{f^k\}$ does not converge. If $f$ is parabolic, $\{f^k\}$ converges to the unique fixed point in the boundary. If $f$ is hyperbolic, $\{f^k\}$ still converges to a fixed point in the boundary, but which one? The answer is

**Proposition 1.3.10:** *Let $f \in \mathrm{Aut}(\Delta)$ be hyperbolic. Then $\{f^k\}$ converges to the fixed point of $f$ farthest from $f^{-1}(0)$.*

*Proof:* Let $a = f^{-1}(0) \neq 0$, and write

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z},$$

for a suitable $\theta \in \mathbf{R}$. Let $\tau_1, \tau_2 \in \partial\Delta$ be the distinct fixed points of $f$, where $\tau_1$ is its Wolff point and $\tau_2$ is the other one. Now

$$\forall z \in \overline{\Delta} \qquad f'(z) = e^{i\theta} \frac{1 - |a|^2}{(1 - \overline{a}z)^2}.$$

We know that $|f'(\tau_1)| < 1 < |f'(\tau_2)|$, by Corollary 1.2.16 and Theorem 1.2.11; hence

$$|\tau_1 - a|^2 > 1 - |a|^2 > |\tau_2 - a|^2,$$

and the assertion follows from Theorem 1.3.9, **q.e.d.**

Due to the importance of the Wolff-Denjoy theorem in this book, we shall now describe a second, totally independent proof.

Let $f \in \mathrm{Hol}(\Delta, \Delta)$. If $f$ is an elliptic automorphism, then $\{f^k\}$ obviously does not converge (and $f$ is either periodic or pseudoperiodic). If $f$ is a hyperbolic or parabolic automorphism, we can transfer everything on $H^+$, and then it is evident that $\{f^k\}$ converges to a fixed point of $f$ in the boundary.

Assume now $f \notin \mathrm{Aut}(\Delta)$. Take two distinct points $z_1$, $z_2 \in \Delta$; then the sequence $\{\omega\big(f^k(z_1), f^k(z_2)\big)\}$ is either eventually zero or strictly decreasing.

Let $h = \lim\limits_{\nu\to\infty} f^{k_\nu}$ be a limit point of $\{f^k\}$. Assume first that $h \in \mathrm{Hol}(\Delta, \Delta)$; we claim that $h$ is constant. Indeed, take $z_1$, $z_2 \in \Delta$ such that $h(z_1) \neq h(z_2)$, if possible. Let $\delta$ be the limit (that we know to exist) of the sequence $\{\omega\big(f^k(z_1), f^k(z_2)\big)\}$; obviously, $\delta = \omega\big(h(z_1), h(z_2)\big) > 0$. But, since $f \notin \mathrm{Aut}(\Delta)$, we have

$$\omega\Big(f\big(h(z_1)\big), f\big(h(z_2)\big)\Big) < \omega\big(h(z_1), h(z_2)\big)$$

and

$$\delta = \omega\big(h(z_1), h(z_2)\big) > \omega\big(f\big(h(z_1)\big), f\big(h(z_2)\big)\big) = \lim_{\nu\to\infty} \omega\big(f^{k_\nu+1}(z_1), f^{k_\nu+1}(z_2)\big) = \delta,$$

contradiction. So $h$ is constant, say $h \equiv \tau$. Now

$$\omega\big(f(\tau), \tau\big) = \lim_{\nu\to\infty} \omega\big(f\big(f^{k_\nu}(\tau)\big), f^{k_\nu}(\tau)\big) = \lim_{\nu\to\infty} \omega\big(f^{k_\nu}\big(f(\tau)\big), f^{k_\nu}(\tau)\big) = 0.$$

Therefore $\tau$ is the unique fixed point of $f$; hence for all $z \in \Delta$

$$\lim_{k\to\infty} \omega\big(\tau, f^k(z)\big) = \lim_{k\to\infty} \omega\big(f^k(\tau), f^k(z)\big) = \lim_{\nu\to\infty} \omega\big(f^{k_\nu}(\tau), f^{k_\nu}(z)\big) = 0,$$

for the sequence $\big\{\omega\big(f^k(\tau), f^k(z)\big)\big\}$ is decreasing, and thus $f^k \to \tau$.

To end the proof, assume then that $\{f^k\}$ has no limit points in $\mathrm{Hol}(\Delta, \Delta)$; therefore every limit point should be a constant in $\partial\Delta$. In particular, $f$ cannot have fixed points.

We claim that for every $z_0 \in \Delta$ there is a subsequence $\{f^{k_\nu}\}$ such that

$$\forall \nu \in \mathbf{N} \qquad \big|f\big(f^{k_\nu}(z_0)\big)\big| > |f^{k_\nu}(z_0)|. \tag{1.3.3}$$

Indeed, if this is not true, there should exist $k_1 \in \mathbf{N}$ such that

$$\forall k \geq k_1 \qquad \big|f\big(f^{k}(z_0)\big)\big| \leq |f^{k}(z_0)|.$$

Then

$$\forall k > k_1 \quad |f^k(z_0)| = \big|f\big(f^{k-1}(z_0)\big)\big| \leq |f^{k-1}(z_0)| \leq \ldots \leq |f^{k_1}(z_0)|,$$

and $\{f^k\}$ cannot have limit points in $\partial\Delta$.

So let $\{f^{k_\nu}\}$ be a subsequence satisfying (1.3.3) for a given $z_0 \in \Delta$; we may assume that $f^{k_\nu} \to \tau \in \partial\Delta$. Let $z_\nu = f^{k_\nu}(z_0)$; then

$$\lim_{\nu\to\infty} z_\nu = \tau = \lim_{\nu\to\infty} f^{k_\nu}\big(f(z_0)\big) = \lim_{\nu\to\infty} f(z_\nu).$$

Hence we can apply Julia's lemma with $\alpha \leq 1$; in other words, $f$ contracts the horocycles centered in $\tau$. But then the argument already used at the beginning of the proof of Wolff's lemma shows that, since $f$ has no fixed points, $\tau$ is univoquely determined. Hence $\tau$ is the unique limit point of $\{f^k\}$, that is $f^k \to \tau$, and we are done.

### 1.3.3 *Hyperbolic Riemann surfaces and multiply connected domains*

In this section we shall study the iterates of a function $f$ defined on a multiply connected hyperbolic domain or, more generally, on a hyperbolic Riemann surface. We begin with a lemma about automorphisms without fixed points:

**Lemma 1.3.11:** *Let $X$ be a hyperbolic Riemann surface, and let $f \in \mathrm{Aut}(X)$ be without fixed points. Assume that $\{f^k\}$ is not compactly divergent. Then $X$ is multiply connected and either $f$ is periodic or $f$ is pseudoperiodic and the closure of $\{f^k\}$ is the connected component at the identity of $\mathrm{Aut}(X)$, which is isomorphic to $\mathbf{S}^1$. Furthermore, the latter possibility can occur only if $X$ is doubly connected.*

*Proof:* If $X$ is simply connected, the sequence of iterates of $f$ is compactly divergent, by Theorem 1.3.9; hence $X$ is multiply connected.

If $X$ is not doubly connected, let $\{f^{k_\nu}\}$ be a converging subsequence, necessarily to an element of $\mathrm{Aut}(X)$, by Corollary 1.1.47. Now, $\mathrm{Aut}(X)$ is discrete, by Theorem 1.2.19; hence the sequence $\{f^{k_\nu}\}$ must contain only a finite number of distinct elements, and so $f$ is periodic.

Finally, if $X$ is doubly connected we can realize it as an annulus $A(r,1)$ with $0 \le r < 1$. Hence $f$ should be a rotation around the origin, and the assertion follows immediately, **q.e.d.**

Then we have a Wolff-Denjoy theorem for hyperbolic Riemann surfaces:

**Theorem 1.3.12:** *Let $X$ be a hyperbolic Riemann surface, and let $f \in \mathrm{Hol}(X,X)$. Then either:*

(i) *$f$ has an attractive fixed point in $X$, or*
(ii) *$f$ is a periodic automorphism, or*
(iii) *$f$ is a pseudoperiodic automorphism, or*
(iv) *the sequence $\{f^k\}$ is compactly divergent.*

*Furthermore, the case* (iii) *can occur only if $X$ is either simply connected (and $f$ has a fixed point) or doubly connected (and $f$ has no fixed points).*

*Proof:* If the sequence $\{f^k\}$ is compactly divergent there is nothing to prove. So assume $\{f^k\}$ is not compactly divergent; in particular, there is a subsequence $\{f^{k_\nu}\}$ converging to a holomorphic function $h \in \mathrm{Hol}(X,X)$. If $f \in \mathrm{Aut}(X)$, the assertion follows from Theorem 1.3.7 and Lemma 1.3.11. If $f$ is not an automorphism, by Theorem 1.3.2 $h$ is constant, $h \equiv z_0 \in X$, say. But then $z_0$ is a fixed point of $f$, and so the assertion follows from Theorem 1.3.7, **q.e.d.**

If $X$ is compact, Theorem 1.3.12 drastically simplifies, becoming:

**Corollary 1.3.13:** *Let $X$ be a compact hyperbolic Riemann surface. Then every function $f \in \mathrm{Hol}(X,X)$ is either constant or a periodic automorphism.*

*Proof:* Assume $f$ non constant. Then $f(X)$ is open and closed in $X$, and thus $f$ is surjective. But then the sequence $\{f^k\}$ can neither be compactly divergent (for $X$ is compact) nor converge uniformly to a point in $X$. Hence, by Theorem 1.3.12 and Corollary 1.2.22, $f$ is a periodic automorphism, **q.e.d.**

Another consequence is the generalization of Corollary 1.1.34, *Ritt's theorem:*

**Proposition 1.3.14:** *Let $X$ be a non-compact Riemann surface, and $f \in \mathrm{Hol}(X,X)$ such that $f(X)$ is relatively compact in $X$. Then $f$ has an attractive fixed point $z_0 \in X$.*

*Proof:* Indeed, $f$ is not an automorphism and $\{f^k\}$ is not compactly divergent, **q.e.d.**

For a generic hyperbolic Riemann surface Theorem 1.3.12 is the best statement we can hope for; on the other hand, for a hyperbolic domain we can do something better. Indeed, let $D \subset \widehat{X}$ be a hyperbolic domain, and take $f \in \mathrm{Hol}(D,D)$. If the sequence $\{f^k\}$ is compactly divergent, by Corollary 1.3.3 every limit point of $\{f^k\}$ should be a constant belonging to $\partial D$. In this case we shall say that $\{f^k\}$ *converges to the boundary*, and Theorem 1.3.12 becomes:

**Theorem 1.3.15:** *Let $D \subset \widehat{X}$ be a hyperbolic domain, and take $f \in \mathrm{Hol}(D,D)$. Then either*
(i) *$f$ has an attractive fixed point in $D$, or*
(ii) *the sequence $\{f^k\}$ converges to the boundary of $D$, and the set of limit points is closed and connected, or*
(iii) *$f$ is a periodic automorphism, or*
(iv) *$f$ is a pseudoperiodic automorphism. This latter possibility can occur only if $D$ is either simply or doubly connected.*

*Proof:* The only thing left to prove is that the set $L$ of limit points of a function as in case (ii) is connected. Choose a connected compact subset $K$ of $D$ such that $f(K) \cap K \neq \emptyset$. Then clearly $f^{k+1}(K) \cap f^k(K) \neq \emptyset$ for all $k \in \mathbf{N}$. It follows that both the set

$$L_\nu = \bigcup_{k=\nu}^{\infty} f^k(K)$$

and its closure are connected for all $\nu \in \mathbf{N}$. But now $L = \bigcap\limits_{\nu=1}^{\infty} \overline{L_\nu}$, and so it is closed and connected, **q.e.d.**

Now we want to show that if, roughly speaking, the boundary of $D$ is not too wild, the set $L$ of limit points in case (ii) actually reduces to a point, and then the sequence of iterates converges. A first case is when the boundary is totally disconnected:

**Corollary 1.3.16:** *Let $D \subset \widehat{X}$ be a hyperbolic domain such that $\partial D$ is totally disconnected. Let $f \in \mathrm{Hol}(D,D)$. Then either $f$ is a periodic automorphism or the sequence of iterates converges to a point of $\overline{D}$.*

*Proof:* Under these assumptions on the boundary, $D$ cannot be either simply or doubly connected. Hence the assertion follows from Theorem 1.3.15, **q.e.d.**

A second, quite more important case is when $D$ is of regular type. To properly deal with this situation we must somehow replace the horocycles.

First of all, we need a lemma on the Poincaré distance:

**Lemma 1.3.17:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type, $\pi\colon \Delta \to D$ its universal covering map, and $\Gamma \subset \mathrm{Aut}(\Delta)$ the automorphism group of the covering. Choose $\tau_0$ contained in a Jordan component of $\partial D$ and a neighbourhood $U$ in $\overline{\Delta}$ of a point $\tilde{\tau}_0 \in \partial\Delta$ such that $\pi$ extends to a homeomorphism between a neighbourhood (in $\overline{\Delta}$) of $\overline{U}$ and its image such that $\pi(\tilde{\tau}_0) = \tau_0$. Then for every $z_0 \in \Delta$ there is a finite subset $\Gamma_0$ of $\Gamma$ such that*

$$\forall w \in U \qquad \omega_D\bigl(\pi(z_0), \pi(w)\bigr) = \min_{\gamma\in\Gamma_0} \omega\bigl(\gamma(z_0), w\bigr).$$

*Proof:* Since $\Gamma$ acts properly discontinuously on $\Delta$, for every $w \in U\cap\Delta$ we can find $\gamma_w \in \Gamma$ so that $\omega_D\bigl(\pi(z_0), \pi(w)\bigr) = \omega\bigl(\gamma_w(z_0), w\bigr)$. Let $Z = \{\gamma_w(z_0) \mid w \in U \cap \Delta\}$; it suffices to show that $\overline{Z} \cap \partial\Delta = \emptyset$, again because $\Gamma$ acts properly discontinuously on $\Delta$. Assume, by contradiction, there is a sequence $\{w_\nu\} \subset U \cap \Delta$ such that $z_\nu = \gamma_{w_\nu}(z_0) \to \sigma_0 \in \partial\Delta$; up to a subsequence we can assume $w_\nu \to \sigma_1 \in \overline{U}$.

First of all, since $\pi$ is injective in a neighbourhood of $\overline{U}$, $\sigma_0$ cannot belong to $\overline{U}$; in particular, $\sigma_0 \neq \sigma_1$. Furthermore, since $\omega(z_\nu, w_\nu) = \omega_D\bigl(\pi(z_0), \pi(w_\nu)\bigr)$, $\sigma_1$ must belong to $\partial\Delta$. Now,

$$\omega(z_\nu, w_\nu) - \omega(0, w_\nu) = \tfrac{1}{2}\log\left[\frac{|1-\bar{z}_\nu w_\nu|^2}{1-|z_\nu|^2}\cdot\left(\frac{1+|\gamma_{z_\nu}(w_\nu)|}{1+|w_\nu|}\right)^2\right]. \tag{1.3.4}$$

Letting $\nu \to +\infty$, the right-hand side of (1.3.4) diverges; on the other hand, the left-hand side is bounded by $\omega_D\bigl(\pi(z_0), \pi(0)\bigr)$, contradiction, **q.e.d.**

The idea is to define horocycles in hyperbolic domains of regular type using something like Proposition 1.2.2. The main step is

**Proposition 1.3.18:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type; fix $z_0 \in D$. Then for every $\tau_0 \in \partial D$ contained in a Jordan component of $\partial D$ and for every $z \in X$ the limit*

$$\lim_{w\to\tau_0} \bigl[\omega_D(z, w) - \omega_D(z_0, w)\bigr]$$

*exists and is finite.*

*Proof:* Let $\pi\colon \Delta \to D$ be the universal covering map of $D$, and denote by $\Gamma \subset \mathrm{Aut}(\Delta)$ the automorphism group of the covering. Since $\tau_0$ belongs to a Jordan component of $\partial D$, we can find (Theorem 1.1.57) a point $\tilde{\tau}_0 \in \partial\Delta$ and a neighbourhood $U$ of $\tilde{\tau}_0$ in $\overline{\Delta}$ such that $\pi$ extends to a homeomorphism of a neighbourhood of $\overline{U}$ with its image such that $\pi(\tilde{\tau}_0) = \tau_0$.

Choose $z \in D$, and fix $\tilde{z} \in \pi^{-1}(z)$ and $\tilde{z}_0 \in \pi^{-1}(z_0)$. By Lemma 1.3.17, there are two finite subsets $\Gamma_z$ and $\Gamma_0$ of $\Gamma$ such that for every $\tilde{w} \in U$ we have

$$\omega_D\bigl(z, \pi(\tilde{w})\bigr) - \omega_D\bigl(z_0, \pi(\tilde{w})\bigr) = \min_{\gamma\in\Gamma_z}\max_{\gamma_0\in\Gamma_0}\bigl\{\omega\bigl(\tilde{z}, \gamma(\tilde{w})\bigr) - \omega\bigl(\tilde{z}_0, \gamma_0(\tilde{w})\bigr)\bigr\}. \tag{1.3.5}$$

Now take $\gamma \in \Gamma_z$ and $\gamma_0 \in \Gamma_0$, and set $a = \gamma^{-1}(0)$ and $a_0 = \gamma_0^{-1}(0)$. Then

$$\begin{aligned}\lim_{\tilde{w}\to\tilde{\tau}_0} &\bigl[\omega\bigl(\tilde{z}, \gamma(\tilde{w})\bigr) - \omega\bigl(\tilde{z}_0, \gamma_0(\tilde{w})\bigr)\bigr] \\ &= \tfrac{1}{2}\log\left[\frac{|a-\tilde{\tau}_0|^2}{1-|a|^2}\cdot\frac{1-|a_0|^2}{|a_0-\tilde{\tau}_0|^2}\cdot\frac{|\tilde{z}-\gamma(\tilde{\tau}_0)|^2}{1-|\tilde{z}|^2}\cdot\frac{1-|\tilde{z}_0|^2}{|\tilde{z}_0-\gamma_0(\tilde{\tau}_0)|^2}\right],\end{aligned} \tag{1.3.6}$$

and so (1.3.5) and (1.3.6) yield the assertion, for $\tilde{w} \to \tilde{\tau}_0$ iff $\pi(\tilde{w}) \to \tau_0$, **q.e.d.**

Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type, and fix $z_0 \in D$ and a point $\tau_0 \in \partial D$ contained in a Jordan component of $\partial D$. Then the *horocycle* $E_{z_0}(\tau_0, R)$ of *center* $\tau_0$, *pole* $z_0$ and *radius* $R > 0$ is given by

$$E_{z_0}(\tau_0, R) = \bigl\{z \in D \bigm| \lim_{w \to \tau_0} \bigl[\omega_D(z, w) - \omega_D(z_0, w)\bigr] < \tfrac{1}{2} \log R\bigr\};$$

note that, by Proposition 1.2.2, if we take $D = \Delta$ and $z_0 = 0$ we get the classical horocycles.

The proof of our main theorem requires only one feature of the horocycles:

**Lemma 1.3.19:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type, fix $z_0 \in D$ and choose $\tau_0 \in \partial D$ contained in a Jordan component of $\partial D$. Then*

$$\forall R > 0 \qquad \overline{E_{z_0}(\tau_0, R)} \cap \partial D = \{\tau_0\}.$$

*Proof:* First of all, $\tau_0 \in \overline{E_{z_0}(\tau_0, R)}$. Indeed, let $\pi\colon \Delta \to D$ be the universal covering map of $D$. Since $\pi$ is a local isometry, it is easy to check that if $\tilde{\sigma}\colon [0, 1) \to \Delta$ is a geodesic for $\omega$ then $\sigma = \pi \circ \tilde{\sigma}$ is a geodesic for $\omega_D$, that is

$$\forall s,\, t \in [0, 1) \qquad \omega_D\bigl(\sigma(s), \sigma(t)\bigr) = \omega(s, t).$$

Now, since $\tau_0$ is in a Jordan component, we can find a point $\tilde{\tau}_0 \in \partial\Delta$ such that $\pi$ extends continuously to a neighbourhood (in $\overline{\Delta}$) of $\tilde{\tau}_0$ with $\pi(\tilde{\tau}_0) = \tau_0$. Choose a point $\tilde{z}_0 \in \pi^{-1}(z_0)$, and let $\tilde{\sigma}_0$ be the geodesic in $\Delta$ connecting $\tilde{z}_0$ with $\tilde{\tau}_0$. Then $\sigma_0 = \pi \circ \tilde{\sigma}_0$ is a geodesic for $\omega_D$ connecting $z_0$ and $\tau_0$; in particular,

$$\lim_{w \to \tau_0} \bigl[\omega_D(z, w) - \omega_D(z_0, w)\bigr] = \lim_{t \to 1} \bigl[\omega_D\bigl(z, \sigma_0(t)\bigr) - \omega(0, t)\bigr]$$

for every $z \in D$. Now set $z_\nu = \sigma_0(1 - 1/\nu)$. Then $z_\nu \to \tau_0$ and

$$\lim_{w \to \tau_0} \bigl[\omega_D(z_\nu, w) - \omega_D(z_0, w)\bigr] = -\omega(0, 1 - 1/\nu) \longrightarrow -\infty$$

as $\nu \to +\infty$, showing that $\tau_0 \in \overline{E_{z_0}(\tau_0, R)}$ for every $R > 0$.

Finally, let $\tau_1 \in \overline{E_{z_0}(\tau_0, R)} \cap \partial D$; we have to prove that $\tau_1 = \tau_0$. Choose a sequence $\{z_\nu\} \subset E_{z_0}(\tau_0, R)$ converging to $\tau_1$. Now, by Lemma 1.3.17 (and Theorem 1.1.57) there is a $\delta < 1$ such that for every $\nu \in \mathbf{N}$ we can find a finite subset $Z_\nu$ of $\pi^{-1}(z_\nu)$ so that

$$\forall t \in (\delta, 1) \qquad \omega_X\bigl(z_\nu, \sigma_0(t)\bigr) = \min_{\tilde{z} \in Z_\nu} \omega\bigl(\tilde{z}, \tilde{\sigma}_0(t)\bigr).$$

In particular, by (1.3.6) used with $\gamma = \gamma_0 = \mathrm{id}_\Delta$

$$\begin{aligned}\min_{\tilde{z} \in Z_\nu} \tfrac{1}{2} \log\left(\frac{|\tilde{z} - \tilde{\tau}_0|^2}{1 - |\tilde{z}|^2} \cdot \frac{1 - |\tilde{z}_0|^2}{|\tilde{z}_0 - \tilde{\tau}_0|^2}\right) &= \min_{\tilde{z} \in Z_\nu} \lim_{t \to 1} \bigl[\omega\bigl(\tilde{z}, \tilde{\sigma}_0(t)\bigr) - \omega(0, t)\bigr] \\ &= \lim_{t \to 1} \bigl[\omega_D\bigl(z_\nu, \sigma_0(t)\bigr) - \omega(0, t)\bigr] \le \tfrac{1}{2} \log R.\end{aligned} \tag{1.3.7}$$

Now for every $\nu \in \mathbf{N}$ choose $\tilde{z}_\nu \in Z_\nu$ realizing the minimum of the left-hand side of (1.3.7). Since $z_\nu \to \tau_1 \in \partial D$, up to a subsequence we can assume $\tilde{z}_\nu \to \tilde{\tau}_1 \in \partial\Delta$. But then (1.3.7) forces $\tilde{\tau}_1 = \tilde{\tau}_0$, and so $\tau_1 = \tau_0$, **q.e.d.**

We do not dwell anymore on the properties of horocycles (we shall thoroughly study them in the second part of this book), because we are finally ready to prove the main theorem of one-variable iteration theory, *Heins' theorem:*

**Theorem 1.3.20:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type, and take a function $f \in \mathrm{Hol}(D, D)$. Then either*
(i) *the sequence $\{f^k\}$ converges to a point $z_0 \in \overline{D}$, and, if $D$ is not simply connected, $f$ is not an automorphism, or*
(ii) *$f$ is a periodic automorphism, or*
(iii) *$f$ is a pseudoperiodic automorphism. This latter possibility can occur only if $D$ is either simply or doubly connected.*

*Proof:* If $D$ is simply connected, it is bounded by a Jordan curve and the universal covering map of $\widehat{X}$ gives a biholomorphism of $D$ with a simply connected plane domain bounded by a Jordan curve, and this biholomorphism is continuous up to the boundary; therefore, quoting Theorem 1.1.28, we can apply the Wolff-Denjoy theorem. Hence we can assume $D$ multiply connected.

If $D$ is not doubly connected, by Theorem 1.2.25 $\mathrm{Aut}(D)$ is finite, and thus every $f \in \mathrm{Aut}(D)$ is periodic. If $D$ is doubly connected, we know by Proposition 1.1.32 that every $f \in \mathrm{Aut}(D)$ is either periodic or pseudoperiodic. In conclusion, by Theorem 1.3.15 it suffices to show that if $\{f^k\}$ converges to the boundary, then the limit point set $L$ contains just one point.

If $\partial D$ has no Jordan components, then $\partial D$ is totally disconnected, and so we can apply Corollary 1.3.16. Assume then $\partial D$ has at least one Jordan component. Then, by the Big Picard Theorem 1.1.51, every $f \in \mathrm{Hol}(D, D)$ extends holomorphically across the point components of $\partial D$. This extension cannot be an automorphism, for $f$ is not; hence, by Theorem 1.3.15, either it has an attractive fixed point (which is necessarily a point component of $\partial D$), or its sequence of iterates is still compactly divergent. In the former case, $\{f^k\}$ converges to a point component of $\partial D$; therefore to finish the proof we can also assume that $D$ is bounded by a finite number of disjoint Jordan curves.

Fix $z_0 \in D$; since $\{f^k\}$ is compactly divergent, $\omega_D\big(f^k(z_0), z_0\big) \to +\infty$ as $k \to +\infty$. Now, we can find a subsequence $\{f^{k_\nu}\}$ such that

$$\forall \nu \in \mathbf{N} \qquad \omega_D\big(f^{k_\nu}(z_0), z_0\big) < \omega_D\big(f^{k_\nu+1}(z_0), z_0\big). \tag{1.3.8}$$

Indeed, let $k_\nu$ denote the largest integer $k$ satisfying $\omega_D\big(f^k(z_0), z_0\big) \le \nu$. Then

$$\forall \nu \in \mathbf{N} \qquad \omega_D\big(f^{k_\nu}(z_0), z_0\big) \le \nu < \omega_D\big(f^{k_\nu+1}(z_0), z_0\big),$$

as claimed.

Up to a subsequence, we can assume that $f^{k_\nu}(z_0) \to \tau_0 \in \partial D$. Now for every $z \in D$ we have

$$\omega_D\big(f^{k_\nu+1}(z_0), f(z)\big) \le \omega_D\big(f^{k_\nu}(z_0), z\big).$$

Therefore, by (1.3.8),

$$\omega_D\big(f(z), f^{k_\nu+1}(z_0)\big) - \omega_D\big(z_0, f^{k_\nu+1}(z_0)\big) \le \omega_D\big(z, f^{k_\nu}(z_0)\big) - \omega_D\big(z_0, f^{k_\nu}(z_0)\big). \tag{1.3.9}$$

Clearly, we have $\omega_D\big(f^{k+1}(z_0), f^k(z_0)\big) \le \omega_D\big(f(z_0), z_0\big)$ for all $k \in \mathbf{N}$; so Proposition 1.1.59 implies that $f^{k_\nu+1}(z_0) \to \tau_0$ as $\nu \to +\infty$. Then we can take the limit as $\nu \to +\infty$ in (1.3.9) obtaining, by Proposition 1.3.18,

$$\forall R > 0 \qquad\qquad f\big(E_{z_0}(\tau_0, R)\big) \subset E_{z_0}(\tau_0, R).$$

It follows that the limit point set $L$ of $\{f^k\}$ is contained in $\overline{E_{z_0}(\tau_0, R)} \cap \partial D = \{\tau_0\}$, by Lemma 1.3.19, and the theorem is proved, **q.e.d.**

We have reached the highest peak of one-variable iteration theory. We have earned a bit of relax; so we end the section with an easy corollary, generalizing Proposition 1.2.23:

**Corollary 1.3.21:** *Let $D \subset \widehat{X}$ be a multiply connected hyperbolic domain of regular type without point components, and realize its fundamental group as a subgroup $\Gamma$ of $\mathrm{Aut}(\Delta)$. Take $f \in \mathrm{Hol}(D, D)$. Then the following statements are equivalent:*
(i) *$f \notin \mathrm{Aut}(D)$;*
(ii) *$f_*\colon \pi_1(D) \to \pi_1(D)$ is nilpotent;*
(iii) *the iterates $\tilde{f}^k$ of any lifting $\tilde{f}$ of $f$ are automorphic under $\Gamma$ for all sufficiently large $k$.*

*Proof:* We already know (Proposition 1.1.21) that (ii) $\iff$ (iii). If $f \in \mathrm{Aut}(D)$, $f_*$ cannot be nilpotent; hence (ii) $\Longrightarrow$ (i). Finally, assume $f \notin \mathrm{Aut}(D)$. Then, by Theorem 1.3.20, the sequence of iterates of $f$ converges to a point of $\overline{D}$. In particular, if $K$ is a compact subset of $D$, there is a large enough $k \in \mathbf{N}$ such that $f^k(K)$ is contained in a contractible subset of $D$.

Now take an element $[\sigma]$ of $\pi_1(D)$. By the previous observation, there is a sufficiently large $k \in \mathbf{N}$ such that $(f_*)^k[\sigma] = [f^k \circ \sigma]$ is trivial. Since $\pi_1(D)$ is finitely generated (by Lemma 1.1.53), this implies that $(f_*)^k$ is trivial for a sufficiently large $k \in \mathbf{N}$, **q.e.d.**

This is not true for generic domains of regular type: take for instance $D = \Delta^*$ and $f(z) = z/2$. Then $f \notin \mathrm{Aut}(\Delta^*)$ but $f_* = \mathrm{id}$.

Corollary 1.3.21 reveals an interesting fact: roughly speaking, holomorphic functions do not like topological complications. Besides the automorphisms (which are finite in number), every other $f \in \mathrm{Hol}(D, D)$ gets rid of topological obstructions in a finite number of steps. This may be another reason for the predominance of simply connected domains in function theory of one complex variable.

## 1.3.4 *Common fixed points*

In this section we shall bring to its natural conclusion the *ménage à trois* between iteration theory, commuting functions and common fixed points. We already saw in two different occasions (Proposition 1.1.13 and Theorem 1.2.18) that particular kinds of commuting functions must have a common fixed point, and we used these facts to prove several results eventually leading to Heins' theorem. Now we shall close the circle proving, by means of iteration theory, *Shields' theorem:*

**Theorem 1.3.22:** *Let $D \subset \widehat{X}$ be a hyperbolic domain of regular type, and $\mathcal{F}$ a family of continuous self-maps of $\overline{D}$ which are holomorphic in $D$ and commute with each other under composition. Assume either*
(i) *$D$ is simply connected, or*
(ii) *$D$ is multiply connected and $\mathcal{F}$ is not contained in* $\mathrm{Aut}(D)$.
*Then $\mathcal{F}$ has a fixed point, that is there exists $\tau \in \overline{D}$ such that $f(\tau) = \tau$ for all $f \in \mathcal{F}$.*

*Proof:* We may obviously assume that $\mathrm{id}_D \notin \mathcal{F}$. If $\mathcal{F}$ contains a constant function, then this constant is fixed by every element of $\mathcal{F}$, due to the commutativity. Hence we may also assume that $\mathcal{F}$ contains no constant functions. By the open map theorem, then, every $f \in \mathcal{F}$ sends $D$ into itself.

Now take $f \in \mathcal{F}$, assuming, in case (ii), that $f$ is not an automorphism. If $f$ has a fixed point $z_0 \in D$, then $z_0$ is unique (for $f \neq \mathrm{id}_D$ in case (i), and because $f$ is not an automorphism in case (ii); see Corollary 1.3.8). If $g$ is another element of $\mathcal{F}$, we have $f\bigl(g(z_0)\bigr) = g\bigl(f(z_0)\bigr) = g(z_0)$; hence $g(z_0) = z_0$ and $z_0$ is a fixed point of $\mathcal{F}$.

Finally, if $f$ has no fixed points in $D$, by Heins' theorem the sequence of iterates of $f$ converges to a point $\tau \in \partial D$. Hence, if $g \in \mathcal{F}$ we have

$$g(\tau) = \lim_{k\to\infty} g \circ f^k = \lim_{k\to\infty} f^k \circ g = \tau,$$

and $\tau$ is a fixed point of $\mathcal{F}$, **q.e.d.**

If $D$ is multiply connected and $\mathcal{F}$ is contained in $\mathrm{Aut}(D)$, $\mathcal{F}$ can have no fixed points. For instance, take $D = A(r,1)$ with $r \in (0,1)$, $f(z) = r/z$, $g(z) = -z$ and $\mathcal{F} = \{f, g\}$.

In Theorem 1.3.22 the continuity at the boundary is necessary just to take care of functions without fixed points in $D$. In the unit disk $\Delta$ we have developed a lot of material about boundary behavior of functions without fixed points; using it, we can remove the hypothesis of boundary continuity, with just a bit more effort.

We start with a digression. In section 1.2.1 we often saw that the existence of a non-tangential limit was forced by some kind of weak assumptions about radial behavior. This is a general fact, as shown in *Lindelöf's theorem:*

**Theorem 1.3.23:** *Let $\gamma\colon [0,1) \to \Delta$ be a continuous curve such that $\gamma(t) \to \sigma \in \partial\Delta$ as $t \to 1$. If $f \in \mathrm{Hol}(\Delta, \Delta)$ is such that*

$$\lim_{t\to 1} f\bigl(\gamma(t)\bigr) = \tau \in \overline{\Delta}$$

*exists, then $f$ has non-tangential limit $\tau$ at $\sigma$.*

*Proof:* Without loss of generality we can suppose $\sigma = 1$ and, up to replacing $f$ by $(f-\tau)/2$, we can also assume $\tau = 0$.

We begin transfering everything to $H^+$, via the Cayley transform. Then we have $f \in \mathrm{Hol}(H^+, \Delta)$ and a curve $\gamma\colon [0,1) \to H^+$ such that $\gamma(t) \to \infty$ and $f\bigl(\gamma(t)\bigr) \to 0$ as $t \to 1$, and we claim that $f(w) \to 0$ as $w \to \infty$ within $K_\varepsilon = \{w \in H^+ \mid \mathrm{Im}\, w > \varepsilon|w|\}$

for all $\varepsilon \in (0,1)$. However, it will be convenient to change the stage once again: let $\Sigma = \{z \in \mathbf{C} \mid |\operatorname{Re} z| < 1\}$ be an infinite strip, and define $\Phi\colon H^+ \to \Sigma$ by

$$\Phi(w) = \frac{i}{\pi} \log w + \frac{1}{2},$$

where log denotes the principal branch of the logarithm. Then $\Phi$ is a biholomorphism of $H^+$ with $\Sigma$ such that $\Phi(i) = 0$, $\operatorname{Im}\Phi(w) \to +\infty$ as $w \to \infty$ and $\operatorname{Im}\Phi(w) \to -\infty$ as $w \to 0$. Furthermore,

$$\Phi(K_\varepsilon) = \left\{ z \in \mathbf{C} \;\middle|\; |\operatorname{Re} z| < \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \right\}.$$

Therefore we have $F \in \operatorname{Hol}(\Sigma, \Delta)$ and a curve $\tilde{\gamma}\colon [0,1) \to \Sigma$ such that $\operatorname{Im}\tilde{\gamma}(t) \to +\infty$ and $F\bigl(\tilde{\gamma}(t)\bigr) \to 0$ as $t \to 1$, and we claim that $F(z) \to 0$ as $\operatorname{Im} z \to +\infty$, uniformly in $|\operatorname{Re} z| \le 1 - \delta$.

Fix $\varepsilon \in (0,1)$, and take $y_0 > \operatorname{Im}\tilde{\gamma}(0)$ so large that $\bigl|F\bigl(\tilde{\gamma}(t)\bigr)\bigr| < \varepsilon$ when $\operatorname{Im}\tilde{\gamma}(t) \ge y_0$. We claim that if $\operatorname{Im} z \ge y_0$ then

$$|F(z)| \le \varepsilon^{(1-|\operatorname{Re} z|)/4}, \tag{1.3.10}$$

and the theorem will obviously follows from (1.3.10).

Fix $z_0 \in \Sigma$ with $\operatorname{Im} z_0 \ge y_0$, and choose $t_0 < 1$ so that $\operatorname{Im}\tilde{\gamma}(t_0) = \operatorname{Im} z_0$ and $\operatorname{Im}\tilde{\gamma}(t) > \operatorname{Im} z_0$ for all $t \in (t_0, 1)$. Let $E$ be the image of $[t_0, 1)$ under $\tilde{\gamma}$, and let $\overline{E}$ be the reflection of $E$ with respect to the axis $\{\operatorname{Im} z = \operatorname{Im} z_0\}$. In particular, $E \cup \overline{E}$ intersects this axis only in $\tilde{\gamma}(t_0)$.

If $z_0 \in E \cup \overline{E}$, (1.3.10) is clear. So assume $z_0 \notin E \cup \overline{E}$; thus there exists a connected component $\Sigma_0$ of $\Sigma \setminus (E \cup \overline{E})$ containing $z_0$, with $\partial\Sigma_0 \subset \partial\Sigma \cup (E \cup \overline{E})$. Furthermore, $\partial\Sigma_0$ cannot intersect both $\{\operatorname{Re} z = 1\}$ and $\{\operatorname{Re} z = -1\}$.

Assume for the moment $\operatorname{Re} z_0 > \operatorname{Re}\tilde{\gamma}(t_0)$, and for every $\eta > 0$ define $G_\eta\colon \Sigma \to \mathbf{C}$ by

$$G_\eta(z) = \frac{F(z)\overline{F(\overline{z} + 2i \operatorname{Im} z_0)}\varepsilon^{(1+z)/2}}{1 + \eta(1+z)}. \tag{1.3.11}$$

Clearly, $|G_\eta(z)| \le 1$ for all $z \in \Sigma$. Moreover, $|F(z)| < \varepsilon$ on $E$, and $|F(\overline{z} + 2i \operatorname{Im} z_0)| < \varepsilon$ on $\overline{E}$; hence $|G_\eta(z)| < \varepsilon$ on $E \cup \overline{E}$. If $\operatorname{Re} z$ is sufficiently close to 1, or if $|\operatorname{Im} z|$ is sufficiently large, then again $|G_\eta(z)| < \varepsilon$. Then we can apply the maximum modulus principle to infer that $|G_\eta(z)| \le \varepsilon$ on $\Sigma_0$, and in particular in $z_0$. Letting $\eta \to 0$ we get therefore

$$|F(z_0)|^2 \le \varepsilon \cdot \varepsilon^{-(1+\operatorname{Re} z)/2} = \varepsilon^{(1-\operatorname{Re} z)/2},$$

and thus (1.3.10) is proved if $\operatorname{Re} z_0 > \operatorname{Re}\tilde{\gamma}(t_0)$. If $\operatorname{Re} z_0 < \operatorname{Re}\tilde{\gamma}(t_0)$, replacing $1+z$ by $1-z$ in (1.3.11) we are led to the same conclusion, and the proof is finished, **q.e.d.**

Using Theorem 1.3.23 we can prove the following extension of Theorem 1.3.22 for functions without fixed points:

**Theorem 1.3.24:** *Let $f$, $g \in \mathrm{Hol}(\Delta,\Delta)$ be such that $f\circ g = g\circ f$. Assume that $f$ has no fixed points, and $g \neq \mathrm{id}_\Delta$. Let $\tau=\tau(f)$ be the Wolff point of $f$. Then $g$ also has no fixed points, and*
$$\operatorname*{\mathit{K}\text{-lim}}_{z\to\tau} g(z) = \tau = \operatorname*{\mathit{K}\text{-lim}}_{z\to\tau} f(z).$$

*Proof:* If $g$ would have a fixed point $z_0\in\Delta$, then, as usual, $z_0$ would be a fixed point for $f$, impossible. By Corollary 1.2.16, it remains to show that $g$ has non-tangential limit $\tau$ at $\tau$. It will suffice, by Theorem 1.3.23, to construct a continuous curve $\gamma\colon[0,1)\to\Delta$ with $\gamma(t)\to\tau$ as $t\to1$ such that $g\bigl(\gamma(t)\bigr)\to\tau$ as $t\to1$.

For $0\le t<1$ let $k(t)$ be the greatest integer less than or equal to $-\log_2(1-t)$. Let $z_0=f(0)$, and for $t\in[0,1)$ define
$$\gamma(t) = f^{k(t)}\bigl(2[1-2^{k(t)}(1-t)]z_0\bigr). \tag{1.3.12}$$

Since $\gamma\bigl([1-2^{-k},1-2^{-k-1}]\bigr)$ is the image by $f^k$ of the segment $S$ from 0 to $f(0)$, it is easily checked that $\gamma$ is continuous. Moreover, $\gamma(t)\to\tau$ as $t\to1$, for $f^k\to\tau$ uniformly on $S$.

But now $f^k\to\tau$ uniformly on $g(S)$; since $g\bigl(f^k(S)\bigr) = f^k\bigl(g(S)\bigr)$, this implies that $g\bigl(\gamma(t)\bigr)\to\tau$ as $t\to1$, **q.e.d.**

Theorem 1.3.24 raises a natural question: if $f$, $g\in\mathrm{Hol}(\Delta,\Delta)$ are without fixed point and commute, do they have the same Wolff point? If it is so, this can be an ultimate generalization of Theorem 1.3.22.

In general, unfortunately, the answer is negative: if $\gamma$ is a hyperbolic automorphism of $\Delta$, then $\gamma$ and $\gamma^{-1}$ commute but $\tau(f)\neq\tau(f^{-1})$. Then our aim will be to show that this is the only exception.

We begin recalling some facts and notations discussed in chapter 1.2. If $f\in\mathrm{Hol}(\Delta,\Delta)$ admits non-tangential limit $\tau\in\partial\Delta$ at $\sigma\in\partial\Delta$, we shall write $f(\sigma)=\tau$. Moreover, we shall denote by $f'(\sigma)$ the non-tangential limit at $\sigma$ of $\bigl(f(z)-\tau\bigr)/\bigl(z-\sigma\bigr)$. By Theorem 1.2.7, $|f'(\sigma)|\in(0,+\infty]$ and, if it is finite, $f'(\sigma)$ coincides with the non-tangential limit of $f'$ at $\sigma$; $f'(\sigma)$ is expressed in terms of the boundary dilatation coefficient $\beta_f(\sigma)$ of $f$ at $\sigma$ by $f'(\sigma)=\tau\overline{\sigma}\beta_f(\sigma)$. Furthermore, $\sigma=\tau$ and $f'(\sigma)\le1$ iff $\tau$ is the Wolff point of $f$ (Corollary 1.2.16).

The usual chain rule holds for angular derivatives too:

**Lemma 1.3.25:** *Let $f$, $g\in\mathrm{Hol}(\Delta,\Delta)$ be both different from* $\mathrm{id}_\Delta$. *Choose $\sigma\in\partial\Delta$ so that $f(\sigma)=\tau\in\partial\Delta$ and $g(\tau)=\eta\in\partial\Delta$. Then*
$$\operatorname*{\mathit{K}\text{-lim}}_{z\to\sigma}\frac{g\bigl(f(z)\bigr)-\eta}{z-\sigma} = g'(\tau)f'(\sigma).$$

*Proof:* First of all, assume $\beta_f(\sigma)$ and $\beta_g(\tau)$ finite. We claim that the curve $\rho(t)=f(t\sigma)$ goes to $\tau$ non-tangentially as $t\to1$. Indeed, $\dot\rho(1)$ is tangent to $\partial\Delta$ in $\tau$ iff $\mathrm{Re}\bigl(\dot\rho(1)\overline{\tau}\bigr)=0$; but
$$\mathrm{Re}\bigl(\dot\rho(1)\overline{\tau}\bigr) = \mathrm{Re}\bigl(f'(\sigma)\sigma\overline{\tau}\bigr) = \beta_f(\sigma)>0$$

by Theorem 1.2.7, and the claim is proved.

In particular, then, we get $g \circ f(z) \to \eta$ as $z \to \sigma$ non-tangentially, and the usual chain rule gives

$$K\text{-}\lim_{z\to\sigma}(g\circ f)'(z) = \big[K\text{-}\lim_{z\to\sigma} g'\big(f(z)\big)\big]\cdot\big[K\text{-}\lim_{z\to\sigma} f'(\sigma)\big] = g'(\tau)f'(\sigma);$$

therefore the assertion follows from Theorem 1.2.7 and Proposition 1.2.8.

Now assume either $g'(\tau)$ or $f'(\sigma)$ infinite. Since

$$\frac{1-\big|g\big(f(z)\big)\big|}{1-|z|} = \frac{1-\big|g\big(f(z)\big)\big|}{1-|f(z)|}\cdot\frac{1-|f(z)|}{1-|z|}$$

and both factors in the right-hand side have strictly positive infimum (by Lemma 1.2.4), if one of them goes to infinity the product does. Hence

$$\liminf_{z\to\sigma}\frac{1-\big|g\big(f(z)\big)\big|}{1-|z|} = +\infty,$$

and the assertion follows from Proposition 1.2.6, **q.e.d.**

We shall also need an arithmetic lemma:

**Lemma 1.3.26:** *Let $a$, $b$, $c$, $d \in (-\infty, +\infty]$ be such that*

$$a+b>0,\qquad c+d\ge 0,\qquad a\le 0,\qquad b>0,\qquad c>0\qquad \text{and}\qquad d\le 0.$$

*Then there are positive integers $h$, $k \in \mathbf{N}$ such that $ha+kc>0$ and $hb+kd>0$.*

*Proof:* If $b$ (or $c$) is equal to $+\infty$ we take $h=1$ and $k$ large ($k=1$ and $h$ large). If $d$ (or $a$) is zero, we again take $h=1$ and $k$ large ($k=1$ and $h$ large). So assume $a$, $b$, $c$ and $d$ finite and not zero. From $a+c|a/c|=0$ we infer

$$0<(a+b)+(c+d)|a/c| = b+d|a/c|.$$

Hence $|a/c|<|b/d|$. Then $h$, $k\in\mathbf{N}$ such that $|a/c|<k/h<|b/d|$ are exactly as we need, **q.e.d.**

And now:

**Theorem 1.3.27:** *Let $f$, $g \in \mathrm{Hol}(\Delta,\Delta)\setminus\{\mathrm{id}_\Delta\}$ be such that $f\circ g = g\circ f$. Then:*
(i) *if $f$ is not a hyperbolic automorphism of $\Delta$, then $\tau(f)=\tau(g)$;*
(ii) *otherwise, $g$ also is a hyperbolic automorphism of $\Delta$, with the same fixed point set as $f$, and either $\tau(f)=\tau(g)$ or $\tau(f^{-1})=\tau(g)$.*

*Proof:* If $f$ is a hyperbolic automorphism of $\Delta$, the assertion follows by Theorem 1.2.18. So, we can assume $f$ and $g$ without fixed points in $\Delta$ (for otherwise it is obvious), and $f$ not a hyperbolic automorphism. Let $\sigma=\tau(f)$ and $\tau=\tau(g)$; hence, by Theorem 1.3.24

$f(\sigma) = g(\sigma) = \sigma$ and $f(\tau) = g(\tau) = \tau$. In particular, by Theorem 1.2.7, $f'(\sigma)$, $f'(\tau)$, $g'(\sigma)$ and $g'(\tau)$ are real (possibly equal to $+\infty$).

Assume, by contradiction, $\tau \neq \sigma$. Since $f$ is not a hyperbolic automorphism of $\Delta$, neither $g$ is (by Theorem 1.2.18) and Theorem 1.2.11 implies

$$f'(\tau)f'(\sigma) > 1 \qquad \text{and} \qquad g'(\tau)g'(\sigma) > 1.$$

Moreover, $f'(\sigma) \leq 1$, $f'(\tau) > 1$, $g'(\sigma) > 1$ and $g'(\tau) \leq 1$. If we apply Lemma 1.3.26 with $a = \log f'(\sigma)$, $b = \log f'(\tau)$, $c = \log g'(\sigma)$ and $d = \log g'(\tau)$, we come up with two positive integers $h$ and $k$ such that

$$(f^h \circ g^k)'(\sigma) > 1 \qquad \text{and} \qquad (f^h \circ g^k)'(\tau) > 1,$$

where we used Lemma 1.3.25. In particular, the Wolff point $\eta \in \partial\Delta$ of $f^h \circ g^k$ is neither $\sigma$ nor $\tau$. Since both $f$ and $g$ commute with $f^h \circ g^k$, Theorem 1.3.24 shows that $f(\eta) = g(\eta) = \eta$. Hence we should have $f'(\eta) > 1$ and $g'(\eta) > 1$; in particular, using Lemma 1.3.25 we find

$$(f^h \circ g^k)'(\eta) = (f')^h(\eta) \cdot (g')^k(\eta) > 1,$$

contradiction, **q.e.d.**

We end this chapter with another application of iteration theory to the construction of fixed points of particular families of holomorphic functions.

By Theorem 1.2.25 and Proposition 1.1.32 the automorphism group of a multiply connected domain of regular type is always compact. On the other hand, we know that $\mathrm{Aut}(\Delta)$ is not compact, though the isotropy group of one point is. Our last result is that a subgroup of $\mathrm{Aut}(\Delta)$ is relatively compact iff it has a fixed point. For the sake of generality, we shall prove something more:

**Theorem 1.3.28:** *Let $G \subset \mathrm{Hol}(\Delta, \Delta)$ be a group under composition. Then $G$ has a fixed point in $\Delta$ iff $G$ is relatively compact in $\mathrm{Hol}(\Delta, \Delta)$. Moreover, in this case $\overline{G}$ is either a single constant function, finite cyclic or the isotropy group of its fixed point. In any case, $G$ is abelian.*

*Proof:* Let $e \in G$ be the identity of $G$. If $e$ is constant, then for all $f \in G$ we have $f = e \circ f = e$, and thus $G = \{e\}$.

If $e$ is not constant, then $e(\Delta)$ is open in $\Delta$, and from $e = e \circ e$ we infer that $e = \mathrm{id}_\Delta$ on the open set $e(\Delta)$, and hence everywhere. This implies that $G$ is a subgroup of $\mathrm{Aut}(\Delta)$.

If $G$ has a fixed point, then it is relatively compact by Corollary 1.1.47. Conversely, assume that $\overline{G} \subset \mathrm{Hol}(\Delta, \Delta)$ is compact. Take $f \in \overline{G}$. By the Wolff-Denjoy theorem, $f$ must have a fixed point in $\Delta$. Up to conjugation, we can assume $f(0) = 0$, and thus $f(z) = e^{i\theta} z$ for some $0 < \theta < 2\pi$. Replacing $f$ by $f^{-1}$ if necessary, we can assume $0 < \theta \leq \pi$.

By compactness, there is $g_0 \in \overline{G}$ maximizing $|g(0)|$. Write

$$g_0(z) = e^{i\phi} \frac{z + a}{1 + \overline{a}z},$$

with $|a| = |g_0(0)|$. Let $k \in \mathbf{Z}$ be such that $\mathrm{Re}(e^{i(k\theta+\phi)}) \geq 0$. Then, letting $\tau = e^{i(k\theta+\phi)}$, we have

$$|1 + \tau |a|^2|^2 < |1 + \tau|^2.$$

Now $(f^k \circ g_0)^2 \in \overline{G}$; since $f^k \circ g_0(0) = \tau a$, we obtain

$$(f^k \circ g_0)^2(0) = \tau a \frac{1 + \tau}{1 + \tau |a|^2}.$$

Hence we should have $a = 0$ since otherwise $|(f^k \circ g_0)^2| > |g_0(0)|$ would contradict the maximality of $|g_0(0)|$. It follows that $\overline{G}$ is contained in the isotropy group of 0, and the rest of the assertion is now evident, **q.e.d.**

An important consequence of Theorem 1.3.28 is that a compact group acting holomorphically on $\Delta$ has a fixed point. This is a feature of the holomorphic structure: there are examples of compact Lie groups acting on cells without fixed points (see Oliver [1979]).

Finally,

**Corollary 1.3.29:** *Let $G \subset \mathrm{Hol}(\Delta, \Delta)$ be a group under composition. If there exists a compact set $K \subset \Delta$ invariant under $G$, then $G$ has a fixed point in $\Delta$.*

*Proof:* In fact, $G$ is clearly relatively compact in $\mathrm{Hol}(\Delta, \Delta)$, **q.e.d.**

## Notes

The first work on iteration theory seems to be Schröder [1870]. He studied the local situation near a fixed point, essentially obtaining Theorem 1.3.7.(i); another proof is in Kœnigs [1883]. Theorem 1.3.2 is proved, in a slightly different form, in Tricomi [1916].

Theorem 1.3.4 is due to Radó [1924], but some special cases were known before. For instance, Bieberbach [1913] proved that the unique automorphism $f$ of a hyperbolic domain $D$ with a fixed point $z_0 \in D$ such that $f'(z_0) > 0$ is the identity. This is important in uniformization theory of multiply connected domains.

Corollary 1.3.5 was originally proved by Aumann and Carathéodory [1934]; our proof is taken from Heins [1941b]. Hervé [1951] computed the supremum in (1.3.1) and described the functions attaining it in doubly connected domains and, partially, in multiply connected domains.

The first papers explicitely devoted to iteration theory on $\Delta$ are Fatou [1919, 1920a, b]. He studied the iteration of rational functions sending both $\Delta$ and $\partial\Delta$ into themselves and, more generally, of functions of the form

$$f(z) = e^{i\theta} \prod_{\nu=0}^{\infty} \frac{z - a_\nu}{1 - \overline{a_\nu} z},$$

where $a_\nu \in \Delta$ for all $\nu \in \mathbf{N}$ and $\sum\limits_{\nu=0}^{\infty} (1 - |a_\nu|) < +\infty$. These functions are known as *Fatou functions, infinite Blaschke products* or even as *inner functions;* see also Valiron [1954].

The history of the Wolff-Denjoy Theorem 1.3.9 is quite interesting. On December 21, 1925, Wolff [1926a] presented a first proof, assuming continuity at the boundary. Few weeks later, on January 18, 1926, Wolff [1926b] suceeded in removing the extra hypothesis, with a brute force approach. But just a few days later, on January 25, 1926, Denjoy [1926] published a completely new proof, based on Fatou's theorem on boundary values of bounded holomorphic functions. Finally, after a couple of months, on April 7, 1926, inspired by Denjoy's proof, Wolff [1926c] discovered Wolff's lemma, and the elegant proof we presented. The second proof we described is due to Vesentini [1983]. Wolff [1929] and Valiron [1931] have studied the asymptotic behavior of $\arg(f^k)$ as $k \to +\infty$, where $f \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points.

Using Theorem 1.1.28, the Wolff-Denjoy theorem can be transferred to simply connected domains bounded by a Jordan curve. In simply connected domains with bad boundary the situation is not so agreeable: see Ferrand [1941].

There is a whole branch of one-variable iteration theory we did not mention: functional equations. Given $f \in \mathrm{Hol}(X, X)$, the idea is to find a function $g\colon X \to X$ solving the equation

$$g \circ f = \phi \circ g, \tag{1.3.13}$$

where $\phi \in \mathrm{Hol}(X, X)$ is a fixed function, usually an automorphism, and $g$ is often required to be a local homeomorphism, and if possible holomorphic. Then the investigation of the behavior of $\{f^k\}$ is somehow reduced to the description of the (known) behavior of $\{\phi^k\}$. This approach is particularly useful in iteration theory of rational and entire functions, where it is particularly studied (1.3.13) for $X \subset \widehat{\mathbf{C}}$, $\phi(z) = \lambda z$ and $f$ with a fixed point at 0 of multiplier $\lambda$; in this case (1.3.13) is called *Schröder equation*, and a solution $g$ is a *Kœnigs function.*

We quote only one result in this area due to Pommerenke [1979] (but cf. also Baker and Pommerenke [1979] and Cowen [1981]), just to give the flavor of the subject. Take $f \in \mathrm{Hol}(H^+, H^+)$ with Wolff point at infinity, and define $z_k = f^k(i)$,

$$q_k = \left|\frac{z_{k+1} - z_k}{z_{k+1} - \overline{z_k}}\right|,$$

and $g_k = (f_k - \mathrm{Re}\, z_k)/\mathrm{Im}\, z_k$. Then $\{g_k\}$ converges to a function $g \in \mathrm{Hol}(H^+, H^+)$ such that $g(i) = i$ and $g \circ f = \phi \circ g$, where $\phi \in \mathrm{Aut}(H^+)$ fixes the point at infinity. Furthermore, if the angular derivative $\beta$ of $f$ at $\infty$ is greater than one, then $\phi$ is hyperbolic; if $\beta = 1$ and $q_k \to \delta > 0$ as $k \to +\infty$, then $\phi$ is parabolic; if $\beta = 1$ and $q_k \to 0$ as $k \to +\infty$, then $\phi = \mathrm{id}_{H^+}$ and $g \equiv i$. Therefore, but for the last case, $f$ behaves as either a hyperbolic or a parabolic automorphism near infinity, according to $\beta > 1$ or $\beta = 1$, exactly as discussed in section 1.3.2.

Section 1.3.3 is inspired by Heins [1941a, 1988]. In Heins [1941a] Theorem 1.3.20 was proved for hyperbolic plane domains bounded by a finite number of Jordan curves, by a clever (and complicated) argument making use of the Julia-Wolff-Carathéodory theorem. The complete statement is in Heins [1988], where it is proved using a different (though essentially equivalent) approach involving Green functions. We preferred to stress the role played by the horocycles, in view of what we shall do in the second part of this book.

Proposition 1.3.14 was stated by Farkas [1884], but the first complete proof is due to Ritt [1920]. Corollary 1.3.21 is taken from H. Cartan [1932].

Theorem 1.3.22 is due to Shields [1964]. This result is a typical feature of the holomorphic structure: there are examples of commuting continuous functions mapping the closed unit interval into itself without common fixed points (see Boyce [1969] and Huneke [1969]).

Theorem 1.3.23 was originally proved by Montel [1912] using normal family techniques, and assuming $\gamma$ to be radial. The general statement (and the idea of using maximum modulus arguments) is due to Lindelöf [1915]; our proof is taken from Rudin [1980]. Scattered in Burckel [1979] there are several other results of this kind; for instance, if $D \subset \widehat{\mathbf{C}}$ is a hyperbolic domain and $f \in \mathrm{Hol}(\Delta, D)$ is bounded along a curve ending at $\sigma \in \partial\Delta$, then $f$ is bounded in every Stolz region at $\sigma$ (Lindelöf [1909]). Finally, Lehto and Vitahren [1957] have proved that Theorem 1.3.23 holds for *normal* functions, i.e., for functions $f \in \mathrm{Hol}(\Delta, \widehat{\mathbf{C}})$ such that $\{f \circ \gamma \mid \gamma \in \mathrm{Aut}(\Delta)\}$ is a normal family.

Theorems 1.3.24 and 1.3.27 are taken from Behan [1973]. Cowen [1984] has given an almost complete characterization of functions in $\mathrm{Hol}(\Delta, \Delta)$ commuting with a given one.

Finally, Theorem 1.3.28 was first proved by Mitchell [1979], with quite different arguments involving elementary facts of semigroup theory. Our proof is due to Burckel [1981].