# Preface

When, some years ago, we started working on a differential geometric study of the structure of strongly convex domains in $\mathbb{C}^n$, we did not expect to end up writing a book on global Finsler geometry. But, along the way, we found ourselves needing several basic results on real and complex Finsler metrics that we were unable to find in the literature (or at least not in the form necessary to us). So we felt compelled to provide proofs — and this is the final result of our work.

Our exposition is very much in the vein of the work of Cartan [C], Chern [Ch1, 2], Bao-Chern [BC] and Kobayashi [K]; in particular the latter gave us the preliminary idea for our approach to smooth complex Finsler metrics. We would also like to say that we would have been very happy to know earlier of [BC], which, although only marginally related to our work, would have been of great help in solving questions which we treated independently.

Our starting point was the study of the existence and global behavior of complex geodesics for intrinsic metrics in complex manifolds. Our goal was to look at this question from a differential geometric point of view, with the hope of possibly reproducing in a wider class of complex manifolds what Lempert [L] was able to prove for strongly convex domains in $\mathbb{C}^n$. The idea was to treat complex geodesics through a point as images of disks through the origin in the tangent space at the point via the exponential map of a complex Finsler metric; thus we were led to study the local and global theory of geodesics of a Finsler metric. As in Hermitian (and Riemannian) geometry, the local theory of geodesics means the study of the first variation of the length integral, and of the associated Euler-Lagrange equation. The global theory, on the other hand, involves the accurate control of the second variation and hence of the curvature, together with Jacobi fields, conjugate points, the Morse index form and the like. In particular, we needed a Finsler version of the Cartan-Hadamard theorem (originally proved by Auslander [Au1, 2]), and a way to apply it in a complex situation.

The main difficulty at this point was that the problems we were interested in involved complex Finsler metrics, and whereas there is a clear understanding of the relationship between the complex geometry and the underlying real geometry of a Hermitian manifold, nothing of the kind was available to us in Finsler geometry. We then started following the tradition of "linearizing" the questions by passing from the study of Finsler metrics on the tangent bundle (real or complex) to the study of the associated Hermitian structure on the tangent bundle of the tangent bundle. At this level it is also possible to describe the correct relationship between the complex and the corresponding real structure of objects like connections and curvatures.

But our approach is different from the traditional one for two main reasons. First of all, we everywhere stress global objects and global definitions (in fact, we are interested in global results), using local coordinates almost uniquely as a computational tool (in a way not too far from the first chapter of Bejancu [B]). But the main difference is another one. Possibly because of our motivations, working in this area we discovered that there might be a danger of carrying out the linearization program previously described too far. In fact, the formal setting naturally leads to very general definitions which make proofs of theorems easier, but do not give much geometrical insight: we had the feeling that working only at the tangent-tangent level was too restrictive, too formal, too far away from the actual geometry of the manifold. For this reason, our point of view now is to stick to notions which really provide informations about the geometry of geodesics on the manifold, and about the curvature of the manifold. This approach leads to "minimal" definitions, which are probably more complicated to state and surely more difficult to handle, but nevertheless more effective and really conveying the geometry of the manifold. For instance, there are many ways of generalizing the notion of Kählerianity to Finsler metrics, but not all of them have non-trivial examples and applications. We shall show how the notions we singled out can be effectively used by illustrating their applications in complex geometric function theory.

The first two chapters of this book are devoted to the exposition of our approach to real and complex Finsler geometry. In the first chapter, after setting the stage introducing the necessary general definitions and objects, we define in a global way the classical Cartan connection, and we discuss the variation formulas of the length integral, the exponential map, Jacobi fields, conjugate points and the Morse index form up to provide a proof of the Cartan-Hadamard and Bonnet theorems for Finsler metrics suitable for our needs in complex geometry. In the exposition we stress the similarities with the standard Riemannian treatment of the subject, as naturally suggested by our global approach.

In the second chapter we study the geometry of complex Finsler metrics. After having adapted the general definitions of chapter 1 to the complex setting, we introduce (following Kobayashi [K]) the Chern-Finsler connection, which is our main tool. We discuss at some length several Kähler conditions, and we introduce the notion of holomorphic curvature of a complex Finsler manifold, showing the equivalence of the differential geometric definition with a variational definition previously used in function theory.

Finally the third chapter contains the results and applications that motivated our work. From a differential geometric point of view, it is devoted to the study of the function theory on Kähler Finsler manifolds with constant nonpositive holomorphic curvature; from a complex analysis point of view, it is devoted to the study of manifolds where there is a Monge-Ampère foliation with exactly the same properties as the one discovered by Lempert in strongly convex domains. In particular we prove the existence of nice foliations and strictly plurisubharmonic exhaustions satisfying the Monge-Ampère equation on Kähler Finsler manifolds with constant nonpositive holomorphic curvature. Furthermore we prove that the only complex manifold admitting such a metric with zero holomorphic curvature is $\mathbb{C}^n$, and we describe a characterization of strongly convex circular domains in terms of differential geometric properties of the Kobayashi metric.

Of course, this book is not intended as a definitive treatise on the subject; on the contrary, it is just the description of an approach to Finsler metrics that we found reasonable and fruitful, but still leaving a lot of open problems. Just to mention a couple of them: the comparison between the complex Finsler geometry and the underlying real one carried out in section 2.6 seems to suggest that the Cartan connection contains terms which have no direct influence on the geometry of the manifold — and so maybe it is not the correct connection to use even in real Finsler geometry. Or: in the third chapter we give a fairly complete description of the complex structure of Kähler Finsler manifolds of constant nonpositive holomorphic curvature, which is satisfying from a geometric function theory point of view, but it still leaves completely open the problem of classifying the metrics with these properties (we remark that there are many more such manifolds and metrics than in the Hermitian case: there are at least all the strongly convex domains in $\mathbb{C}^n$ endowed with the Kobayashi metric, thanks to Lempert's work [L]) — and in fact it is even still far from being completed the classification of simply connected real Finsler manifolds with constant (horizontal flag) real curvature. Or: it follows from chapter 3 that the only part of Lempert's results actually depending on the strong convexity of the domain is the smoothness of the Kobayashi metric. It would be then interesting to construct directly a smooth weakly Kähler Finsler metric of constant holomorphic curvature $-4$ on any strongly convex domain; then this metric would automatically be the Kobayashi metric of the domain, and we would have recovered the full extent of Lempert's work.

So we hope that the possibly new perspectives on Finsler geometry introduced in this book will eventually lead to new results in this field; and in particular in geometric function theory of complex Finsler manifolds, where all this work started.