Formula di Bochner

Teorema 1. *Sia Ω un aperto in $\mathbb{R}^d$ ed $u \in C^\infty(\Omega)$. Allora,*

$$\Delta\big(|\nabla u|^2\big) = 2\nabla(\Delta u)\cdot\nabla u + 2\|Hess(u)\|_2^2 \qquad \text{dove} \qquad \|Hess(u)\|_2^2 := \sum_{i=1}^{d}\sum_{j=1}^{d}\big(\partial_{ij}u\big)^2.$$

In particolare, se u è armonica in Ω, allora

$$\Delta\big(|\nabla u|^2\big) \ge 0 \quad \text{in} \quad \Omega.$$

Corollario 2. *Sia $h \in C^\infty(B_\rho)$ una funzione armonica sula palla $B_\rho \subset \mathbb{R}^d$. Allora,*

$$⨍_{B_r} |\nabla h|^2\,dx \le ⨍_{B_R} |\nabla h|^2\,dx \qquad \text{per ogni} \qquad 0 < r \le R < \rho.$$