

Hausdorff measure and surface measure

Definition 1 (Projections and graphs). *Let $d \geq 2$.*

- *For every set $E \subset \mathbb{R}^d$ we denote by $\pi(E)$ the projection of E on $\mathbb{R}^{d-1}$, that is*

$$\pi(E) := \left\{ x' \in \mathbb{R}^{d-1} : (x', x_d) \in E \text{ for some } x_d \in \mathbb{R} \right\}.$$

- *Given a set $A \subset \mathbb{R}^{d-1}$ and a function $f : A \rightarrow \mathbb{R}$, we denote by $\Gamma(f; A)$ the graph of f over A , that is,*

$$\Gamma(f; A) := \left\{ (x, f(x)) : x \in A \right\}.$$

PROJECTIONS AND HAUSDORFF MEASURE

Lemma 2. *Let $d \geq 2$. Then, for every set $U \subset \mathbb{R}^d$, we have*

$$\text{diam}(\pi(U)) \leq \text{diam}(U).$$

Proof. Suppose that $x', y' \in \pi(U)$. Let $x_d, y_d \in \mathbb{R}$ be such that

$$x := (x', x_d) \in U \quad \text{and} \quad y := (y', y_d) \in U.$$

Then,

$$|x' - y'| \leq \sqrt{|x' - y'|^2 + (x_d - y_d)^2} = |x - y| \leq \text{diam}(U).$$

Taking the supremum over all $x', y' \in \pi(U)$, we get the claim. □

Proposition 3. *Let $d \geq 2$. Then, for every set $E \subset \mathbb{R}^d$, we have*

$$\mathcal{H}^{d-1}(\pi(E)) \leq \mathcal{H}^{d-1}(E).$$

Proof. Suppose that

$$\mathcal{H}^{d-1}(E) = \sup_{\delta > 0} \mathcal{H}_\delta^{d-1}(E) < +\infty.$$

Fix $\varepsilon > 0$.

By the definition of $\mathcal{H}^{d-1}$, for every $\delta > 0$, we can find a covering $\{U_i\}_{i \geq 1}$ of E with sets $U_i \subset \mathbb{R}^{d-1}$ such that

$$\text{diam}(U_i) < \delta \quad \text{for all } i \geq 1,$$

and

$$\omega_{d-1} \sum_{i \geq 1} \left(\frac{\text{diam}(U_i)}{2} \right)^{d-1} \leq \mathcal{H}^{d-1}(E) + \varepsilon.$$

The family of projections $\{\pi(U_i)\}_{i \geq 1}$ is a covering of $\pi(E)$ and is such that

$$\text{diam}(\pi(U_i)) < \delta \quad \text{for all } i \geq 1.$$

Then, by the definition of $\mathcal{H}_\delta^{d-1}$, we have

$$\begin{aligned} \mathcal{H}_\delta^{d-1}(\pi(E)) &\leq \omega_{d-1} \sum_{i \geq 1} \left(\frac{\text{diam}(\pi(U_i))}{2} \right)^{d-1} \\ &\leq \omega_{d-1} \sum_{i \geq 1} \left(\frac{\text{diam}(U_i)}{2} \right)^{d-1} \leq \mathcal{H}^{d-1}(E) + \varepsilon. \end{aligned}$$

Taking the supremum over $\delta > 0$ we have

$$\mathcal{H}^{d-1}(\pi(E)) = \sup_{\delta > 0} \mathcal{H}_\delta^{d-1}(\pi(E)) \leq \mathcal{H}^{d-1}(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get the claim. □

More in general, we have the following

Proposition 4. *Let $d \geq 1$, $K \subset \mathbb{R}^d$ be a closed convex set, and $\pi_K : \mathbb{R}^d \rightarrow K$ be the projection on K . Then:*

$$\text{diam}(\pi_K(U)) \leq \text{diam}(U) \quad \text{for every set } U \subset \mathbb{R}^d;$$

$$\mathcal{H}^s(\pi_K(E)) \leq \mathcal{H}^s(E) \quad \text{for every set } E \subset \mathbb{R}^d \text{ and every } s \in [0, d].$$

HAUSDORFF MEASURE AND LIPSCHITZ MAPS

Lemma 5. Let $d \geq 2$. Let $A \subset \mathbb{R}^d$ and let

$$\Phi : A \rightarrow \mathbb{R}^d$$

be an L -Lipschitz map:

$$|\Phi(x) - \Phi(y)| \leq L|x - y| \quad \text{for all } x, y \in A.$$

Then, for all $s > 0$ we have

$$\mathcal{H}^s(\Phi(A)) \leq L^s \mathcal{H}^s(A).$$

Proof. Suppose that $\mathcal{H}^s(A) < +\infty$. Since Fix $\delta > 0$ and $\varepsilon > 0$. By the definition of $\mathcal{H}_\delta^s(A)$ we can find a covering of A with sets $\{U_i\}_{i \geq 1}$ such that:

$$\text{diam}(U_i) \leq \delta \quad \text{for all } i \geq 1,$$

and

$$\mathcal{H}_\delta^s(A) \leq \sum_{i \geq 1} \omega_s \left(\frac{\text{diam}(U_i)}{2} \right)^s \leq \mathcal{H}_\delta^s(A) + \varepsilon.$$

Consider the family of sets $\tilde{U}_i := \Phi(U_i)$, for $i \geq 1$. We have that:

- $\{\tilde{U}_i\}_{i \geq 1}$ is a covering of $\Phi(A)$;
- $\text{diam}(\tilde{U}_i) \leq L\delta$ for all $i \geq 1$.

Then, we can use the family $\{\tilde{U}_i\}_{i \geq 1}$ to estimate $\mathcal{H}_{L\delta}^s(\Phi(A))$. Thus,

$$\mathcal{H}_{L\delta}^s(\Phi(A)) \leq \sum_{i \geq 1} \omega_s \left(\frac{\text{diam}(\tilde{U}_i)}{2} \right)^s \leq L^s \sum_{i \geq 1} \omega_s \left(\frac{\text{diam}(U_i)}{2} \right)^s \leq L^s (\mathcal{H}_\delta^s(A) + \varepsilon).$$

Since

$$\mathcal{H}^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A),$$

we have that

$$\mathcal{H}_\delta^s(A) \leq \mathcal{H}^s(A),$$

so we get

$$\mathcal{H}_{L\delta}^s(\Phi(A)) \leq L^s (\mathcal{H}^s(A) + \varepsilon).$$

Taking the limit as $\delta \rightarrow 0$, we get

$$\mathcal{H}^s(\Phi(A)) \leq L^s (\mathcal{H}^s(A) + \varepsilon).$$

Finally, since $\varepsilon > 0$ is arbitrary, we get

$$\mathcal{H}^s(\Phi(A)) \leq L^s \mathcal{H}^s(A).$$

□

LIPSCHITZ GRAPHS AND HAUSDORFF MEASURE.

PART I. A ROUGH ESTIMATE ON THE HAUSDORFF MEASURE OF A LIPSCHITZ GRAPHS.

Lemma 6. Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be an L -Lipschitz function, for some $L > 0$. Then:

$$\text{diam}(\Gamma(f; U)) \leq \sqrt{1 + L^2} \text{diam}(U) \quad \text{for all sets } U \subset \mathbb{R}^{d-1}.$$

Proof. Let $x = (x', f(x')) \in \Gamma(f; U)$ and $y = (y', f(y')) \in \Gamma(f; U)$. Then, since

$$|f(x') - f(y')| \leq L|x' - y'|,$$

we get that

$$\begin{aligned} |x - y| &= \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \\ &\leq \sqrt{1 + L^2} |x' - y'| \\ &\leq \sqrt{1 + L^2} \text{diam}(U). \end{aligned}$$

Taking the supremum over all $x, y \in \Gamma(f; U)$ we get the claim.

□

Proposition 7. *Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be an L -Lipschitz function, for some $L > 0$. Then, for every set $A \subset \mathbb{R}^{d-1}$ we have*

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \leq (1 + L^2)^{\frac{d-1}{2}} \mathcal{H}^{d-1}(A),$$

where $\Gamma(f; A)$ is the graph of f over A . In particular, since $\mathcal{H}^{d-1} = \mathcal{L}^{d-1}$ on $\mathbb{R}^{d-1}$, for every Lebesgue measurable set $A \subset \mathbb{R}^{d-1}$, we have

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \leq (1 + L^2)^{\frac{d-1}{2}} \mathcal{L}^{d-1}(A).$$

Proof. Suppose that

$$\mathcal{H}^{d-1}(A) = \sup_{\delta > 0} \mathcal{H}_\delta^{d-1}(A) < +\infty.$$

Then, for every $\delta > 0$, we can find a covering $\{U_i\}_{i \geq 1}$ of A with sets $U_i \subset \mathbb{R}^{d-1}$ such that

$$\text{diam}(U_i) < \frac{\delta}{\sqrt{1 + L^2}} \quad \text{for all } i \geq 1,$$

and

$$\omega_{d-1} \sum_{i \geq 1} \left(\frac{\text{diam}(U_i)}{2} \right)^{d-1} \leq \mathcal{H}^{d-1}(A) + \varepsilon.$$

We notice that $\Gamma(f; U_i)$, $i \geq 1$, is a covering of $\Gamma(f; A)$ and is such that:

$$\text{diam}(\Gamma(f; U_i)) \leq \sqrt{1 + L^2} \text{diam}(U_i) < \sqrt{1 + L^2} \frac{\delta}{\sqrt{1 + L^2}} = \delta.$$

By the definition of $\mathcal{H}_\delta^{d-1}(\Gamma(f; A))$, we have

$$\begin{aligned} \mathcal{H}_\delta^{d-1}(\Gamma(f; A)) &\leq \omega_{d-1} \sum_{i \geq 1} \left(\frac{\text{diam}(\Gamma(f; U_i))}{2} \right)^{d-1} \\ &\leq \omega_{d-1} (1 + L^2)^{\frac{d-1}{2}} \sum_{i \geq 1} \left(\frac{\text{diam}(U_i)}{2} \right)^{d-1} \\ &\leq (1 + L^2)^{\frac{d-1}{2}} (\mathcal{H}^{d-1}(A) + \varepsilon). \end{aligned}$$

Taking the supremum in $\delta > 0$, we get

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \leq (1 + L^2)^{\frac{d-1}{2}} (\mathcal{H}^{d-1}(A) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get the claim. $\square$

Corollary 8. *Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be an L -Lipschitz function, for some $L > 0$. Then, for every Lebesgue measurable set $A \subset \mathbb{R}^{d-1}$, we have*

$$\mathcal{L}^{d-1}(A) = 0 \quad \Leftrightarrow \quad \mathcal{H}^{d-1}(\Gamma(f; A)) = 0.$$

LIPSCHITZ GRAPHS AND HAUSDORFF MEASURE.

PART II. THE HAUSDORFF MEASURE OF THE GRAPH OF A LINEAR FUNCTION.

Lemma 9. *Suppose that $d \geq 2$ and $s \geq 0$. Then, for every $A' \subset \mathbb{R}^{d-1}$, we have*

$$\mathcal{H}^s(A') = \mathcal{H}^s(A' \times \{0\}),$$

where:

- $\mathcal{H}^s(A')$ is computed with the distance in $\mathbb{R}^{d-1}$ via coverings with sets in $\mathbb{R}^{d-1}$;
- $\mathcal{H}^s(A' \times \{0\})$, is calculated with the distance in $\mathbb{R}^d$ via coverings with subsets of $\mathbb{R}^d$.

Proof. We proceed in two steps.

Step 1. We first prove the inequality $\mathcal{H}^s(A' \times \{0\}) \leq \mathcal{H}^s(A')$.

Let $\delta > 0$ and let $\{U'_i\}_{i \geq 1}$ be a family of subsets of $\mathbb{R}^{d-1}$ such that:

- $\text{diam}(U'_i) \leq \delta$ for every $i \geq 1$.
- A' is contained in the union of U'_i , $i \geq 1$.

Then, the family of sets $U_i := U'_i \times \{0\}$ is a family of subsets of $\mathbb{R}^d$ such that:

- $\text{diam}(U_i) \leq \delta$ for every $i \geq 1$.
- $A' \times \{0\}$ is contained in the union of $U_i = U'_i \times \{0\}$, $i \geq 1$.

In particular, we have

$$\mathcal{H}_\delta^s(A' \times \{0\}) \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\text{diam}(U'_i \times \{0\})}{2} \right)^s \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\text{diam}(U'_i)}{2} \right)^s.$$

Taking the infimum on the right, over all coverings $\{U'_i\}_{i \geq 1}$ of A' with sets of diameter at most δ , we get

$$\mathcal{H}_\delta^s(A' \times \{0\}) \leq \mathcal{H}_\delta^s(A').$$

Passing to the limit as $\delta \rightarrow 0$, we get

$$\mathcal{H}^s(A' \times \{0\}) \leq \mathcal{H}^s(A'),$$

which concludes the proof of the inequality in the first step.

Step 2. We next prove the inequality $\mathcal{H}^s(A') \leq \mathcal{H}^s(A' \times \{0\})$.

Fix $\delta > 0$. Let $\{U_i\}_{i \geq 1}$ be a family of subsets of $\mathbb{R}^d$ such that:

- $\text{diam}(U_i) \leq \delta$ for every $i \geq 1$.
- A is contained in the union of U_i , $i \geq 1$.

Consider the family of projections $U'_i := \pi(U_i) \subset \mathbb{R}^{d-1}$, of the sets U_i on $\mathbb{R}^{d-1} = \mathbb{R}^{d-1} \times \{0\}$. Then:

- $\text{diam}(U'_i) \leq \text{diam}(U_i) \leq \delta$ for every $i \geq 1$.
- A' is contained in the union of U'_i , $i \geq 1$.

In particular, we have

$$\mathcal{H}_\delta^s(A') \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\text{diam}(U'_i)}{2} \right)^s \leq \sum_{i=1}^{+\infty} \omega_s \left(\frac{\text{diam}(U_i)}{2} \right)^s.$$

Taking the infimum on the right, over all coverings $\{U_i\}_{i \geq 1}$ of A with sets of diameter at most δ , we get

$$\mathcal{H}_\delta^s(A') \leq \mathcal{H}_\delta^s(A' \times \{0\}).$$

Passing to the limit as $\delta \rightarrow 0$, we get

$$\mathcal{H}^s(A') \leq \mathcal{H}^s(A' \times \{0\}),$$

which concludes the proof. □

Proposition 10. Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a function of the form

$$f(x) = c + x \cdot V,$$

for some $a \in \mathbb{R}$ and some fixed vector $V \in \mathbb{R}^{d-1} \setminus \{0\}$. Then, for every measurable set $A \subset \mathbb{R}^{d-1}$ we have

$$\mathcal{H}^{d-1}(\Gamma(f; A)) = \sqrt{1 + |V|^2} \mathcal{L}^{d-1}(A).$$

Proof. Since the $(d-1)$ -Hausdorff measure in $\mathbb{R}^d$ and the Lebesgue measure in $\mathcal{L}^d$ are both invariant with respect to translations and rotations, we can suppose that

$$c = 0 \quad \text{and} \quad V = a e_{d-1},$$

for some real constant $a > 0$.

Then, the set $\Gamma(f; A)$ is contained in the plane $\Gamma(f; \mathbb{R}^{d-1})$ and is given by:

$$\Gamma(f; A) := \left\{ (x'', x, ax) \in \mathbb{R}^{d-2} \times \mathbb{R} \times \mathbb{R} : (x'', x) \in A \right\}.$$

Let now $\mathcal{R}$ be a rotation sending the plane $\Gamma(f; \mathbb{R}^{d-1})$ in $\mathbb{R}^{d-1} \times \{0\}$. Then,

$$\mathcal{H}^{d-1}(\Gamma(f; A)) = \mathcal{H}^{d-1}(\mathcal{R}(\Gamma(f; A))).$$

On the other hand, for all $(x'', x, y) \in \Gamma(f; A)$, we have that

$$\mathcal{R}(x'', x, y) = (x'', \sqrt{x^2 + y^2}, 0).$$

Thus,

$$\mathcal{R}(\Gamma(f; A)) = \left\{ (x'', \sqrt{1 + a^2} x, 0) : (x'', x) \in A \right\}.$$

Since $\mathcal{L}^{d-1} = \mathcal{H}^{d-1}$ on $\mathbb{R}^{d-1}$, we get the claim by the Fubini's theorem. □

LIPSCHITZ GRAPHS AND HAUSDORFF MEASURE.
PART III. THE HAUSDORFF MEASURE OF THE GRAPH OF A LIPSCHITZ FUNCTION.

In this section we prove the following theorem.

Theorem 11 (Area formula for C^1 graphs). *Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a C^1 function with bounded gradient $\nabla f \in L^\infty(\mathbb{R}^{d-1}; \mathbb{R}^{d-1})$. Then, for every measurable set $A \subset \mathbb{R}^{d-1}$ we have*

$$(1) \quad \mathcal{H}^{d-1}(\Gamma(f; A)) = \int_A \sqrt{1 + |\nabla f|^2} dx,$$

where $\Gamma(f; A)$ is the graph of f over A .

Theorem 12 (A covering theorem). *Let $A \subset \mathbb{R}^d$ be a set of finite Lebesgue measure. For every $x \in A$, let $\mathcal{F}_x$ be a family of balls of the form*

$$\mathcal{F}_x = \left\{ \overline{B}_{r_{x,n}}(x) : n \in \mathbb{N} \right\} \quad \text{with} \quad \lim_{n \rightarrow +\infty} r_{x,n} = 0.$$

Then, there is a countable set of centers

$$\{x_k\}_{k \geq 1} \subset A,$$

and radii

$$r_k := r_{x_k, n(x_k)}, \quad k \geq 1,$$

such that:

- $\overline{B}_{r_k}(x_k) \in \mathcal{F}_{x_k}$ for every $k \geq 1$;
- the balls $\overline{B}_{r_k}(x_k)$ are disjoint;
- $\mathcal{L}^d \left(A \setminus \left(\bigcup_{k \geq 1} \overline{B}_{r_k}(x_k) \right) \right) = 0$.

Proof. The proof follows by iterating the following lemma. □

Lemma 13. *Let $A \subset \mathbb{R}^d$ be a set of finite Lebesgue measure. Suppose that, for every $x \in A$, there is a family of balls*

$$\mathcal{F}_x = \left\{ \overline{B}_{r(x,n)}(x) : n \in \mathbb{N} \text{ such that } \lim_{n \rightarrow +\infty} r(x,n) = 0 \right\}.$$

Then, for every open set $\Omega \subset \mathbb{R}^d$ containing A , there is a finite set of centers

$$x_k \in A, \quad k = 1, \dots, m,$$

and radii

$$r_k := r(x_k, n(x_k)), \quad k = 1, \dots, m,$$

such that:

- $\overline{B}_{r_k}(x_k) \in \mathcal{F}_{x_k}$ for every $k = 1, \dots, m$;
- the balls $\overline{B}_{r_k}(x_k)$ are disjoint and contained in Ω ;
- there is a dimensional constant $\varepsilon_d \in (0, 1)$ such that

$$\mathcal{L}^d \left(A \setminus \left(\bigcup_{k=1}^m \overline{B}_{r_k}(x_k) \right) \right) \leq (1 - \varepsilon_d) \mathcal{L}^d(A).$$

Proof. We fix a dimensional constant $\kappa = \kappa_d \in (0, 1)$, which will be chosen at the end of the proof. Let $U \subset \mathbb{R}^d$ be an open set such that

$$A \subset U \subset \Omega \quad \text{and} \quad \mathcal{L}^d(U \setminus A) \leq \kappa \mathcal{L}^d(A).$$

For every $x \in A$ pick a ball

$$\overline{B}_{r(x,n(x))}(x) \in \mathcal{F}_x \quad \text{such that} \quad \overline{B}_{r(x,n(x))}(x) \subset U.$$

Let $\mathcal{F}$ be the family of closed balls

$$\mathcal{F} = \left\{ \overline{B}_{r(x,n(x))}(x) : x \in A \right\}.$$

By the Besicovitch's covering theorem, there are N families of balls

$$\mathcal{F}_1, \dots, \mathcal{F}_N,$$

such that:

- $N \leq N_d$, where N_d is a dimensional constant;
- each family of balls $\mathcal{F}_j$, $j = 1, \dots, N$, is a subset (subfamily) of $\mathcal{F}$;
- the union of the balls in $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_N$ covers A ;
- every subfamily $\mathcal{F}_j$ consists of (at most) countably many disjoint closed balls $\{B_{j,k}\}_{k \geq 1}$.

By construction we have

$$A \subset \bigcup_{j=1}^N \left(\bigcup_{k \geq 1} B_{j,k} \right) \subset U.$$

Then, we can find j such that

$$\frac{1}{N} \mathcal{L}^d(A) \leq \mathcal{L}^d \left(\bigcup_{k \geq 1} B_{j,k} \right) \leq \mathcal{L}^d(U).$$

Since the balls $B_{j,k}$, $k \geq 1$, are all disjoint, we have

$$\mathcal{L}^d \left(\bigcup_{B_{j,k} \in \mathcal{F}_j} B_{j,k} \right) = \sum_{k=1}^{+\infty} \mathcal{L}^d(B_{j,k}).$$

Then, we can find a (large) finite number $m > 0$, such that

$$\frac{1}{2N_d} \mathcal{L}^d(A) \leq \frac{1}{2N} \mathcal{L}^d(A) \leq \mathcal{L}^d \left(\bigcup_{k=1}^m B_{j,k} \right) = \sum_{k=1}^m \mathcal{L}^d(B_{j,k}) \leq \mathcal{L}^d(U).$$

Now, using the identity

$$\mathcal{L}^d \left(\bigcup_{k=1}^m B_{j,k} \right) + \mathcal{L}^d \left(A \setminus \left(\bigcup_{k=1}^m B_{j,k} \right) \right) = \mathcal{L}^d \left(A \cup \left(\bigcup_{k=1}^m B_{j,k} \right) \right) = \mathcal{L}^d(A) + \mathcal{L}^d \left(\left(\bigcup_{k=1}^m B_{j,k} \right) \setminus A \right),$$

we get that

$$\begin{aligned} \mathcal{L}^d \left(A \setminus \left(\bigcup_{k=1}^m B_{j,k} \right) \right) &= \mathcal{L}^d(A) + \mathcal{L}^d \left(\left(\bigcup_{k=1}^m B_{j,k} \right) \setminus A \right) - \mathcal{L}^d \left(\bigcup_{k=1}^m B_{j,k} \right) \\ &\leq \mathcal{L}^d(A) + \mathcal{L}^d(U \setminus A) - \mathcal{L}^d \left(\bigcup_{k=1}^m B_{j,k} \right) \\ &\leq \mathcal{L}^d(A) + \mathcal{L}^d(U \setminus A) - \frac{1}{2N_d} \mathcal{L}^d(A) \\ &\leq \mathcal{L}^d(A) + \kappa \mathcal{L}^d(A) - \frac{1}{2N_d} \mathcal{L}^d(A). \end{aligned}$$

Choosing

$$\kappa = \frac{1}{4N_d} = \varepsilon_d,$$

we get that

$$\mathcal{L}^d \left(A \setminus \left(\bigcup_{k=1}^m B_{j,k} \right) \right) \leq \mathcal{L}^d(A) + \frac{1}{4N_d} \mathcal{L}^d(A) - \frac{1}{2N_d} \mathcal{L}^d(A) = (1 - \varepsilon_d) \mathcal{L}^d(A),$$

which concludes the proof. $\square$

Lemma 14 (Control on the projections on tangent planes). *Let $L : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a function of the form*

$$L(x') = c + x' \cdot V,$$

for some $c \in \mathbb{R}$ and some $V \in \mathbb{R}^{d-1}$. Let P_L be the plane $P_L := \Gamma(L; \mathbb{R}^{d-1})$ and $\pi_L : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the projection on P_L . Let $B'_r(x'_0) \subset \mathbb{R}^d$ and let $f : B'_r(x'_0) \rightarrow \mathbb{R}$ be a continuous function such that

$$\|f - L\|_{L^\infty(B'_r(x'_0))} \leq \varepsilon r.$$

Then,

$$\Gamma(L; B'_{(1-\varepsilon)r}(x'_0)) \subset \pi_L\left(\Gamma(f; B'_r(x'_0))\right) \subset \Gamma(L; B'_{(1+\varepsilon)r}(x'_0)).$$

Proof. Up to translations (in $\mathbb{R}^d$) and rotations (in $\mathbb{R}^{d-1}$), we can assume that:

$$x_0 = 0, \quad c = 0 \quad \text{and} \quad V = ae_{d-1},$$

for some $a \geq 0$. Then, for every $x' = (x'', x_{d-1}) \in \mathbb{R}^{d-2} \times \mathbb{R}$, we have

$$L(x'', x_{d-1}) = ax_{d-1},$$

and

$$P_L := \left\{ (x'', x_{d-1}, ax_{d-1}) : x'' \in \mathbb{R}^{d-2}, x_{d-1} \in \mathbb{R} \right\}.$$

For every $(x'', s, t) \in \mathbb{R}^{d-2} \times \mathbb{R} \times \mathbb{R}$, the map π_L is given by

$$\pi_L(x'', s, t) = \left(x'', \frac{s + ta}{1 + a^2}, \frac{a(s + ta)}{1 + a^2} \right).$$

Thus, for every $(x'', x_{d-1}) \in B'_r$ we have

$$\pi_L(x'', x_{d-1}, f(x'', x_{d-1})) = \left(x'', \frac{x_{d-1} + af(x'', x_{d-1})}{1 + a^2}, \frac{a(x_{d-1} + af(x'', x_{d-1}))}{1 + a^2} \right),$$

and so

$$\begin{aligned} & \left| \pi_L(x'', x_{d-1}, f(x'', x_{d-1})) - (x'', x_{d-1}, ax_{d-1}) \right| \\ &= \sqrt{\left(\frac{x_{d-1} + af(x'', x_{d-1})}{1 + a^2} - x_{d-1} \right)^2 + \left(\frac{a(x_{d-1} + af(x'', x_{d-1}))}{1 + a^2} - ax_{d-1} \right)^2} \\ &= \frac{1}{\sqrt{1 + a^2}} \left| (x_{d-1} + af(x'', x_{d-1})) - (1 + a^2)x_{d-1} \right| \\ &= \frac{a}{\sqrt{1 + a^2}} |f(x'', x_{d-1}) - ax_{d-1}| \\ &\leq |f(x'', x_{d-1}) - ax_{d-1}| = |f(x'', x_{d-1}) - L(x'', x_{d-1})| \leq \varepsilon r. \end{aligned}$$

Let π be the projection on $\mathbb{R}^{d-1}$. It is sufficient to show that

$$(2) \quad B'_{(1-\varepsilon)r} \subset \pi\left(\pi_L(x'', x_{d-1}, f(x'', x_{d-1}))\right) \subset B'_{(1+\varepsilon)r}.$$

We first notice that, thanks to the inequality

$$\left| \pi_L(x'', x_{d-1}, f(x'', x_{d-1})) - (x'', x_{d-1}, ax_{d-1}) \right| \leq \varepsilon r,$$

we have

$$\left| \pi\left(\pi_L(x'', x_{d-1}, f(x'', x_{d-1}))\right) - \pi(x'', x_{d-1}, ax_{d-1}) \right| \leq \varepsilon r.$$

Now, since

$$\pi(x'', x_{d-1}, ax_{d-1}) = (x'', x_{d-1}) \in B'_r,$$

we get that

$$\pi\left(\pi_L(x'', x_{d-1}, f(x'', x_{d-1}))\right) \subset B'_{(1+\varepsilon)r},$$

which proves the second inclusion in (2). In order to prove the other inclusion, we fix some $x'' \in \mathbb{R}^{d-2}$ with $|x''| < r$ and we consider the segment

$$I_{x''} := \left(-\sqrt{r^2 - |x''|^2}, \sqrt{r^2 - |x''|^2} \right).$$

Suppose next that

$$|x''| \leq r\sqrt{1 - \varepsilon^2}.$$

Then,

$$\sqrt{r^2 - |x''|^2} \geq \varepsilon r,$$

so we have the following inclusion:

$$(-\varepsilon r, \varepsilon r) \subset I_{x''}.$$

Now, by the continuity of f we know that, for each fixed $x'' \in (-r, r)$, the set

$$\left\{ \pi \left(\pi_L(x'', x_{d-1}, f(x'', x_{d-1})) \right) : x_{d-1} \in I_{x''} \right\}$$

contains the set

$$\{x''\} \times J_{x''},$$

where $J_{x''} = [\alpha(x''), \beta(x'')]$ is the interval with endpoints

$$\alpha(x'') := \frac{-\sqrt{r^2 - |x''|^2} + af(x'', -\sqrt{r^2 - |x''|^2})}{1 + a^2} \quad \text{and} \quad \beta(x'') := \frac{\sqrt{r^2 - |x''|^2} + af(x'', \sqrt{r^2 - |x''|^2})}{1 + a^2}.$$

Now, since

$$\left| \frac{x_{d-1} + af(x'', x_{d-1})}{1 + a^2} - x_{d-1} \right| = \frac{a}{1 + a^2} |f(x'', x_{d-1}) - ax_{d-1}| \leq \frac{\varepsilon r}{2},$$

we get that $J_{x''}$ contains the interval

$$K_{x''} := \left(-\sqrt{r^2 - |x''|^2} + \frac{\varepsilon r}{2}, \sqrt{r^2 - |x''|^2} - \frac{\varepsilon r}{2} \right),$$

which is non-empty when $|x''| \leq r\sqrt{1 - \varepsilon^2}$. Moreover, thanks to the inequality

$$\sqrt{r^2 - |x''|^2} - \frac{\varepsilon r}{2} \geq \sqrt{r^2(1 - \varepsilon)^2 - |x''|^2},$$

which easily follows from the following

$$\begin{aligned} \sqrt{r^2 - |x''|^2} - \sqrt{r^2(1 - \varepsilon)^2 - |x''|^2} &= \frac{(1 - (1 - \varepsilon)^2)r^2}{\sqrt{r^2 - |x''|^2} + \sqrt{r^2(1 - \varepsilon)^2 - |x''|^2}} \\ &\geq \frac{2\varepsilon r^2}{\sqrt{r^2 - |x''|^2} + \sqrt{r^2(1 - \varepsilon)^2 - |x''|^2}} \geq \frac{\varepsilon r^2}{\sqrt{r^2 - |x''|^2}} \geq \varepsilon r, \end{aligned}$$

we get that

$$\left(-\sqrt{r^2(1 - \varepsilon)^2 - |x''|^2}, \sqrt{r^2(1 - \varepsilon)^2 - |x''|^2} \right) \subset K_{x''} \subset J_{x''},$$

for all $|x''| \leq r(1 - \varepsilon)$. Taking the union of all these segments, for all $x'' \in B''_{(1-\varepsilon)r} \subset \mathbb{R}^{d-2}$, we get the inclusion

$$B'_{r(1-\varepsilon)} \subset \pi \left(\pi_L(x'', x_{d-1}, f(x'', x_{d-1})) \right),$$

which concludes the proof of (2). $\square$

Proof of the area formula for C^1 graphs. Let A be a Lebesgue measurable set in $\mathbb{R}^{d-1}$.

If $\mathcal{L}^{d-1}(A) = +\infty$, then also $\mathcal{H}^{d-1}(A) = +\infty$. Since A is the projection of $\Gamma(f; A)$ on $\mathbb{R}^{d-1}$,

$$\pi(\Gamma(f; A)) = A,$$

we get that

$$\mathcal{H}^{d-1}(A) \leq \mathcal{H}^{d-1}(\Gamma(f; A)).$$

Thus, when $\mathcal{L}^{d-1}(A) = +\infty$, we have:

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \geq \mathcal{H}^{d-1}(A) = \mathcal{L}^{d-1}(A) = +\infty,$$

and at the same time

$$\int_A \sqrt{1 + |\nabla f|^2} dx \geq \mathcal{L}^{d-1}(A) = +\infty,$$

so (3) holds trivially as both sides are $+\infty$.

In what follows, we will assume that $\mathcal{L}^{d-1}(A) < +\infty$.

We proceed in two steps.

Step 1. $\int_A \sqrt{1 + |\nabla f|^2} dx \leq \mathcal{H}^{d-1}(\Gamma(f; A)).$

Fix $\varepsilon > 0$. Fix an open set Ω containing A and such that $\mathcal{L}^{d-1}(\Omega \setminus A) \leq \varepsilon \mathcal{L}^{d-1}(A)$. For every $x \in A$ we select a family of closed balls $\mathcal{F}_x$ with center x such that:

- $\overline{B}_r(x) \subset \Omega$ for all $\overline{B}_r(x) \in \mathcal{F}_x$;
- $|f(y) - f(x) - (y - x) \cdot \nabla f(x)| \leq \varepsilon |y - x| \leq \varepsilon r$ for all $y \in \overline{B}_r(x)$ and all balls $\overline{B}_r(x) \in \mathcal{F}_x$;
- for every ball $\overline{B}_r(x)$, we have

$$\left| \sqrt{1 + |\nabla f(x)|^2} - \frac{1}{\mathcal{L}^{d-1}(B_r)} \int_{A \cap B_r(x)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \right| \leq \varepsilon.$$

Thanks to Theorem 12, we can find a countable family of disjoint closed balls $\{\overline{B}_{r_i}(x_i)\}_{i \geq 1}$ such that the following holds

- $x_i \in A$ and $\overline{B}_{r_i}(x_i) \subset \Omega$ for all $i \geq 1$;
- setting

$$U := \bigcup_{i \geq 1} \overline{B}_{r_i}(x_i),$$

we have that

$$\mathcal{L}^{d-1}(A \setminus U) = 0 \quad \text{and} \quad \mathcal{L}^{d-1}(U \setminus A) \leq \varepsilon \mathcal{L}^{d-1}(A);$$

- setting $L_i(y) := f(x_i) + (y - x_i) \cdot \nabla f(x_i)$, we have

$$\|f - L_i\|_{L^\infty(B_{r_i}(x_i))} \leq \varepsilon r_i \quad \text{for all } i \geq 1;$$

- for all $i \geq 1$, we have

$$\left| \sqrt{1 + |\nabla f(x_i)|^2} - \frac{1}{\mathcal{L}^{d-1}(B_{r_i})} \int_{A \cap B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \right| \leq \varepsilon.$$

Then, thanks to Lemma 14, we have

$$\mathcal{H}^{d-1}(\Gamma(f; B_{r_i}(x_i))) \geq \mathcal{H}^{d-1}(\pi_i(\Gamma(f; B_{r_i}(x_i)))) \geq \mathcal{H}^{d-1}(\Gamma(L_i; B_{r_i(1-\varepsilon)}(x_i))).$$

Now, thanks to Proposition 10, we have that

$$\mathcal{H}^{d-1}(\Gamma(L_i; B_{r_i(1-\varepsilon)}(x_i))) = \sqrt{1 + |\nabla L_i|^2} \mathcal{L}^{d-1}(B_{r_i(1-\varepsilon)}(x_i)) = (1 - \varepsilon)^{d-1} \sqrt{1 + |\nabla L_i|^2} \mathcal{L}^{d-1}(B_{r_i}).$$

By the choice of L_i , we have that $\nabla L_i = \nabla f(x_i)$, so we have

$$\mathcal{H}^{d-1}(\Gamma(L_i; B_{r_i(1-\varepsilon)}(x_i))) = (1 - \varepsilon)^{d-1} \sqrt{1 + |\nabla f(x_i)|^2} \mathcal{L}^{d-1}(B_{r_i}),$$

which finally gives the inequality

$$\mathcal{H}^{d-1}(\Gamma(f; B_{r_i}(x_i))) \geq (1 - \varepsilon)^{d-1} \sqrt{1 + |\nabla f(x_i)|^2} \mathcal{L}^{d-1}(B_{r_i}).$$

Now, since x_i is a Lebesgue point for both A and $\sqrt{1 + |\nabla f|^2}$, we have

$$\left| \sqrt{1 + |\nabla f(x_i)|^2} - \frac{1}{\mathcal{L}^{d-1}(B_{r_i})} \int_{A \cap B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \right| \leq \varepsilon,$$

so

$$\frac{1}{\mathcal{L}^{d-1}(B_{r_i})} \int_{A \cap B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \leq \varepsilon + \sqrt{1 + |\nabla f(x_i)|^2} \leq (1 + \varepsilon) \sqrt{1 + |\nabla f(x_i)|^2},$$

which finally gives

$$\mathcal{H}^{d-1}(\Gamma(f; B_{r_i}(x_i))) \geq (1 - \varepsilon)^{d-2} \int_{A \cap B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1}.$$

Now, summing up over all $i \geq 1$, we obtain

$$\begin{aligned}
\mathcal{H}^{d-1}(\Gamma(f; U)) &= \sum_{i \geq 1} \mathcal{H}^{d-1}(\Gamma(f; \overline{B}_{r_i}(x_i))) \\
&\geq (1 - \varepsilon)^{d-2} \sum_{i \geq 1} \int_{A \cap \overline{B}_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \\
&= (1 - \varepsilon)^{d-2} \int_{A \cap U} \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1} \\
&= (1 - \varepsilon)^{d-2} \int_A \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1}.
\end{aligned}$$

Now, we only need to observe that, thanks to Proposition 7, we have

$$\begin{aligned}
\mathcal{H}^{d-1}(\Gamma(f; U)) &= \mathcal{H}^{d-1}(\Gamma(f; A \cap U)) + \mathcal{H}^{d-1}(\Gamma(f; U \setminus A)) \\
&= \mathcal{H}^{d-1}(\Gamma(f; A)) + \mathcal{H}^{d-1}(\Gamma(f; U \setminus A)) \\
&\leq \mathcal{H}^{d-1}(\Gamma(f; A)) + (1 + L^2)^{\frac{d-1}{2}} \mathcal{L}^{d-1}(U \setminus A) \\
&\leq \mathcal{H}^{d-1}(\Gamma(f; A)) + \varepsilon (1 + L^2)^{\frac{d-1}{2}} \mathcal{L}^{d-1}(A) \\
&\leq \mathcal{H}^{d-1}(\Gamma(f; A)) + \varepsilon (1 + L^2)^{\frac{d-1}{2}} \int_A \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1}.
\end{aligned}$$

Putting together all these estimates, we get

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \geq \left((1 - \varepsilon)^{d-2} - \varepsilon (1 + L^2)^{\frac{d-1}{2}} \right) \int_A \sqrt{1 + |\nabla f|^2} d\mathcal{L}^{d-1}.$$

Since $\varepsilon > 0$ is arbitrary, the proof of Step 1 is concluded.

Step 2. $\mathcal{H}^{d-1}(\Gamma(f; A)) \leq \int_A \sqrt{1 + |\nabla f|^2} dx$.

The construction is similar, yet different with respect to the one from the previous step. Fix $\varepsilon > 0$. Fix an open set Ω containing A and such that

$$\int_{\Omega \setminus A} \sqrt{1 + |\nabla f|^2} dx \leq \varepsilon \int_A \sqrt{1 + |\nabla f|^2} dx.$$

For every $x \in A$ we select a family of closed balls $\mathcal{F}_x$ with center x such that:

- $\overline{B}_r(x) \subset \Omega$ for all $\overline{B}_r(x) \in \mathcal{F}_x$;
- $|\nabla f(y) - \nabla f(x)| \leq \varepsilon$ for all $y \in \overline{B}_r(x)$ and all balls $\overline{B}_r(x) \in \mathcal{F}_x$.

By Theorem 12, there is a countable family of disjoint closed balls $\{\overline{B}_{r_i}(x_i)\}_{i \geq 1}$ such that the following holds:

- $x_i \in A$ and $\overline{B}_{r_i}(x_i) \subset \Omega$ for all $i \geq 1$;
- setting $U := \bigcup_{i \geq 1} \overline{B}_{r_i}(x_i)$, we have that $\mathcal{L}^{d-1}(A \setminus U) = 0$;
- setting $L_i(y) := f(x_i) + (y - x_i) \cdot \nabla f(x_i)$, we have

$$\|\nabla f - \nabla L_i\|_{L^\infty(B_{r_i}(x_i))} \leq \varepsilon \quad \text{for all } i \geq 1.$$

For each $i \geq 1$ we define the map

$$\Phi_i : \Gamma(L_i; \overline{B}_{r_i}(x_i)) \rightarrow \Gamma(f; \overline{B}_{r_i}(x_i)),$$

as follows:

$$\Phi_i((x', L_i(x'))) = (x', f(x')) \quad \text{for all } x' \in \overline{B}_{r_i}(x_i).$$

We will next prove that Φ_i is Lipschitz. Indeed, for all $x', y' \in \overline{B}_{r_i}(x_i)$, we have

$$\begin{aligned}
\left| (x', f(x')) - (y', f(y')) \right| &= \sqrt{|x' - y'|^2 + |f(x') - f(y')|^2} \\
&= \sqrt{|x' - y'|^2 + |(x' - y') \cdot \nabla f(tx' + (1-t)y')|^2},
\end{aligned}$$

for some $t = t_{x', y'} \in (0, 1)$. Now, since $tx' + (1-t)y' \in B_{r_i}(x_i)$, we have that

$$|\nabla f(x_i) - \nabla f(tx' + (1-t)y')| \leq \varepsilon,$$

so we get

$$\begin{aligned}
|x' - y'|^2 + |(x' - y') \cdot \nabla f(tx' + (1-t)y')|^2 &\leq |x' - y'|^2 + |(x' - y') \cdot \nabla L_i|^2 \\
&\quad + 2\left((x' - y') \cdot \nabla L_i\right)\left((x' - y') \cdot (\nabla L_i - \nabla f(tx' + (1-t)y'))\right) \\
&\quad + \left|(x' - y') \cdot (\nabla L_i - \nabla f(tx' + (1-t)y'))\right|^2 \\
&\leq |x' - y'|^2 + |(x' - y') \cdot \nabla L_i|^2 \\
&\quad + 2|(x' - y') \cdot \nabla L_i| |x' - y'| \varepsilon \\
&\quad + |x' - y'|^2 \varepsilon^2 \\
&\leq |x' - y'|^2 + |(x' - y') \cdot \nabla L_i|^2 \\
&\quad + \varepsilon\left(|(x' - y') \cdot \nabla L_i|^2 + |x' - y'|^2\right) \\
&\quad + |x' - y'|^2 \varepsilon^2 \\
&\leq (1 + \varepsilon)^2\left(|x' - y'|^2 + |(x' - y') \cdot \nabla L_i|^2\right),
\end{aligned}$$

so that

$$\begin{aligned}
\left|(x', f(x')) - (y', f(y'))\right| &\leq (1 + \varepsilon)\sqrt{|x' - y'|^2 + |(x' - y') \cdot \nabla L_i|^2} \\
&= \left|(x', L_i(x')) - (y', L_i(y'))\right|,
\end{aligned}$$

which proves that Φ_i is $(1 + \varepsilon)$ -Lipschitz continuous. Thus,

$$\begin{aligned}
\mathcal{H}^{d-1}\left(\Gamma(f; \overline{B}_{r_i}(x_i))\right) &\leq \mathcal{H}^{d-1}\left(\Phi_i\left(\Gamma(L_i; \overline{B}_{r_i}(x_i))\right)\right) \\
&\leq (1 + \varepsilon)^{d-1} \mathcal{H}^{d-1}\left(\Gamma(L_i; \overline{B}_{r_i}(x_i))\right) \\
&= (1 + \varepsilon)^{d-1} \int_{B_{r_i}(x_i)} \sqrt{1 + |\nabla L_i|^2} dx.
\end{aligned}$$

Now, since

$$\left|\sqrt{1 + |V|^2} - \sqrt{1 + |W|^2}\right| = \left|\frac{|V|^2 - |W|^2}{\sqrt{1 + |V|^2} + \sqrt{1 + |W|^2}}\right| \leq |V - W| \quad \text{for all } V, W \in \mathbb{R}^{d-1},$$

we have that

$$\int_{B_{r_i}(x_i)} \sqrt{1 + |\nabla L_i|^2} dx \leq (1 + \varepsilon) \int_{B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} dx,$$

so we finally obtain

$$\mathcal{H}^{d-1}\left(\Gamma(f; \overline{B}_{r_i}(x_i))\right) \leq (1 + \varepsilon)^d \int_{B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} dx.$$

Now, summing over all $i \geq 1$, we get

$$\begin{aligned}
\mathcal{H}^{d-1}\left(\Gamma(f; A)\right) &\leq \sum_{i=1}^{+\infty} \mathcal{H}^{d-1}\left(\Gamma(f; \overline{B}_{r_i}(x_i))\right) \\
&\leq \sum_{i=1}^{+\infty} (1 + \varepsilon)^d \int_{B_{r_i}(x_i)} \sqrt{1 + |\nabla f|^2} dx \\
&= (1 + \varepsilon)^d \int_U \sqrt{1 + |\nabla f|^2} dx \\
&= (1 + \varepsilon)^d \int_A \sqrt{1 + |\nabla f|^2} dx + (1 + \varepsilon)^d \int_{\Omega \setminus A} \sqrt{1 + |\nabla f|^2} dx \\
&= (1 + \varepsilon)^{d+1} \int_A \sqrt{1 + |\nabla f|^2} dx,
\end{aligned}$$

which concludes the proof since $\varepsilon > 0$ is arbitrary. $\square$

LIPSCHITZ GRAPHS AND HAUSDORFF MEASURE.
PART IV. THE HAUSDORFF MEASURE OF THE GRAPH OF A LIPSCHITZ FUNCTION.

In this section we prove the following theorem.

Theorem 15 (Area formula for Lipschitz graphs). *Let $d \geq 2$ and let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be an L -Lipschitz function, for some $L > 0$. Then, for every measurable set $A \subset \mathbb{R}^{d-1}$ we have*

$$(3) \quad \mathcal{H}^{d-1}(\Gamma(f; A)) = \int_A \sqrt{1 + |\nabla f|^2} \, dx,$$

where $\Gamma(f; A)$ is the graph of f over A .

The key for the proof of the area formula is the following lemma.

Lemma 16. *Let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a Lipschitz function and let $A \subset \mathbb{R}^{d-1}$ be a Borel set of points at which f is differentiable. Then, for every $\varepsilon \in (0, 1/2)$, there is a countable collection of disjoint Borel sets $\{E_k\}_{k \geq 1}$ in $\mathbb{R}^{d-1}$ and vectors $\{V_k\}_{k \geq 1}$ in $\mathbb{R}^{d-1}$ such that:*

- (i) $\text{diam}(E_k) \leq 2$ for all $k \geq 1$;
- (ii) $E_k \subset A$ for all $k \geq 1$;
- (iii) $A = \bigcup_{k \geq 1} E_k$;
- (iv) $|V_k - \nabla f(x')| \leq \varepsilon$ for all $x' \in E_k$;
- (v) for all $x', y' \in E_k$, it holds

$$(4) \quad \begin{aligned} & (1 - 2\varepsilon) \sqrt{|x' - y'|^2 + (V_k \cdot (x' - y'))^2} \\ & \leq \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \\ & \leq (1 + 2\varepsilon) \sqrt{|x' - y'|^2 + (V_k \cdot (x' - y'))^2}. \end{aligned}$$

Proof. Let $\mathcal{C}$ be a dense subset of A . For each $(p', V, n) \in \mathcal{C} \times \mathbb{Q}^{d-1} \times \mathbb{N}$, define the set

$$E(p', V, n) \subset A,$$

as follows. We say that $x' \in E(p', V, n)$, if:

- $x' \in B'_{1/n}(p')$;
- $|V - \nabla f(x')| \leq \varepsilon$;
- for all $y' \in B'_{2/n}(x')$ it holds:

$$|f(y') - f(x') - \nabla f(x') \cdot (y' - x')| \leq \varepsilon |y' - x'|.$$

Step 1. We claim that, if $(p', V, n) \in \mathcal{C} \times \mathbb{Q}^{d-1} \times \mathbb{N}$, then

$$\text{for all } x' \in E(p', V, n) \text{ and all } y' \in B'_{2/n}(x'),$$

we have

$$(5) \quad \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \leq (1 + 2\varepsilon) \sqrt{|x' - y'|^2 + (V \cdot (x' - y'))^2}.$$

Indeed,

$$\begin{aligned} \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} & \leq \sqrt{|x' - y'|^2 + \left((x' - y') \cdot \nabla f(x') + \varepsilon |x' - y'| \right)^2} \\ & \leq \sqrt{|x' - y'|^2 + \left((x' - y') \cdot V + (x' - y') \cdot (\nabla f(x') - V) + \varepsilon |x' - y'| \right)^2}. \end{aligned}$$

Now, since

$$\begin{aligned}
\left((x' - y') \cdot \nabla f(x') + \varepsilon |x' - y'| \right)^2 &\leq \left((x' - y') \cdot V + (x' - y') \cdot (\nabla f(x') - V) + \varepsilon |x' - y'| \right)^2 \\
&\leq \left((x' - y') \cdot V \right)^2 \\
&\quad + 2 \left((x' - y') \cdot V \right) \left((x' - y') \cdot (\nabla f(x') - V) + \varepsilon |x' - y'| \right) \\
&\quad + \left((x' - y') \cdot (\nabla f(x') - V) + \varepsilon |x' - y'| \right)^2 \\
&\leq |(x' - y') \cdot V|^2 \\
&\quad + 4\varepsilon |(x' - y') \cdot V| |x' - y'| \\
&\quad + 4\varepsilon^2 |x' - y'|^2 \\
&\leq |(x' - y') \cdot V|^2 \\
&\quad + 2\varepsilon \left(|(x' - y') \cdot V|^2 + |x' - y'|^2 \right) \\
&\quad + 4\varepsilon^2 |x' - y'|^2,
\end{aligned}$$

we get precisely (5).

Step 2. We claim that, if $(p', V, n) \in \mathcal{C} \times \mathbb{Q}^{d-1} \times \mathbb{N}$, then

$$\text{for all } x' \in E(p', V, n) \text{ and all } y' \in B'_{2/n}(x'),$$

we have

$$(6) \quad \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \geq (1 - 2\varepsilon) \sqrt{|x' - y'|^2 + (V \cdot (x' - y'))^2}.$$

Indeed, since

$$\begin{aligned}
|f(x') - f(y')| &\geq |(y' - x') \cdot \nabla f(x')| - |f(x') - f(y') - (y' - x') \cdot \nabla f(x')| \\
&\geq |(y' - x') \cdot \nabla f(x')| - \varepsilon |y' - x'| \\
&\geq |(y' - x') \cdot V| - |(y' - x') \cdot (V - \nabla f(x'))| - \varepsilon |y' - x'| \\
&\geq |(y' - x') \cdot V| - 2\varepsilon |y' - x'|,
\end{aligned}$$

we get

$$\begin{aligned}
&\sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \\
&\geq \sqrt{|x' - y'|^2 + \left(|(y' - x') \cdot V| - 2\varepsilon |y' - x'| \right)^2} \\
&= \sqrt{|x' - y'|^2 + |(y' - x') \cdot V|^2 - 4\varepsilon |(y' - x') \cdot V| |x' - y'| + 4\varepsilon^2 |x' - y'|^2} \\
&\geq \sqrt{|x' - y'|^2 + |(y' - x') \cdot V|^2 - 2\varepsilon \left(|x' - y'|^2 + |(y' - x') \cdot V|^2 \right) + 4\varepsilon^2 |x' - y'|^2} \\
&\geq \sqrt{|x' - y'|^2 + |(y' - x') \cdot V|^2 - 4\varepsilon \left(|x' - y'|^2 + |(y' - x') \cdot V|^2 \right) + 4\varepsilon^2 \left(|x' - y'|^2 + |(y' - x') \cdot V|^2 \right)} \\
&= (1 - 2\varepsilon) \sqrt{|x' - y'|^2 + |(y' - x') \cdot V|^2},
\end{aligned}$$

where in the last inequality we used that

$$-2\varepsilon \left(|x' - y'|^2 + |(y' - x') \cdot V|^2 \right) + 4\varepsilon^2 |(y' - x') \cdot V|^2 \leq 0,$$

which is true since $\varepsilon \leq 1/2$. This concludes the proof of (6).

Step 3. We claim that, if $(p', V, n) \in \mathcal{C} \times \mathbb{Q}^{d-1} \times \mathbb{N}$, then for all $x', y' \in E(p', V, n)$ it holds

$$(7) \quad \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \leq (1 + 2\varepsilon) \sqrt{|x' - y'|^2 + (V \cdot (x' - y'))^2},$$

and xxx

$$(8) \quad \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \geq (1 - 2\varepsilon) \sqrt{|x' - y'|^2 + (V \cdot (x' - y'))^2}.$$

This follows from the previous steps. Indeed, since $x', y' \in E(p', V, n)$, we have that $x', y' \in B'_{1/n}(p')$. But then, $y' \in B'_{2/n}(x')$, so (7) follows from Step 1, while (8) is a consequence of Step 2.

Step 4. We claim that every $x' \in A$ at which f is differentiable lies in some set $E(p', V, n)$. Indeed, since

$$f(y') - f(x') - (y' - x') \cdot \nabla f(x') = o(|x' - y'|),$$

we can find some $n \in \mathbb{N}$ large enough such that

$$\left| f(y') - f(x') - (y' - x') \cdot \nabla f(x') \right| \leq \varepsilon |x' - y'| \quad \text{for all } y' \in B'_{2/n}(x').$$

Now, simply pick some $p' \in C \cap B'_{1/n}(x')$ and some $V \in \mathbb{Q}^{d-1}$ sufficiently close to $\nabla f(x')$.

Step 5. Since the family of sets

$$\{E(p', V, n)\}_{(p', V, n) \in C \times \mathbb{Q}^{d-1} \times \mathbb{N}},$$

is countable we can order it in a sequence $\{\tilde{E}_k\}_{k \geq 1}$, with

$$\tilde{E}_k = E(p'_k, V_k, n_k),$$

for some $p'_k \in C$, $V_k \in \mathbb{Q}^{d-1}$, $n_k \in \mathbb{N}$.

We define the family $\{E_k\}_{k \geq 1}$ as follows. We set $E_1 = \tilde{E}_1$ and

$$E_k := \tilde{E}_k \setminus \left(\bigcup_{j=1}^{k-1} E_j \right) \quad \text{for all } k \geq 2.$$

By construction, we know that each of the sets E_k is contained in some ball $B'_{1/n_k}(p'_k)$, so clearly $\text{diam}(E_k) \leq 2$. The estimate in (iv) follows from the definition of the sets $E(p'_k, V_k, n_k)$, while the last estimate (4) follows from (5) and (6). This concludes the proof of Lemma 16. $\square$

Proof of the area formula for Lipschitz graphs (Theorem 15).

Let A be a Lebesgue measurable set in $\mathbb{R}^{d-1}$.

If $\mathcal{L}^{d-1}(A) = +\infty$, then also $\mathcal{H}^{d-1}(A) = +\infty$. Since A is the projection of $\Gamma(f; A)$ on $\mathbb{R}^{d-1}$,

$$\pi(\Gamma(f; A)) = A,$$

we get that

$$\mathcal{H}^{d-1}(A) \leq \mathcal{H}^{d-1}(\Gamma(f; A)).$$

Thus, when $\mathcal{L}^{d-1}(A) = +\infty$, we have:

$$\mathcal{H}^{d-1}(\Gamma(f; A)) \geq \mathcal{H}^{d-1}(A) = \mathcal{L}^{d-1}(A) = +\infty,$$

and at the same time

$$\int_A \sqrt{1 + |\nabla f|^2} dx \geq \mathcal{L}^{d-1}(A) = +\infty,$$

so (3) holds trivially as both sides are $+\infty$.

In what follows, we will assume that $\mathcal{L}^{d-1}(A) < +\infty$.

Let $\tilde{A} \subset A$ be a Borel set in $\mathbb{R}^{d-1}$ such that $\mathcal{L}^{d-1}(A \setminus \tilde{A}) = 0$ and

$$f \text{ is differentiable at all points } x \in \tilde{A}.$$

Fix $\varepsilon > 0$. Thanks to Lemma 16, there are countable families of Borel sets $E_k \subset \tilde{A}$ and linear functions

$$L_k : \mathbb{R}^{d-1} \rightarrow \mathbb{R}, \quad L_k(x') = V_k \cdot x' \quad \text{with } V_k \in \mathbb{R}^{d-1},$$

such that:

- $\text{diam}(E_k) \leq 2$ for all $k \geq 1$;
- E_k are disjoint and $\tilde{A} = \bigcup_{k \geq 1} E_k$;
- $|\nabla L_k - \nabla f(x')| \leq \varepsilon$ for all $x' \in E_k$;
- for all $x', y' \in E_k$, it holds

$$\begin{aligned} (1 - 2\varepsilon) \sqrt{|x' - y'|^2 + (L_k(x' - y'))^2} \\ \leq \sqrt{|x' - y'|^2 + (f(x') - f(y'))^2} \\ \leq (1 + 2\varepsilon) \sqrt{|x' - y'|^2 + (L_k(x' - y'))^2}. \end{aligned}$$

Then, since $\mathcal{L}^{d-1}(A \setminus \tilde{A}) = 0$, we have

$$\mathcal{H}^{d-1}(\Gamma(f; A)) = \sum_{k \geq 1} \mathcal{H}^{d-1}(\Gamma(f; E_k)) \quad \text{and} \quad \int_A \sqrt{1 + |\nabla f|^2} dx = \sum_{k \geq 1} \int_{E_k} \sqrt{1 + |\nabla f|^2} dx.$$

Now, thanks to Lemma 5, we have that for each k

$$(1 - 2\varepsilon)^{d-1} \mathcal{H}^{d-1}(\Gamma(L_k; E_k)) \leq \mathcal{H}^{d-1}(\Gamma(f; E_k)) \leq (1 + 2\varepsilon)^{d-1} \mathcal{H}^{d-1}(\Gamma(L_k; E_k)).$$

On the other hand, Proposition 10 gives that

$$\mathcal{H}^{d-1}(\Gamma(L_k; E_k)) = \int_{E_k} \sqrt{1 + |\nabla L_k|^2} dx,$$

while since

$$\sqrt{1 + |\nabla L_k|^2} - \sqrt{1 + |\nabla f|^2} = \frac{\nabla(f + L_k) \cdot \nabla(f - L_k)}{\sqrt{1 + |\nabla L_k|^2} + \sqrt{1 + |\nabla f|^2}},$$

we get that

$$|\sqrt{1 + |\nabla L_k|^2} - \sqrt{1 + |\nabla f|^2}| \leq |\nabla f - \nabla L_k| \leq \varepsilon \quad \text{on } E_k,$$

so that

$$\begin{aligned} \mathcal{H}^{d-1}(\Gamma(f; A)) &= \sum_{k \geq 1} \mathcal{H}^{d-1}(\Gamma(f; E_k)) \\ &\leq (1 + 2\varepsilon)^{d-1} \sum_{k \geq 1} \mathcal{H}^{d-1}(\Gamma(L_k; E_k)) \\ &= (1 + 2\varepsilon)^{d-1} \sum_{k \geq 1} \int_{E_k} \sqrt{1 + |\nabla L_k|^2} dx \\ &\leq (1 + 2\varepsilon)^{d-1} \sum_{k \geq 1} (1 + \varepsilon) \int_{E_k} \sqrt{1 + |\nabla f|^2} dx \leq (1 + 2\varepsilon)^d \int_A \sqrt{1 + |\nabla f|^2} dx, \end{aligned}$$

and conversely

$$\begin{aligned} \mathcal{H}^{d-1}(\Gamma(f; A)) &= \sum_{k \geq 1} \mathcal{H}^{d-1}(\Gamma(f; E_k)) \\ &\geq (1 - 2\varepsilon)^{d-1} \sum_{k \geq 1} \mathcal{H}^{d-1}(\Gamma(L_k; E_k)) \\ &= (1 - 2\varepsilon)^{d-1} \sum_{k \geq 1} \int_{E_k} \sqrt{1 + |\nabla L_k|^2} dx \\ &\geq (1 - 2\varepsilon)^{d-1} \sum_{k \geq 1} (1 - \varepsilon) \int_{E_k} \sqrt{1 + |\nabla f|^2} dx \geq (1 - 2\varepsilon)^d \int_A \sqrt{1 + |\nabla f|^2} dx. \end{aligned}$$

Sending $\varepsilon \rightarrow 0$, we get the claim. $\square$