Exercices

Exercice 1. *Let u_n be a sequence in $L^2(\mathbb{R})$ converging weakly to some $u \in L^2(\mathbb{R})$. Prove that*

$$\int_{\mathbb{R}} e^{-x^2} u^2(x)\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}} e^{-x^2} u_n^2(x)\,dx.$$

Exercice 2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let*

$$T : \mathcal{B}_1 \to \mathcal{B}_2$$

be a bounded linear map in the sense that

$$\sup\Big\{\|Tx\|_{\mathcal{B}_2} \ : \ x \in \mathcal{B}_1,\ \|x\|_{\mathcal{B}_1} \le 1\Big\} < +\infty.$$

Prove that, if $x_n \in \mathcal{B}_1$ converges weakly to x in $\mathcal{B}_1$, then $T(x_n)$ converges weakly to $T(x)$ in $\mathcal{B}_2$

Exercice 3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let*

$$T : \mathcal{B}_1 \to \mathcal{B}_2$$

be a bounded linear map in the sense that

$$\sup\Big\{\|Tx\|_{\mathcal{B}_2} \ : \ x \in \mathcal{B}_1,\ \|x\|_{\mathcal{B}_1} \le 1\Big\} < +\infty.$$

Prove that the adjoint map

$$T_* : \mathcal{B}_2' \to \mathcal{B}_1'$$

defined as

$$< T_*(\xi), x >_{\mathcal{B}_1',\mathcal{B}_1} = < \xi, T(x) >_{\mathcal{B}_2',\mathcal{B}_2}$$

is bounded in the sense that

$$\sup\Big\{\|T\xi\|_{\mathcal{B}_1'} \ : \ \xi \in \mathcal{B}_2',\ \|\xi\|_{\mathcal{B}_2'} \le 1\Big\} < +\infty.$$

Exercice 4. *Let $\mathcal{I}$ be a bounded interval in $\mathbb{R}$. Let $p \in (0, +\infty)$ and let $q := \frac{p}{p-1}$.*

Consider the space $\mathcal{W}$ of all $u \in L^p$ for which there exists $v \in L^q$ such that

$$\int_I e^{-x^2} u(x)\varphi'(x)\,dx = -\int_I v(x)\varphi(x)\,dx.$$

1. *Prove that $\mathcal{W}$ is a Banach space with the norm $\|u\|_{L^p} + \|v\|_{L^q}$.*
2. *For $p \le 2$ prove that if $\mathcal{W} = W^{1,p}(I)$.*
3. *Is is true that $\mathcal{W} = W^{1,p}(I)$ when $p > 2$?*

Exercice 5. *Consider the operator*

$$T : C_c^\infty(0,1) \to \mathbb{R}\ , \qquad T(u) = \int_0^1 \frac{u(x)}{\sqrt{x}}\,dx.$$

1. *Prove that the operator T cannot be extended to a bounded linear operator on $L^2(0,1)$.*
2. *Prove that the operator T can be extended to a bounded linear operator on $W^{1,2}(0,1)$*
3. *Prove that the operator T can be extended to a bounded linear operator on $W_0^{1,2}(0,1)$ endowed with the norm*

$$\|u\|^2_{W_0^{1,2}(0,1)} = \int_0^1 (u')^2\,dx,$$

and compute the norm

$$\sup\Big\{T(u) \ : \ u \in W^{1,2}(0,1),\ \|u\|_{W_0^{1,2}(0,1)} \le 1\Big\}.$$

Exercice 6. *Let $I = (0,1)$. Consider the bilinear map*

$$Q : W^{1,2}(I) \times W^{1,2}(I) \to \mathbb{R},$$

defined, for every $u, v \in W^{1,2}(I)$, as:

$$Q(u,v) = u(0)v(0) + \int_0^1 u'(x)v'(x)\,dx.$$

Prove that

$$Q(u,u) > 0 \quad \textit{for all} \quad u \in W^{1,2}(I) \setminus \{0\},$$

and that $W^{1,2}(I)$ endowed with the scalar product Q is complete.

Exercice 7. *Let $I = (0,1)$. Consider the space*

$$W := \Big\{ u \in W^{1,2}(I) \ : \ \int_0^1 u(x)\,dx = 0 \Big\}.$$

(a) *Consider the bilinear map*

$$Q : W \times W \to \mathbb{R},$$

defined, for every $u, v \in W$, as:

$$Q(u,v) = \int_0^1 u'(x)v'(x)\,dx.$$

Prove that

$$Q(u,u) > 0 \quad \textit{for all} \quad u \in W \setminus \{0\},$$

(b) *Prove that the space W endowed with the scalar product Q is complete.*

(c) *Prove that there exists a function $u \in W^{1,2}(I)$ such that*

$$\int_0^1 u(x)\,dx = 0 \ , \qquad \int_0^1 u^2(x)\,dx = 1$$

and

$$\int_0^1 (u'(x))^2\,dx = \inf \Big\{ \int_0^1 (v'(x))^2\,dx \ : \ v \in W^{1,2}(I), \ \int_0^1 v(x)\,dx = 0, \ \int_0^1 v^2(x)\,dx = 1 \Big\}.$$

Exercice 8. *Consider the interval $I = (0,1)$ and the space $W_0^{1,2}(I)$ endowed with the norm*

$$\|u\|_{W^{1,2}(I)} = \Big(\int_0^1 (u'(x))^2\,dx \Big)^{1/2}.$$

For every $f, g \in L^2(I)$, we define the map

$$T_{f,g} : W^{1,2}(I) \to \mathbb{R}$$

$$T_{f,g}(u) = \int_0^1 f(x)u(x)\,dx + \int_0^1 g(x)v(x)\,dx.$$

(a) *Prove that for every $(f,g) \in L^2(I) \times L^2(I)$, there is a function $h \in L^2(I)$ such that.*

(b) *For $f = 1$ and $g = 1$, find the function h and compute the norm of the operator $T_{f,g}$.*

Exercice 9. *Consider the interval $I = (0,1)$ and the space $W^{1,2}(I)$ endowed with the norm*

$$\|u\|_{W^{1,2}(I)} = \Big(\int_0^1 u^2(x)\,dx + \int_0^1 (u'(x))^2\,dx \Big)^{1/2}.$$

For every $f, g \in L^2(I)$, we define the map

$$T_{f,g} : W^{1,2}(I) \to \mathbb{R}$$

$$T_{f,g}(u) = \int_0^1 f(x)u(x)\,dx + \int_0^1 g(x)v(x)\,dx.$$

(a) *Prove that if $v \in H_0^1(I)$, then*

$$T_{f,g} = T_{f+v',g+v}.$$

(b) *Prove that if $(f_1, g_1) \in L^2(I) \times L^2(I)$ and $(f_2, g_2) \in L^2(I) \times L^2(I)$ are such that*

$$T_{f_1,g_1} = T_{f_2,g_2},$$

then there is a function $v \in H_0^1(I)$ such that

$$f_2 = f_1 + v' \quad \textit{and} \quad g_2 = g_1 + v.$$

(c) *Prove that, for every $(f,g) \in L^2(I) \times L^2(I)$, there is a solution to the variational problem*

$$\min \Big\{ \int_0^1 |f + v'|^2 + \int_0^1 |g + v|^2\,dx : \ v \in W_0^{1,2}(I) \Big\}.$$