Disuguaglianza di Gagliardo-Nirenberg-Sobolev

Lemma 1. *Per ogni* $\varphi \in C_c^1(\mathbb{R}^d)$ *si ha*

$$\left(\int_{\mathbb{R}^d} \varphi^{\frac{d}{d-1}}\,dx\right)^{\frac{d-1}{d}} \le \int_{\mathbb{R}^d} |\nabla\varphi|\,dx.$$

Dimostrazione. In dimensione 2, abbiamo le disuguaglianze

$$|\varphi|(x,y) \le \int_{-\infty}^{+\infty} |\nabla\varphi|(x,t)\,dt \qquad \text{e} \qquad |\varphi|(x,y) \le \int_{-\infty}^{+\infty} |\nabla\varphi|(s,y)\,ds.$$

In questo modo

$$\varphi^2(x,y) \le \left(\int_{-\infty}^{+\infty} |\nabla\varphi|(x,t)\,dt\right)\left(\int_{-\infty}^{+\infty} |\nabla\varphi|(s,y)\,ds\right).$$

Integrando in x e in y, otteniamo

$$\int_{\mathbb{R}^2} \varphi^2(x,y)\,dx\,dy \le \left(\int_{\mathbb{R}^2} |\nabla\varphi|(x,y)\,dx\,dy\right)^2.$$

In dimensione 3, abbiamo le disuguaglianze

$$|\varphi|(x,y,z) \le \int_{-\infty}^{+\infty} |\nabla\varphi|(x,y,T)\,dT\ ,$$
$$|\varphi|(x,y,z) \le \int_{-\infty}^{+\infty} |\nabla\varphi|(x,S,z)\,dS\ ,$$
$$|\varphi|(x,y,z) \le \int_{-\infty}^{+\infty} |\nabla\varphi|(R,y,z)\,dR\ .$$

In questo modo

$$|\varphi|^{3/2}(x,y,z) \le \left(\int_{-\infty}^{+\infty} |\nabla\varphi|(x,y,T)\,dT\right)^{1/2}\left(\int_{-\infty}^{+\infty} |\nabla\varphi|(x,S,z)\,dS\right)^{1/2}\left(\int_{-\infty}^{+\infty} |\nabla\varphi|(R,y,z)\,dR\right)^{1/2}.$$

Selezioniamo la prima coppia di integrali sulla destra e integriamo in x.

$$\int_{\mathbb{R}} \left(\int_{-\infty}^{+\infty} |\nabla\varphi|(x,y,T)\,dT\right)^{1/2}\left(\int_{-\infty}^{+\infty} |\nabla\varphi|(x,S,z)\,dS\right)^{1/2} dx$$
$$\le \left(\int_{\mathbb{R}}\int_{-\infty}^{+\infty} |\nabla\varphi|(x,y,T)\,dT\,dx\right)^{1/2}\left(\int_{\mathbb{R}}\int_{-\infty}^{+\infty} |\nabla\varphi|(x,S,z)\,dS\,dx\right)^{1/2}.$$

Quindi,

$$\int_{\mathbb{R}} |\varphi|^{3/2}(x,y,z)\,dx \le \left(\iint |\nabla\varphi|(x,y,T)\,dT\,dx\right)^{1/2}\left(\iint |\nabla\varphi|(x,S,z)\,dS\,dx\right)^{1/2}\left(\int |\nabla\varphi|(R,y,z)\,dR\right)^{1/2}.$$

Osserviamo che, integrando in y, abbiamo

$$\int_{\mathbb{R}} \left(\iint |\nabla\varphi|(x,y,T)\,dT\,dx\right)^{1/2}\left(\int |\nabla\varphi|(R,y,z)\,dR\right)^{1/2} dy$$
$$\le \left(\int_{\mathbb{R}}\iint |\nabla\varphi|(x,y,T)\,dT\,dx\,dy\right)^{1/2}\left(\int_{\mathbb{R}}\int |\nabla\varphi|(R,y,z)\,dR\,dy\right)^{1/2}.$$

Quindi,

$$\iint |\varphi|^{3/2}(x,y,z)\,dx\,dy$$
$$\leq \left(\iiint |\nabla\varphi|(x,y,T)\,dT\,dx\,dy\right)^{1/2}\left(\iint |\nabla\varphi|(x,S,z)\,dS\,dx\right)^{1/2}\left(\iint |\nabla\varphi|(R,y,z)\,dR\,dy\right)^{1/2}.$$

Ora, integrando in z,

$$\int_{\mathbb{R}}\left(\iint |\nabla\varphi|(x,S,z)\,dS\,dx\right)^{1/2}\left(\iint |\nabla\varphi|(R,y,z)\,dR\,dy\right)^{1/2}dz$$
$$\leq \left(\int_{\mathbb{R}}\iint |\nabla\varphi|(x,S,z)\,dS\,dx\,dz\right)^{1/2}\left(\int_{\mathbb{R}}\iint |\nabla\varphi|(R,y,z)\,dR\,dy\,dz\right)^{1/2}.$$

Quindi,

$$\iiint |\varphi|^{3/2}(x,y,z)\,dx\,dy\,dz$$
$$\leq \left(\iiint |\nabla\varphi|(x,y,T)\,dT\,dx\,dy\right)^{1/2}\left(\iint |\nabla\varphi|(x,S,z)\,dS\,dx\,dz\right)^{1/2}\left(\iint |\nabla\varphi|(R,y,z)\,dR\,dy\,dz\right)^{1/2}$$
$$= \left(\iiint |\nabla\varphi|(x,y,z)\,dx\,dy\,dz\right)^{3/2}.$$

In dimensione $d \geq 4$ la dimostrazione è analoga. □

Lemma 2. *Sia $\varphi \in C_c^\infty(\mathbb{R}^d)$ e $1 < p < d$. Allora,*

$$\left(\int \varphi^{\frac{pd}{d-p}}\,dx\right)^{\frac{d-p}{pd}} \leq C_{d,p}\left(\int |\nabla\varphi|^p\,dx\right)^{1/p},$$

dove

$$C_{d,p} = \left(\frac{p(d-1)}{d-p}\right)^{\frac{p-1}{p}}$$

Dimostrazione. Applicheremo il lemma precedente alla funzione φ^α. Allora,

$$\begin{aligned}\int_{\mathbb{R}^d}|\varphi|^{\frac{\alpha d}{d-1}}\,dx &\leq \left(\int_{\mathbb{R}^d}|\nabla(\varphi^\alpha)|\,dx\right)^{\frac{d}{d-1}}\\ &= \alpha^{\frac{d}{d-1}}\left(\int_{\mathbb{R}^d}|\varphi^{\alpha-1}\nabla\varphi|\,dx\right)^{\frac{d}{d-1}}\\ &\leq \alpha^{\frac{d}{d-1}}\left(\int_{\mathbb{R}^d}|\varphi|^{\frac{p}{p-1}(\alpha-1)}\,dx\right)^{\frac{p-1}{p}\frac{d}{d-1}}\left(\int_{\mathbb{R}^d}|\nabla\varphi|^p\,dx\right)^{\frac{1}{p}\frac{d}{d-1}}.\end{aligned}$$

ora, scegliendo

$$\alpha = \frac{p(d-1)}{d-p},$$

abbiamo che $\dfrac{\alpha d}{d-1} = \dfrac{p}{p-1}(\alpha-1) = \dfrac{pd}{d-p}$. □

Teorema 3 (Gagliardo-Nirenberg-Sobolev)**.** *Siano $d \geq 2$ e $p \in (1,d)$. Allora, esiste una costante dimensionale $C = C(d,p)$ tale che per ogni $u \in W^{1,p}(\mathbb{R}^d)$*

$$\left(\int_{\mathbb{R}^d} u^{p^*}\,dx\right)^{1/p^*} \leq C_{d,p}\left(\int_{\mathbb{R}^d}|\nabla u|^p\,dx\right)^{1/p} \qquad \text{dove} \qquad p^* := \frac{pd}{d-p}.$$

In particolare, $W^{1,p}(\mathbb{R}^d) \subset L^{p^*}(\mathbb{R}^d)$ e l'immersione

$$i : W^{1,p}(\mathbb{R}^d) \to L^{p^*}(\mathbb{R}^d), \quad i(u) = u,$$

è una mappa lineare continua.