# Dual spaces

THE SPACE $\mathcal{B}'$ OF BOUNDED LINEAR OPERATORS

**Theorem 1.** *Let $\mathcal{B}$ be a Banach space. Let $\mathcal{B}'$ be the collection of all bounded linear operators on $\mathcal{B}$. Moreover, for every $T \in \mathcal{B}'$, let*

$$\|T\|_{\mathcal{B}'} := \sup\Big\{T(f) \ : \ f \in \mathcal{B}\,,\ \|f\|_{\mathcal{B}} \le 1\Big\}.$$

*Then, $(\mathcal{B}', \|\cdot\|_{\mathcal{B}'})$ is a Banach space.*

*Proof.*
**Step 1.** $\mathcal{B}'$ is a linear space. Indeed, it is immediate to check that if

$$S, T : \mathcal{B} \to \mathbb{R}$$

are linear and continuous operators on $\mathcal{B}$, then for all $\alpha, \beta \in \mathbb{R}$, the map

$$\alpha T + \beta S : \mathcal{B} \to \mathbb{R}$$

is linear and continuous.

**Step 2.** $\|\cdot\|_{\mathcal{B}'}$ is a norm on $\mathcal{B}'$. We first notice that

$$\|T\|_{\mathcal{B}'} \quad \Rightarrow \quad T(f) = 0 \quad \text{for all} \quad f \in \mathcal{B} \quad \Rightarrow \quad T = 0,$$

and that

$$\begin{aligned}\|\alpha T\|_{\mathcal{B}'} &= \sup\Big\{\alpha T(f) \ : \ f \in \mathcal{B}\,,\ \|f\|_{\mathcal{B}} \le 1\Big\}\\ &= |\alpha| \sup\Big\{T(f) \ : \ f \in \mathcal{B}\,,\ \|f\|_{\mathcal{B}} \le 1\Big\} = |\alpha| \|T\|_{\mathcal{B}'}.\end{aligned}$$

Finally, for the triangular inequality we notice that

$$|(T+S)(f)| \le |T(f)| + |S(f)| \le \Big(\|T\|_{\mathcal{B}'} + \|S\|_{\mathcal{B}'}\Big)\|f\|_{\mathcal{B}} \quad \text{for all} \quad f \in \mathcal{B}.$$

Taking the supremum over all $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \le 1$, we get

$$\|T + S\|_{\mathcal{B}'} \le \|T\|_{\mathcal{B}'} + \|S\|_{\mathcal{B}'}.$$

**Step 3.** $(\mathcal{B}', \|\cdot\|_{\mathcal{B}'})$ is complete. Consider a sequence $T_n$ in $\mathcal{B}'$, which is Cauchy with respect to the norm $\|\cdot\|_{\mathcal{B}'}$, that is, for all $\varepsilon > 0$ there exists $N \ge 0$ such that

$$\|T_n - T_m\|_{\mathcal{B}'} \le \varepsilon \qquad \text{for all} \qquad n, m \ge N\,.$$

In particular, for all $f \in \mathcal{B}$, we have

$$|T_n(f) - T_m(f)| \le \|T_n - T_m\|_{\mathcal{B}'}\|f\|_{\mathcal{B}} \le \varepsilon\|f\|_{\mathcal{B}}.$$

Thus, $T_n(f)$ is a Cauchy sequence in $\mathbb{R}$. Let

$$T : \mathcal{B} \to \mathbb{R}$$

be the map defined as

$$T(f) := \lim_{n \to +\infty} T_n(f) \quad \text{for all} \quad f \in \mathcal{B}.$$

It remains to prove that:

(★) $T$ is linear and bounded;

(★★) $\|T_n - T\|_{\mathcal{B}'} \to 0$.

In order to prove (★), we first notice that, since $\|\cdot\|_{\mathcal{B}'}$ is a norm, we have

$$\Big|\|T_n\|_{\mathcal{B}'} - \|T_m\|_{\mathcal{B}'}\Big| \le \|T_n - T_m\|_{\mathcal{B}'}.$$

Thus, the sequence of norms $\|T_n\|_{\mathcal{B}'}$ is a Cauchy sequence in $\mathbb{R}$. Then, the limit

$$L := \lim_{n\to\infty} \|T_n\|_{\mathcal{B}'},$$

exists and is finite. Now, for all $f \in \mathcal{B}$, we have

$$|T(f)| \le \lim_{n\to\infty} \|T_n\|_{\mathcal{B}'}\|f\|_{\mathcal{B}} = L\|f\|_{\mathcal{B}},$$

which proves that $T$ is bounded and $\|T\|_{\mathcal{B}'} \le L$.

We next prove (★★). Let $\varepsilon > 0$ be fixed. We can find $N \ge 1$ such that

$$\|T_n - T_m\|_{\mathcal{B}'} \le \varepsilon \qquad \text{for all} \qquad n, m \ge N\,.$$

Then, for all $f \in \mathcal{B}$ and all $m \ge n \ge N$, we have

$$|T_n(f) - T(f)| \le |T(f) - T_m(f)| + |T_n(f) - T_m(f)| \le |T(f) - T_m(f)| + \varepsilon\|f\|_{L^\infty}.$$

Letting $m \to \infty$, we get that for all $f \in \mathcal{B}$ and all $n \ge N$

$$|T_n(f) - T(f)| \le \varepsilon\|f\|_{\mathcal{B}},$$

which implies that

$$\|T_n - T\|_{\mathcal{B}'} \le \varepsilon \qquad \text{for all} \qquad n \ge N.$$

□

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## The bidual $\mathcal{B}''$

Fix a vector $f \in \mathcal{B}$ and consider the map

$$\delta_f : \mathcal{B}' \to \mathbb{R},$$

defined through the identity

$$\delta_f(T) = T(f) \quad \text{for all} \quad T \in \mathcal{B}'.$$

By construction, $\delta_f$ is linear. Moreover, since

$$|\delta_f(T)| = |T(f)| \le \|T\|_{\mathcal{B}'}\|f\|_{\mathcal{B}} \quad \text{for all} \quad T \in \mathcal{B}',$$

we get that $\delta_f$ is a bounded operator and for its operator norm

$$\|\delta_f\|_{\mathcal{B}''} := \sup\Big\{\delta_f(T) \ : \ T \in \mathcal{B}',\ \|T\|_{\mathcal{B}'} \le 1\Big\},$$

we have

$$\|\delta_f\|_{\mathcal{B}''} \le \|f\|_{\mathcal{B}}.$$

Moreover, by the Hahn-Banach theorem, there is an operator $T_f \in \mathcal{B}'$ such that

$$\|T_f\|_{\mathcal{B}'} = 1 \quad \text{and} \quad |T_f(f)| = \|f\|_{\mathcal{B}}.$$

This gives the opposite inequality

$$\|\delta_f\|_{\mathcal{B}''} \geq |T_f(f)| = \|f\|_{\mathcal{B}},$$

which finally leads to

$$\|\delta_f\|_{\mathcal{B}''} = \|f\|_{\mathcal{B}}.$$

Thus, we have obtained by the inclusion operator

$$I : \mathcal{B} \to \mathcal{B}'' \ , \qquad I(f) = \delta_f,$$

has the following properties:

- $I$ is linear;
- $I$ is injective;
- $I(\mathcal{B})$ is a closed linear subspace of $\mathcal{B}''$;
- $I$ is an isometry between the Banach spaces $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(I(\mathcal{B}), \|\cdot\|_{\mathcal{B}''})$.

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## Reflexive spaces

**Definition 2.** *A Banach space $\mathcal{B}$ is reflexive, if the inclusion operator $I$ is surjective, that is, if $I(\mathcal{B}) = \mathcal{B}''$. For a reflexive space $\mathcal{B}$, we will simply write*

$$\mathcal{B}'' = \mathcal{B}.$$

**Remark 3.** *For any measurable set $\Omega \subset \mathbb{R}^d$ and any $p \in (1, +\infty)$, the space $L^p(\Omega)$ is reflexive.*

**Proposition 4.** *The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are NOT reflexive.*

*Proof.* We first recall that

$$(L^1(\Omega))' = L^\infty(\Omega),$$

and that $(L^\infty(\Omega))'$ contains $L^1(\Omega)$, but also contains elements, which are not in $L^1(\Omega)$. This immediately gives that

$$(L^1(\Omega))'' \neq L^1(\Omega),$$

so $L^1(\Omega)$ is not reflexive.

Next, suppose by contradiction that $L^\infty(\Omega)$ is reflexive: $(L^\infty(\Omega))'' = L^\infty(\Omega)$. Since $L^1(\Omega)$ is (can be identified with) a proper closed subspace of $(L^\infty(\Omega))' = (L^1(\Omega))''$, by the second version of the Hahn-Banach's theorem we can find a bounded linear operator

$$\xi : (L^\infty(\Omega))' \to \mathbb{R}$$

such that

$$\xi \neq 0 \quad \text{and} \quad \xi \equiv 0 \text{ on } L^1(\Omega).$$

Now, since by assumption $(L^\infty(\Omega))'' = L^\infty(\Omega)$ the operator $\xi$ should be of the form

$$\xi = \delta_f : (L^\infty(\Omega))' \to \mathbb{R}\ , \quad \delta_f(T) = T(f),$$

for some $f \in L^\infty(\Omega)$. We next notice that $\delta_f$ acts on the subspace $L^1(\Omega) \subset (L^\infty(\Omega))'$ as follows: for every $T_g \in (L^\infty(\Omega))'$ of the form

$$T_g(\phi) = \int_\Omega g(x)\phi(x)\,dx \quad \text{for all} \quad \phi \in L^\infty(\Omega),$$

where $g \in L^1(\Omega)$ is fixed, we have

$$\delta_f(T_g) = T_g(f) = \int_\Omega f(x)g(x)\,dx.$$

Since the operator $\xi = \delta_f$ is vanishing on the space of all such operators $T_g$, we have that $f \equiv 0$. But then $\xi \equiv 0$, which is a contradiction. $\square$