About the dual space of $L^\infty(\Omega)$

A special class of bounded linear functionals on $L^\infty(\Omega)$

Proposizione 1. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Fixed a function $g \in L^1(\Omega)$, consider the linear functional*

$$T_g : L^\infty(\Omega) \to \mathbb{R}$$

defined as

$$T_g(f) = \int_\Omega f(x)g(x)\,dx \quad \text{for all} \quad f \in L^\infty(\Omega).$$

Then, T_g is a bounded operator and its norm

$$\|T_g\| := \sup\Big\{T_g(f) \ : \ f \in L^\infty(\Omega)\,,\ \|f\|_{L^\infty(\Omega)} \le 1\Big\},$$

is given by

$$\|T_g\| = \|g\|_{L^1(\Omega)}.$$

Proof. The boundedness of T_g follows from the inequality

$$|T_g(f)| = \left|\int_\Omega f(x)g(x)\,dx\right| \le \|f\|_{L^\infty(\Omega)}\|g\|_{L^1(\Omega)} \quad \text{for all} \quad f \in L^\infty(\Omega),$$

which also gives that

$$\|T_g\| \le \|g\|_{L^1(\Omega)}.$$

In order to prove that we have an equality, we consider an increasing sequence of sets

$$\omega_1 \subset \omega_2 \subset \cdots \subset \omega_n \subset \cdots \subset \Omega$$

such that

$$\omega_n \subset \Omega \quad \text{and} \quad |\omega_n| < \infty \quad \text{for all} \quad n \ge 1.$$

Fix $n > 1$ and consider as a test function

$$f_n(x) := \text{sign}\,(g(x))\,\mathbb{1}_{\omega_n}(x),$$

where the sign function is defined as

$$\text{sign} : \mathbb{R} \to \mathbb{R}\ , \qquad \text{sign}\, g := \begin{cases} 1 & \text{if} \quad g > 0, \\ 0 & \text{if} \quad g = 0, \\ -1 & \text{if} \quad g < 0. \end{cases}$$

Then,

$$T_g(f_n) = \int_\Omega f_n(x)g(x)\,dx = \int_{\omega_n} \text{sign}\,(g(x))g(x)\,dx = \int_{\omega_n} |g(x)|\,dx,$$

which implies that

$$\sup\Big\{T_g(f) \ : \ f \in L^\infty(\Omega)\,,\ \|f\|_{L^\infty(\Omega)} \le 1\Big\} \ge T_g(f_n) \ge \int_{\omega_n} |g(x)|\,dx.$$

Now, by, the dominated convergence theorem, we have that

$$\mathbb{1}_{\omega_n}|g| \to |g| \quad \text{in} \quad L^1(\Omega).$$

Thus, taking the limit as $n \to +\infty$, we get

$$\|T_g\| \ge \lim_{n\to+\infty} \int_{\omega_n} |g(x)|\,dx = \|g\|_{L^1(\Omega)},$$

which concludes the proof. □

The dual space of $L^\infty(\Omega)$ is not $L^1(\Omega)$

Let $\Omega = \mathbb{R}$. Consider the space

$$C_0(\mathbb{R}) = \Big\{ f \in C(\mathbb{R}) \ : \ \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0 \Big\}.$$

Then $C_0(\mathbb{R})$ is a closed subspace of the Banach space $L^\infty(\mathbb{R})$. Moreover, the functional

$$\delta_0 : C_0(\mathbb{R}) \to \mathbb{R} :, \qquad \delta_0(f) = f(0),$$

is a bounded linear functional on $C_0(\mathbb{R})$ with norm

$$\Big\{ \delta_0(f) \ : \ f \in C_0(\mathbb{R}), \ \|f\|_{L^\infty} \leq 1 \Big\} = 1.$$

By the Hahn-Banach's theorem, δ_0 can be extended to a bounded linear functional

$$T : L^\infty(\mathbb{R}) \to \mathbb{R}$$

with norm 1. On the other hand, it is immediate to check that there is no function $g \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} g(x)\phi(x)\,dx = \phi(0) \quad \text{for all} \quad \phi \in C_0(\mathbb{R}).$$