The dual space of $L^1(\Omega)$

A special class of functionals on $L^1(\Omega)$

Proposizione 1. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Fixed a function $g \in L^\infty(\Omega)$, consider the linear functional*

$$T_g : L^1(\Omega) \to \mathbb{R}$$

defined as

$$T_g(f) = \int_\Omega f(x)g(x)\,dx \quad \textit{for all} \quad f \in L^1(\Omega).$$

Then, T_g is a bounded operator and its norm

$$\|T_g\| := \sup\Big\{T_g(f) \ : \ f \in L^1(\Omega)\,,\ \|f\|_{L^1(\Omega)} \le 1\Big\},$$

is given by

$$\|T_g\| = \|g\|_{L^\infty(\Omega)}.$$

Proof. The fact that T_g is bounded simply follows from the inequality

$$|T_g(f)| = \left|\int_\Omega f(x)g(x)\,dx\right| \le \|f\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)} \quad \text{for all} \quad f \in L^1(\Omega),$$

which also proves that

$$\|T_g\| \le \|g\|_{L^\infty(\Omega)}.$$

In order to prove that an equality holds, we consider a sequence

$$t_n \uparrow \|g\|_{L^\infty(\Omega)} \quad \text{such that} \quad |\{|g| > t_n\}| \ne 0.$$

Without loss of generality, we can suppose that

$$t_n \uparrow \|g\|_{L^\infty(\Omega)} \quad \text{such that} \quad |\{g > t_n\}| \ne 0.$$

Now, fix $n \ge 1$, choose any set of finite measure $\omega \subset \{g > t_n\}$, and take as test function

$$f := \frac{1}{|\omega|}\mathbb{1}_\omega.$$

Then,

$$T_g(f) = \int_\Omega g(x)f(x)\,dx = \frac{1}{\omega}\int_\omega g(x)\,dx \ge t_n.$$

This implies that

$$\|T_g\| := \sup\Big\{T_g(f) \ : \ f \in L^1(\Omega)\,,\ \|f\|_{L^1(\Omega)} \le 1\Big\} \ge t_n.$$

Since, this holds for every n, we get that

$$\|T_g\| \ge \|g\|_{L^\infty(\Omega)},$$

which concludes the proof. □

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set and let*

$$T : L^1(\Omega) \to \mathbb{R}$$

be a bounded linear functional on $L^1(\Omega)$. Then, there is a unique $g \in L^\infty(\Omega)$ such that

$$T(f) = \int_\Omega f(x)g(x)\,dx \quad \textit{for all} \quad f \in L^1(\Omega)\,.$$

Proof. In what follows we fix a constant $C > 0$ such that

$$|T(f)| \le C\|f\|_{L^1(\Omega)} \quad \text{for all} \quad f \in L^1(\Omega).$$

We proceed in two steps.

Step 1. We first consider the case $|\Omega| < +\infty$.
Construction of g. Since we have the inclusion

$$L^2(\Omega) \subset L^1(\Omega),$$

and the inequality

$$\|f\|_{L^1(\Omega)} \le |\Omega|^{1/2}\|f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega),$$

we have that the functional

$$T : L^2(\Omega) \to \mathbb{R}$$

is a bounded linear functional on $L^2(\Omega)$ and it holds

$$|T(f)| \le C|\Omega|^{1/2}\|f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in L^2(\Omega).$$

Thus, we can find a function $g \in L^2(\Omega)$ such that

$$T(f) = \int_\Omega f(x)g(x)\,dx \quad \text{for all} \quad f \in L^2(\Omega).$$

Boundedness of g. We will show that $g \le C$. Suppose that there exists a level $t > 0$ such that

$$t > C \quad \text{and} \quad |\{g > t\} \cap \Omega| > 0,$$

and consider the function

$$f := \mathbb{1}_{\{g>t\}\cap\Omega}.$$

Observe that

$$f \in L^1(\Omega) \quad \text{and} \quad \|f\|_{L^1(\Omega)} = \int_\Omega f(x)\,dx = |\{g > t\} \cap \Omega|.$$

We also have that $f \in L^2(\Omega)$, so by the definition of g, we get

$$T(f) = \int_\Omega f(x)g(x)\,dx = \int_{\{g>t\}\cap\Omega} g(x)\,dx \ge t|\{g > t\} \cap \Omega|.$$

On the other hand, since T is bounded on L^1, we get

$$T(f) \le C\|f\|_{L^1(\Omega)} = C|\{g > t\} \cap \Omega|.$$

Thus, we have obtained

$$t|\{g > t\} \cap \Omega| \le T(f) \le CC|\{g > t\} \cap \Omega|,$$

which is a contradiction. This implies that

$$g(x) \leq C \quad \text{for Lebesgue almost-every} \quad x \in \Omega.$$

Analogously, by taking as test function $f = \mathbb{1}_{\{g<t\}\cap\Omega}$ with $t < -C$, we get that

$$g(x) \geq -C \quad \text{for Lebesgue almost-every} \quad x \in \Omega.$$

This proves that

$$g \in L^\infty(\Omega) \quad \text{and} \quad \|g\|_{L^\infty} \leq C.$$

Uniqueness of g. Suppose that there are two distinct functions $g_1, g_2 \in L^\infty(\Omega)$ such that

$$\int_\Omega f(x)g_1(x)\,dx = T(f) = \int_\Omega f(x)g_2(x)\,dx \quad \text{for all} \quad f \in L^1(\Omega).$$

Then, taking

$$g = g_1 - g_2 \in L^\infty(\Omega),$$

we have

$$\int_\Omega f(x)g(x)\,dx = \int_\Omega f(g_1 - g_2)\,dx = 0 \quad \text{for all} \quad f \in L^1(\Omega).$$

Taking as a test function $f = g$ we get that

$$\int_\Omega |g(x)|^2\,dx = 0,$$

which proves that

$$g_1(x) = g_2(x) \quad \text{for almost-every} \quad x \in \Omega.$$

Step 2. Suppose now that $|\Omega| = +\infty$. Consider the sequence of sets

$$\Omega_n = B_n \cap \Omega,$$

and of the corresponding extension maps

$$\pi_n : L^1(\Omega_n) \to L^1(\Omega)\ , \qquad \pi_n(f)(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega_n \\ 0 & \text{otherwise.} \end{cases}$$

For every $n \geq 1$, the operator

$$T_n : L^1(\Omega_n) \to \mathbb{R}\ , \qquad T_n(f) = T(\pi_n(f)),$$

is a bounded linear operator on $L^1(\Omega_n)$ and it holds

$$|T_n(f)| \leq C\|f\|_{L^1(\Omega_n)} \quad \text{for all} \quad f \in L^1(\Omega_n).$$

By Step 1, we can find a unique $g_n \in L^\infty(\Omega_n)$ such that

$$\|g_n\|_{L^\infty(\Omega_n)} \leq C \quad \text{and} \quad T_n(f) = \int_{\Omega_n} f(x)g_n(x)\,dx \quad \text{for all} \quad f \in L^1(\Omega_n).$$

Moreover, by the definition of T we have that

$$g_{n+1} = g_n \quad \text{on} \quad \Omega_n.$$

Thus, we can define the function $g : \Omega \to \mathbb{R}$ as follows

$$g(x) = g_n(x) \quad \text{for all} \quad x \in \Omega_n \quad \text{and all} \quad n \geq 1.$$

By construction we have that

$$g \in L^\infty(\Omega)\ , \quad \|g\|_{L^\infty(\Omega)} \leq 1\ , \quad \text{and}$$

$$T(\pi_n f) = \int_\Omega (\pi_n f)(x)g(x)\,dx \quad \text{for all} \quad f \in L^1(\Omega).$$

Since $\pi_n f \to f$ in $L^1(\Omega)$ we get that

$$T(f) = \int_\Omega f(x)g(x)\,dx \quad \text{for all} \quad f \in L^1(\Omega),$$

which concludes the proof. □