First variation of the L^p norm

Lemma 1. *Let $p > 2$. Then, the function*

$$F : \mathbb{R} \to \mathbb{R} \ , \quad F(x) = |x|^p,$$

is in $C^2(\mathbb{R})$ and

$$F'(x) = px|x|^{p-2} \quad \text{and} \quad F''(x) = p(p-1)|x|^{p-2}.$$

Moreover, for all $x, y \in \mathbb{R}$, we have

$$\left|\frac{1}{t}\big(F(x+ty) - F(x) - tyF'(x)\big)\right| \le p(p-1)|t||y|^2\big(|x|+|y|\big)^{p-2}.$$

Proof. It is sufficient to check that

$$\frac{1}{t}\big(F(x+ty) - F(x) - tyF'(x)\big) = y\Big(F'(x+sy) - F'(x)\Big) = s|y|^2F''(x+\sigma y),$$

where $s = s(t,x,y)$ and $\sigma = \sigma(t,x,y)$ are such that $0 < |\sigma| < |s| < |t|$. □

Lemma 2. *Let $p \ge 1$, $\Omega \subset \mathbb{R}^d$ be a measurable set, and let*

$$\alpha > 0,\ \beta > 0 \text{ be such that } \alpha + \beta = 1.$$

Then,

$$|u|^\alpha |v|^\beta \in L^p \quad \text{for every} \quad u, v \in L^p(\Omega).$$

Proof. By the Young's inequality we have

$$|u|^{p\alpha}|v|^{p\beta} \le \alpha|u|^p + \beta|v|^p.$$

Thus

$$\||u|^\alpha|v|^\beta\|_{L^p}^p = \int_\Omega |u|^{p\alpha}|v|^{p\beta}\,dx \le \alpha\int_\Omega |u|^p + \beta\int_\Omega |v|^p < +\infty$$

□

Proposition 3. *Let $p > 2$ and $\Omega \subset \mathbb{R}^d$ be a measurable set. Then, for every $w, \phi \in L^p(\Omega)$ the function*

$$f(t) := \int_\Omega |w + t\phi|^p,$$

is differentiable at $t = 0$ and its derivative is given by

$$f'(0) = p\int_\Omega \phi w|w|^{p-2}\,dx.$$

Proof. By Lemma 2, we get that

$$\phi|w|^{p-1} \in L^1(\Omega) \quad \text{and} \quad \phi^2|w|^{p-2} \in L^1(\Omega).$$

By applying Lemma 1, for every t and every $x \in \Omega$, we have

$$\left|\frac{1}{t}\Big(|w+t\phi|^p - |w|^p - tp\phi w|w|^{p-2}\Big)\right| \le p(p-1)|\phi|^2\big(|\phi|+|w|\big)^{p-2} \in L^1(\Omega),$$

while for every $x \in \Omega$ we have

$$\frac{1}{t}\Big(|w(x)+t\phi(x)|^p - |w(x)|^p - tp\phi(x)w(x)|w(x)|^{p-2}\Big) \to 0.$$

Thus, by the dominated convergence theorem, we have

$$\lim_{t\to 0}\int_\Omega \frac{1}{t}\Big(|w+t\phi|^p - |w|^p - tp\phi w|w|^{p-2}\Big)\,dx = 0,$$

which concludes the proof. □