Linear operators on Banach spaces

Definition 1. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$. We say that $T:\mathcal{B}\to\mathbb{R}$ is a continuous linear functional if:*

- *T is linear:*
$$T(\alpha x+\beta y)=\alpha T(x)+\beta T(y),$$
for all $x,y\in\mathcal{B}$ and all $\alpha,\beta\in\mathbb{R}$;
- *T is continuous:*
$$x_n\to x \quad in \quad \mathcal{B} \qquad \Rightarrow \qquad T(x_n)\to T(x) \quad in \quad \mathbb{R}\,.$$

Theorem 2. *Let $T:\mathcal{B}\to\mathbb{R}$ be a linear functional on the Banach space $\mathcal{B}$. Then,the following are equivalent:*

(1) *T is continuous;*

(2) *T is bounded:*
$$\|T\|:=\sup\Big\{T(x)\ :\ x\in\mathcal{B}\,,\ \|x\|_{\mathcal{B}}\le 1\Big\}<+\infty\,.$$

Exercice 3. *Let $\mathcal{B}$ be a Banach space and let $T:\mathcal{B}\to\mathbb{R}$ be a linear bounded operator on $\mathcal{B}$. Is it true that there is $x\in\mathcal{B}$ such that*
$$\|x\|_{\mathcal{B}}=1 \qquad and \qquad T(x)=\sup\Big\{T(y)\ :\ y\in\mathcal{B},\ \|y\|_{\mathcal{B}}\le 1\Big\}?$$

Exercice 4. *Let $\mathcal{B}$ be a Banach space and let $T:\mathcal{B}\to\mathbb{R}$ be a linear bounded operator on $\mathcal{B}$. Suppose that there is $x\in\mathcal{B}$ such that*
$$\|x\|_{\mathcal{B}}=1 \qquad and \qquad T(x)=\sup\Big\{T(y)\ :\ y\in\mathcal{B},\ \|y\|_{\mathcal{B}}\le 1\Big\}.$$
Is it true that an element x with this property has to be unique?

For hints about Ex.3 and Ex.4, check the next page.

Exercice 5. *Let $p\in(1,+\infty)$, $q:=\frac{p}{p-1}\in(1,+\infty)$, and let Ω be a measurable set in $\mathbb{R}^d$. Fixed a function $g\in L^p(\Omega)$ consider the operator*
$$T_g:L^p(\Omega)\to\mathbb{R}\ ,\qquad T_g(f)=\int_\Omega f(x)g(x)\,dx.$$

(1) Prove that T_g is bounded and compute the operator norm
$$\|T_g\|:=\sup\Big\{T_g(f)\ :\ f\in L^p(\Omega)\,,\ \|f\|_{L^p}\le 1\Big\}\,.$$

(2) Find a function $f\in L^p(\Omega)$ such that
$$\|f\|_{L^p}=1 \quad and \quad T_g(f)=\|T_g\|.$$

Exercice 6. *Let $\mathcal{H}$ be an Hilbert space withe scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{\mathcal{H}}$. Fixed a vector $v\in\mathcal{H}$ consider the operator*
$$T_v:\mathcal{H}\to\mathbb{R}\ ,\qquad T_v(x)=\langle v,x\rangle.$$

(1) Prove that T_v is bounded and compute the operator norm

$$\|T_v\| := \sup\Big\{T_v(x) \ : \ v \in \mathcal{H}\,, \ \|x\|_{\mathcal{H}} \le 1\Big\}.$$

(2) Find a vector $x \in \mathcal{H}$ such that

$$\|x\|_{\mathcal{H}} = 1 \quad \textit{and} \quad T_v(x) = \|T_v\|.$$

Exercice 7. *Consider the Banach space*

$$C_0(\mathbb{R}) = \Big\{f : \mathbb{R} \to \mathbb{R}\ ,\ f - \textit{continuous and bounded,} \lim_{|x|\to+\infty} f(x) = 0\Big\},$$

equipped with the norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be such that

$$g > 0 \quad \textit{and} \quad \int_{-\infty}^{+\infty} g(x)\,dx = 1.$$

Consider the operator

$$T_g : C_0(\mathbb{R}) \to \mathbb{R}\ , \qquad T_g(f) = \int_\Omega f(x)g(x)\,dx.$$

(1) Prove that T_g is bounded and compute the operator norm

$$\|T_g\| := \sup\Big\{T_g(f) \ : \ f \in C_0(\mathbb{R})\,, \ \|f\|_\infty \le 1\Big\}.$$

(2) Prove that there is no function $f \in C_0(\mathbb{R})$ such that

$$\|f\|_\infty = 1 \quad \textit{and} \quad T_g(f) = \|T_g\|.$$

Exercice 8. *Consider the Banach space*

$$C_0(\mathbb{R}) = \Big\{f : \mathbb{R} \to \mathbb{R}\ ,\ f - \textit{continuous and bounded,} \lim_{|x|\to+\infty} f(x) = 0\Big\},$$

equipped with the norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

Given $x_0 \in \mathbb{R}$, consider the evaluation operator

$$\delta_{x_0} : C_0(\mathbb{R}) \to \mathbb{R}\ , \qquad \delta_{x_0}(f) = f(x_0).$$

(1) Prove that δ_{x_0} is bounded and compute the operator norm

$$\|\delta_{x_0}\| := \sup\Big\{\delta_{x_0}(f) \ : \ f \in C_0(\mathbb{R})\,, \ \|f\|_\infty \le 1\Big\}.$$

(2) Is there a function $f \in C_0(\mathbb{R})$ such that

$$\|f\|_\infty = 1 \quad \textit{and} \quad \delta_{x_0}(f) = \|\delta_{x_0}\| \ ?$$

Exercice 9 (Written exam 2002, Paolo Acquistapace)**.** *For every $n \in \mathbb{N}$ define the functional T_n on the Banach space $L^\infty(0, +\infty)$ as follows:*

$$T_n(f) = n\left[\int_0^1 x^n f(x)\,dx + \int_1^{+\infty} e^{-nx} f(x)\,dx\right].$$

Compute the operator norm

$$\|T_n\| := \sup\Big\{T_n(f) \ : \ f \in L^\infty(0, +\infty)\,, \ \|f\|_\infty \le 1\Big\}.$$

Linear operators from a Banach space to another Banach space

Definition 10. *Let V and W be Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. We say that $T : V \to W$ is a continuous linear functional if:*

- *T is linear:*
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$
for all $x, y \in V$ and all $\alpha, \beta \in \mathbb{R}$;

- *T is continuous:*
$$x_n \to x \quad in \quad V \qquad \Rightarrow \qquad T(x_n) \to T(x) \quad in \quad W\,.$$

Theorem 11. *Let V and W be Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, and let $T : V \to W$ be a linear map. Then,the following are equivalent:*

(1) *T is continuous;*

(2) *T is bounded:*
$$\|T\|_{\mathcal{L}(V,W)} := \sup\Big\{\|T(x)\|_W \ : \ x \in V\,,\ \|x\|_V \le 1\Big\} < +\infty\,.$$

Exercice 12. *Prove Theorem* 11.

Exercice 13. *Fix a function $g \in L^\infty(\mathbb{R})$. Consider the operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as*
$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})\ , \qquad T(f) = fg.$$
Prove that T is bounded and compute its norm
$$\|T\| := \sup\Big\{\|Tf\|_{L^2} \ : \ f \in L^2\,,\ \|f\|_{L^2} \le 1\Big\}.$$

Exercice 14. *Fix an element $a \in \ell^\infty$. For every $x \in \ell^2$, define $Tx \in \ell^2$ as follows:*
$$(Tx)_n = a_n x_n \quad for\ all \quad n \ge 1.$$
Prove that T is a bounded linear operator from ℓ^2 to ℓ^2 and compute its norm
$$\|T\| := \sup\Big\{\|Tx\|_2 \ : \ x \in \ell^2\,,\ \|x\|_2 \le 1\Big\}.$$

Exercice 15. *Consider the Hilbert space ℓ^2 and the shift operator*
$$T : \ell^2 \to \ell^2\ , \quad (Tx)_n = x_{n+1}.$$
Prove that T is a bounded linear operator from ℓ^2 to ℓ^2 and compute its norm
$$\|T\| := \sup\Big\{\|Tx\|_2 \ : \ x \in \ell^2\,,\ \|x\|_2 \le 1\Big\}.$$

Exercice 16. *Consider the Hilbert space ℓ^2 and the operator*
$$T : \ell^2 \to \ell^2\ , \quad (Tx)_n = x_{2n}.$$
Prove that T is a bounded linear operator from ℓ^2 to ℓ^2 and compute its norm
$$\|T\| := \sup\Big\{\|Tx\|_2 \ : \ x \in \ell^2\,,\ \|x\|_2 \le 1\Big\}.$$

Exercice 17 (Written exam 2003, Paolo Acquistapace). *Consider the Banach space* $C([0,1])$, *of continuous functions on* $[0,1]$, *equipped with the norm* $\|\cdot\|_{L^\infty}$. *For every* $f \in C([0,1])$ *define the function*

$$Tf : [0,1] \to \mathbb{R}$$

as follows

$$Tf(x) = \int_0^x (x^2 - t^2) f(t)\, dt \quad \text{for all} \quad x \in [0,1].$$

Prove that T *is a bounded linear operator from* $C([0,1])$ *to* $C([0,1])$ *and compute the norm*

$$\|T\| := \sup \left\{ \|T(f)\|_{L^\infty} \ : \ f \in C([0,1]) \, , \ \|f\|_{L^\infty} \leq 1 \right\} .$$

Exercice 18 (Written exam 2003, Paolo Acquistapace). *Consider the Banach space* $C([a,b])$, *of continuous functions on* $[a,b]$, *equipped with the norm* $\|\cdot\|_{L^\infty(a,b)}$. *For every* $f \in C([a,b])$ *define*

$$(Tf)(x) = \int_a^x \frac{f(t)}{\sqrt{x-t}}\, dt \quad \text{for all} \quad x \in [a,b].$$

Prove that T *is a bounded linear operator from* $C([a,b])$ *to* $C([a,b])$ *and compute the norm*

$$\|T\| := \sup \left\{ \|T(f)\|_{L^\infty} \ : \ f \in C([a,b]) \, , \ \|f\|_{L^\infty} \leq 1 \right\} .$$

Exercice 19 (Written exam, Pietro Majer). *Let* $p \in [1,+\infty]$. *For all* $u \in L^p(0,\pi)$ *and* $x \in [0,\pi]$, *define*

$$(Tu)(x) = \int_0^{\sin x} u(s)\, ds$$

Prove that T *is a bounded linear operator from* $L^p(0,\pi)$ *in* $L^p(0,\pi)$ *and compute (or estimate from above) its norm*

$$\|T\| := \sup \left\{ \|T(f)\|_{L^p} \ : \ f \in L^p(0,\pi) \, , \ \|f\|_{L^p} \leq 1 \right\} .$$

Exercice 20 (Written exam, Pietro Majer). *Consider the Hilbert space* $L^2([0,\pi])$. *For all* $u \in L^2([0,\pi])$ *and* $x \in [0,\pi]$, *define the Volterra's operator*

$$(Tu)(x) = \int_0^x u(s)\, ds$$

Prove that T *is a bounded linear operator from* $L^2([0,\pi])$ *to* $L^2([0,\pi])$ *and compute (or estimate from above) its norm*

$$\|T\| := \sup \left\{ \|T(f)\|_{L^2} \ : \ f \in L^2([0,\pi]) \, , \ \|f\|_{L^2} \leq 1 \right\} .$$

Exercice 21 (Written exam 2023, Pietro Majer). *Consider the Hilbert space* $L^2([0,1])$. *For all* $u \in L^2([0,1])$ *and* $x \in [0,1]$, *define*

$$(Tu)(x) = \int_0^1 \cos(\log xy)\, u(y)\, dy$$

Prove that T *is a bounded linear operator from* $L^2([0,\pi])$ *to* $L^2([0,\pi])$ *and compute*

$$\sup_{\|u\|=1} \|T(u)\| \ , \qquad \sup_{\|u\|=1} \langle T(u), u \rangle \ , \qquad \inf_{\|u\|=1} \langle T(u), u \rangle .$$

Exercice 22 (Written exam 2011, Francesca Prinari). *Let* $p \in [1,+\infty]$. *Consider the Banach space* $L^p([0,1])$. *For all* $u \in L^p([0,1])$ *and* $x \in [0,1]$, *define*

$$(Tu)(x) = xu(x).$$

Prove that T *is a bounded linear operator from* $L^p([0,1])$ *to* $L^p([0,1])$ *and compute its norm.*

Exercice 23 (Written exam 2011, Francesca Prinari). *Let c_0 be the Banach space*

$$c_0 = \Big\{ a = (a_n)_{n\geq 1} \ : \ a_n \in \mathbb{R} \text{ for all } n \geq 1, \ \lim_{n\to\infty} a_n = 0 \Big\},$$

equipped with the norm

$$\|a\|_\infty = \sup_n |a_n|.$$

Consider the linear operator

$$T : c_0 \to c_0 \ , \qquad (Ta)_n = a_{n+1} - a_n.$$

Prove that T is bounded and compute its norm.

Exercice 24 (Written exam 2011, Francesca Prinari). *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator with the following property:*

for every bounded linear operator $\ell : Y \to \mathbb{R}$, the operator $\ell \circ T : X \to \mathbb{R}$ is bounded and linear.

Prove that T is bounded.

Exercice 25. *Check the following exercice booklet by Prof. Prinari:*

https://www.unife.it/scienze/lm.matematica/insegnamenti/analisi-funzionale/materiale-didattico/raccoltaesamianfunz.pdf