Banach and Hilbert spaces

Definition 1 (Banach space). *We say that $\mathcal{B}$ is a Banach space, if:*

- *$\mathcal{B}$ is a vector space over the real numbers;*
- *$\mathcal{B}$ is a normed vector space in the sense that there is a norm $p : \mathcal{B} \to \mathbb{R}$ such that:*
  - *$p(x) \geq 0$ for all $x \in \mathcal{B}$;*
  - *if $p(x) = 0$ for some $x \in \mathcal{B}$, then $x = 0$;*
  - *$p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{B}$;*
  - *$p(\alpha x) = |\alpha| p(x)$ for all $x \in \mathcal{B}$ and all $\alpha \in \mathbb{R}$.*

  We will often use the notation
  $$\|x\|_{\mathcal{B}} := p(x).$$
- *$\mathcal{B}$ is complete, that is, every Cauchy sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{B}$ (equipped with the norm $\|\cdot\|_{\mathcal{B}}$) admits a limit $x \in \mathcal{B}$.*

Definition 2 (Hilbert space). *We say that $\mathcal{H}$ is a Hilbert space, if:*

- *$\mathcal{H}$ is a vector space over the real numbers;*
- *$\mathcal{H}$ is equipped with a scalar product*
  $$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R};$$
- *$\langle$ is complete with respect to the norm*
  $$\|x\|_{\mathcal{H}} = \sqrt{\langle x, x \rangle}.$$

Example 3. *The following spaces are Banach spaces:*

- *$L^p(\mathbb{R})$ for $p \in [1, +\infty]$;*
- *ℓ^p for $p \in [1, +\infty]$;*
- *$C(I)$, where I is an interval in $\mathbb{R}$, equipped with the norm*
  $$\|f\|_\infty = \sup_{x \in I} |f(x)|;$$
- *$C_0(\mathbb{R})$, which is the subspace of $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to \pm\infty} f(x) = 0$.*
- *$C^1(I)$, where I is an interval in $\mathbb{R}$, equipped with the norm*
  $$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty = \sup_{x \in I} |f(x)| + \sup_{x \in I} |f'(x)|.$$

Exercice 4. *For every interval $I \subset \mathbb{R}$ consider the space $C(I)$ of bounded continuous functions $f : I \to \mathbb{R}$. Prove that $C(I)$ equipped with the norm*
$$\|f\|_\infty = \sup_{x \in I} |f(x)|;$$
is a Banach space.

Exercice 5. *Consider the space $C_0(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Prove that $C_0(\mathbb{R})$ equipped with the norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|;$$

is a Banach space.

Exercice 6. *For every interval $I \subset \mathbb{R}$ consider the space $C^1(I)$ of bounded differentiable functions $f : I \to \mathbb{R}$ with continuous derivatives f' . Prove that $C^1(I)$, equipped with the norm*

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty = \sup_{x\in I} |f(x)| + \sup_{x\in I} |f'(x)|,$$

is a Banach space.

Exercice 7. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$. Prove that $\|\cdot\|$ is induced by a scalar product*

$$\langle\cdot,\cdot\rangle : \mathcal{H}?times\mathcal{H} \to \mathbb{R},$$

if and only if

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 \quad \text{for all} \quad x, y \in \mathcal{B}.$$

Moreover, the scalar product is given by

$$\langle x, y\rangle = \left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2.$$

Exercice 8 (P.Acquistapace, esame scritto 2003)**.** *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$. Prove that $\|\cdot\|$ is induced by a scalar product if and only if:*

for all $n \geq 2$ and for all n-uple of vectors $x_1, \dots, x_n \in \mathcal{B}$ it holds:

$$\sum_{1\leq j<k\leq n} \|x_j - x_k\|^2 + \left\|\sum_{j=1}^n x_j\right\|^2 = n\sum_{j=1}^n \|x_j\|^2.$$

Exercice 9 (P.Acquistapace, esame scritto 2004)**.** *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$ and let $x \in \mathcal{B} \setminus \{0\}$ be fixed.*

(1) Prove that for all $v \in X$ it holds

$$-\|v\| \leq \liminf_{t\to 0} \frac{\|x+tv\| - \|x\|}{t} \leq \limsup_{t\to 0} \frac{\|x+tv\| - \|x\|}{t} \leq \|v\|.$$

(2) Find an example (of $\mathcal{B}, x, v$) such that

$$\liminf_{t\to 0} \frac{\|x+tv\| - \|x\|}{t} < \limsup_{t\to 0} \frac{\|x+tv\| - \|x\|}{t}.$$

(3) Show that if the norm $\|\cdot\|$ is induced by a scalar product, then the limit

$$\lim_{t\to 0} \frac{\|x+tv\| - \|x\|}{t}$$

exists.

(4) Suppose that the limit

$$\lim_{t\to 0} \frac{\|x+tv\| - \|x\|}{t}$$

exists for all $x, v \in \mathcal{B} \setminus \{0\}$. Is it true that $\mathcal{B}$ is Hilbert?

Exercice 10. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$.
Is it true that there exists a norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H}$ such that:*

- *$\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent in the sense that*

$$c\|\cdot\|_{\mathcal{H}} \le \|\cdot\|_{\mathcal{B}} \le C\|\cdot\|_{\mathcal{H}}$$

  for some constants $0 < c < C < +\infty$;

- *$\|\cdot\|_{\mathcal{H}}$ is induced by a scalar product?*

Exercice 11. *Let $\mathcal{H}$ be an Hilbert space and let*

$$\mathcal{K} \subset \mathcal{H} \text{ be a closed convex subset of } \mathcal{H}.$$

Prove that there for every $x \in \mathcal{H}$ there is a unique $P(x) \in \mathcal{K}$ such that

$$\|x - P(x)\| = \inf_{y\in K} \|x - y\|.$$

Prove that the map $P : \mathcal{H} \to \mathcal{K}$ is 1-Lipschitz.

Exercice 12. *Let $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{K} = \{f \in \mathcal{H} \; : \; f = 0 \text{ a.e. on } \mathbb{R} \setminus [0,1]\}$. Given a function $f \in L^2(\mathbb{R})$ find the projection $P(f)$ of f on $\mathcal{K}$.*

Exercice 13. *Let $\mathcal{H} = L^2([0,1])$ and $\mathcal{K} = \{f \in \mathcal{H} \; : \; \int_0^1 f(t)\,dt = 0\}$. Given a function $f \in L^2([0,1])$ find the projection $P(f)$ of f on $\mathcal{K}$.*

Exercice 14. *Let $\mathcal{H} = L^2([0,1])$ and let $\mathcal{K} = \{f \in \mathcal{H} \; : \; \|f\| \le 1\}$. Given a function $f \in L^2([0,1])$ find the projection $P(f)$ of f on $\mathcal{K}$.*

Question 15. *Let $\mathcal{H}$ be an Hilbert space and let $\mathcal{K} \subset \mathcal{H}$ be a subset of $\mathcal{H}$ satisfying the following property:*

$$\text{for every } x \in \mathcal{H} \text{ there is a unique } P(x) \in \mathcal{K} \text{ such that } \|x - P(x)\| = \inf_{y\in K} \|x - y\|.$$

Is it true that $\mathcal{K}$ is closed and convex?

Exercice 16. *Let $\mathcal{K}$ be a closed convex subset of $L^p(I)$, where I is an interval and $p \in (0, +\infty)$. Prove that there for every $f \in L^p(I)$ there is a unique $P(f) \in \mathcal{K}$ such that*

$$\|f - P(f)\| = \inf_{g\in K} \|f - g\|_{L^p(I)}.$$

Exercice 17. *Let $\mathcal{H} = L^p([0,1])$ and let $\mathcal{K} = \{f \in \mathcal{H} \; : \; \|f\| \le 1\}$. Given a function $f \in L^2([0,1])$ find the projection $P(f)$ of f on $\mathcal{K}$.*

Definition 18 (Uniformly convex Banach space). *We say that a Banach space $\mathcal{B}$ with norm $\|\cdot\|$ is uniformy convex if:*

$$\text{for all } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that the, for all } x, y \in \mathcal{B}, \text{ following holds :}$$

$$\text{if } \|x\| \le 1,\ \|y\| \le 1 \text{ and } \|x - y\| > \varepsilon, \text{ then } \left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

Exercice 19. *Let $\mathcal{B}$ be a uniformly convex Banach space and let $\mathcal{K}$ be a closed convex subset of $\mathcal{B}$. Prove that there for every $x \in \mathcal{B}$ there is a unique $P(x) \in \mathcal{K}$ such that*

$$\|x - P(x)\| = \inf_{y\in K} \|x - y\|.$$

Exercice 20. *Find a Banach space $\mathcal{B}$, a closed convex subset $\mathcal{K} \subset \mathcal{B}$ and a point $x \in \mathcal{B}$ such that the infimum $\inf_{y\in K} \|x - y\|$ is not achieved.*