Ultrafilters, Determinacy, and Large Cardinals

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Natural numbers, real numbers, transfinite ordinals, cardinals.

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Precisely, quantifiers of φ restricted to range over sets in M.

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card($\mathcal{P}(\kappa)$) denoted 2^κ. The very next cardinal above κ denoted κ^+. Are they the same?

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Singular Cardinal Hypothesis says no.

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Regular: Suppose $f\colon \alpha \to \kappa$, with $\alpha < \kappa$. Then $\pi(f) = f$, so $\pi(f)$ is bounded in $\pi(\kappa)$, so f is bounded in κ.

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$\kappa = \text{crit}(\pi)$ is called a *measurable cardinal*.

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One axiom in a rich hierarchy.

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The hypothesis asserts that if κ is singular and $(\forall\tau < \kappa)2^\tau < \kappa$, then $2^\kappa = \kappa^+$.

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A filter $G \subseteq \mathbb{P}$ is *generic* over M if $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ which belong to M.

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Gives an <u>extension</u> $M[G]$ of M.

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$\langle t, B\rangle \leq \langle s, A\rangle$ iff t extends s, $B \subseteq A$, and $t - s$ takes values in A.

Called *Prikry forcing*.

Suppose $G \subseteq \mathbb{P}$ is generic over M.

Let $g = \bigcup\{s \mid (\exists A)\langle s, A\rangle \in G\}$.

Then $g\colon \omega \to \kappa$ is unbounded in κ.

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Using second coordinates and $\mathcal{U}$ being an ultrafilter get:

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Singular cardinal hypothesis fails in $M[G]$.

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If z belongs to A then player I wins.
If z does not belong to A then player II wins.

$G_\omega(A)$ is *determined* if one of the players has a winning strategy.

(A *strategy* is a complete recipe that instructs the player precisely how to play in each conceivable situation.)

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But determinacy for *definable* sets is: (1) true; and (2) useful.

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The others require large cardinal axioms.

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Determinacy in turn implies the existence of many ultrafilters.

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For $d \in \mathcal{D}$ define $A_d = \{e \mid e \geq_T d\}$.

$\mathcal{F} = \{X \mid X \supseteq A_d \text{ for some } d \in \mathcal{D}\}$ is a filter.

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Player I wins if $[x \oplus y] \in Z$. Player II wins if $[x \oplus y] \in \mathcal{D} - Z$.

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If σ is a winning strategy for I, then $A_{[\sigma]} \subseteq Z$.

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Proof: Let $Z \subseteq \mathcal{D}$. Consider the following game.

$$\begin{array}{c|cccccc} I & x_0 & & x_1 & & \ldots & \\ \hline II & & y_0 & & y_1 & & \ldots \end{array}$$

Player I wins if $[x \oplus y] \in Z$. Player II wins if $[x \oplus y] \in \mathcal{D} - Z$.

If σ is a winning strategy for I, then $A_{[\sigma]} \subseteq Z$.

If τ is a winning strategy for II, then $A_{[\tau]} \subseteq \mathcal{D} - Z$.

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Can we have $\boldsymbol{\delta}^1_n \geq \aleph_2$?

Theorem (Steel–Van Wesep–Woodin) Assume $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$. Then it is consistent (with $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ and AC) that $\boldsymbol{\delta}^1_2 = \aleph_2$.

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Since in $\mathsf{L}(\mathbb{R})$ (where AC fails) $\boldsymbol{\delta}^1_2$ is equal to $\aleph_2$, get that in the extension $\boldsymbol{\delta}^1_2 = \aleph_2$.

Theorem (Neeman, Woodin) Assume $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$. Then it is consistent (with $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ and the axiom of choice) that $\boldsymbol{\delta}^1_3 = \aleph_2$.

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This time produce an extension in which $(\aleph_1)^{\mathsf{L}(\mathbb{R})}$ and $(\aleph_{\omega+1})^{\mathsf{L}(\mathbb{R})}$ remain cardinals, but $(\aleph_n)^{\mathsf{L}(\mathbb{R})}$ for $2 \leq n \leq \omega$ do not.

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The construction of these ultrafilters is done not using games, but using directed systems of ultrapowers of countable models of AC.