New Generalized Functions

Defined by nonstandard discrete
Functions and difference quotients

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Abstract

By using nonstandard analysis, we define new generalized functions as discrete functions, and their derivatives are defined as difference quotients.

For Gevrey's ultradistributions, including Schwartz' distributions, we prove that difference quotients are indeed good replacements of generalized derivatives.

Relations of our new generalized functions with Sobolev theory are presented. It is expected that this theory will be useful for nonlinear partial differential equations with distributional data, via difference method.

Why

Theory of distributions of Schwartz and Sobolev led to revolutionary progress in linear partial differential equations, whereas there are essential difficulties in using it in nonlinear problems.

The aims of new generalized function theories of Columbeau and others, e.g. H.A.Biagioni and M.Oberguggenberger in the framework of standard analysis;

Todorov, we in the framework of nonstandard analysis are all towards nonlinear problems.

Why

Schwartz defined distributions as linear continuous functionals on spaces of test functions. While his distributions can be represented by ordinary functions in the framework of nonstandard analysis.

There are lots of nonstandard representations for a distribution, and it was shown by Li,Banghe in the study of moiré problem that different nonstandard representations of a given distribution themselves have independent physical meanings.

Thus our essential point of view is to regard nonstandard functions as new generalized functions. This makes distribution theory more precise and includes more content.

Why

For a continuously differentiable function, its derivative can be represented by difference quotient with infinitesimal increments. And it is well-known that the finite difference method is at least one of the most commonly used method in solving problems of linear or nonlinear partial differential equations.

To represent new generalized functions by discrete function, we should use difference quotient to replace derivatives.

We will prove that even for Gevrey's ultradistributions which are much wider than Schwartz' distributions, this replacement is reasonable.

Relations of our new generalized functions defined by nonstandard discrete functions with ordinary functions, e.g. L^p functions, will be given. Some embedding theorems of Sobolev type will be proved.

Related Work

There were related works of Kessler and Kinoshita. They proved that distributions can be represented by nonstandard discrete functions. Here we prove that it is also true for ultradistributions, by using complete different method.

Kessler proved that a distribution in dimension one with a representative which is invariant under infinitesimal transformations must be a Radon measure. This interesting result is generalized to any dimension here. Kinoshita has also studied the representation of L^p functions.

Mention

Also that the idea of nonstandard discrete functional analysis has already been widely used by S. Albeverio and his collaborators in quantum mechanics and quantum field theory.

Mention

For applications of nonstandard analysis in stochastic processes, it is usually to take the time discrete.

This method has been fruitful (cf. Cutland). If we consider the generalized stochastic processes, i.e. their sample paths are Schwartz's distributions, or more general, Gevrey's ultradistributions. e.g. in the case of white noise processes, generalized derivative by difference quotients. Hence the results of this paper are expected to be useful in this situation.

Symbols

Ω open set in $\mathbb{R}^m$. $Ns(\Omega)$ the set of near -standard points in ${}^*\Omega$.

N the set of nonnegative integers, and **Z** the set of integers.

Fix positive infinitesimals $h_1, h_2, \cdots, h_m$.

Take $J_i \in {}^*\mathbf{N}$ such that $J_i h_i$ is infinite. Let

$$\tilde{J}_i = \{j_i \,/\, j_i \in {}^*\mathbf{Z}, -J_i \leq j_i \leq J_i \}$$

$$J = \tilde{J}_1 \times \cdots \times \tilde{J}_m$$

Definition of $G_h(\Omega)$–NGF on Ω of type h

Two internal functions

$$u, v : J \to {}^*\mathbb{C} = {}^*\mathbb{R} + \sqrt{-1}\,{}^*\mathbb{R}$$

are $\Omega-$equivalent with respect to $h = (h_1, h_2, \cdots, h_m)$,

if for any $j = (j_1, j_2, \cdots, j_m) \in J$ with $(j_1 h_1, \cdots, j_m h_m) \in Ns(\Omega)$,

$$u(j) = v(j)$$

An equivalent class $[u]$ is a new generalized function (i.e. $u \in G_h(\Omega)$).

Definition: $\delta^\alpha u$ the difference quotient of u with index α

For $u \in G(\Omega)$, we may regard u as an internal function on J which represents it.

$$(\Delta_i u)(j_1, \cdots, j_m) = u(j_1, \cdots, j_{i-1}, j_i+1, j_{i+1}, \cdots, j_m) - u(j_1, \cdots, j_m)$$

then $\Delta_i u$ is well defined on an internal subset of J containing $Ns(\Omega)$.

Thus $\Delta_i u$ as an element in $G(\Omega)$ is well-defined.

For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbf{N}^m$, let

$$\Delta^\alpha = \Delta_1^{\alpha_1} \cdots \Delta_m^{\alpha_m}, \quad h^\alpha = h_1^{\alpha_1} \cdots h_m^{\alpha_m},$$

$$\delta^\alpha u = \frac{\Delta^\alpha u}{h^\alpha}$$

Proposition 1

$G(\Omega)$ is an algebra over ${}^*\mathbb{C}$ with difference quotient operators of any index $\alpha \in \mathbf{N}^m$.

If f is a standard continuous function on Ω, then ${}^*f|_{Ns(\Omega)}$ is finite. For any positive infinity H, there is an internal function $u : J \to {}^*\mathbb{C}$, such that

$$|u(j)| < H \text{ for any } j \in J \text{ and}$$

$$u(j) = f^*(jh), \quad \text{if } jh \in Ns(\Omega)$$

$jh = (j_1 h_1, \cdots, j_m h_m)$.

H any positive infinity.

$G_H^n(\Omega)$: H a positive infinity and $n \in \mathbf{N}$, say $u \in G(\Omega)$ is
$H-$limited NGF of order n,

if for any $\alpha \in \mathbf{N}^m$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m \leq n$,

$$|\delta^\alpha u| < H$$

$G_H^n(\Omega)$ not an algebra.

$$G_H^\infty(\Omega) = \bigcap_{n \in N} G_H^n(\Omega)$$

an algebra over the field ${}^*\mathbb{C}$.

Theorem 1.

For any positive infinity H, and
$U \in \mathscr{D}^{(s)'}(\Omega)$,

there is a $u \in G_H^\infty(\Omega)$

such that

u is a nice representative of U.

Schwartzs space $\mathscr{D}(\Omega)$

$$\mathscr{D}(\Omega) = \varinjlim_{K \subset\subset \Omega} \mathscr{D}_K$$

is the strict inductive limit of $\mathscr{D}_K$.

$\mathscr{D}_K = \bigcap_{n \in \mathbf{N}} \mathscr{D}_K^n$ is a Frechet space with countable norms $\{||\phi||_n \ / \ n \in \mathbf{N}\}$.

$\mathscr{D}_K^n$: space of all complex-valued functions on $\mathbf{R}^m$ with support in a compact set K and continuous derivatives up to order $n \in \mathbf{N}$.

$\mathscr{D}_K^n$ is a Banach spaces with norm

$$||\phi||_n = \max_{|\alpha| \le n} \max_{x \in K} \{|D^\alpha \phi(x)|\}$$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}.$$

Gevrey space $\mathscr{D}^{(s)}(\Omega)$, $1 < s < \infty$, $s \in \mathbb{R}$

$$\mathscr{D}^{(s)}(\Omega) = \varinjlim_{K \subset\subset \Omega} \mathscr{D}_K^{(s)} \text{ strict inductive limit.}$$

$$\mathscr{D}_K^{(s)} = \bigcap_{n \in \mathbf{N}} \mathscr{D}_K^{(s),n}$$

$\mathscr{D}_K^{(s),n}$ the space of all $\phi \in \mathscr{D}_K$ such that

$$\sup_x |D^\alpha \phi(x)| \;/\; n^{-|\alpha|} |\alpha|!^s \to 0 \text{ as } |\alpha| \to \infty$$

$\mathscr{D}_K^{(s),n}$ is a Banach space with norm

$$||\phi||_{(s),n} = \sup_{x,\alpha} \{ \, |D^\alpha \phi(x)| \;/\; n^{-|\alpha|} |\alpha|!^s \}$$

Dual space $\mathscr{D}^{\Delta'}(\Omega)$

$\mathscr{D}(\Omega)$ and $\mathscr{D}^{(s)}(\Omega)$ are Montel spaces. Hence their dual space $\mathscr{D}'(\Omega)$ and $\mathscr{D}^{(s)'}(\Omega)$ with strong topology share the following nice properties: let $\Delta = (s)$ or empty, then a sequence $f_n \to 0$ in $\mathscr{D}^{\Delta'}(\Omega)$ iff for any $\phi \in \mathscr{D}^{\Delta}(\Omega)$, $< f_n, \phi > \to 0$, and a sequence $\phi_n \to 0$ in $\mathscr{D}^{\Delta}(\Omega)$ iff there is a $K \subset\subset \Omega$ such that all $\phi_n \in \mathscr{D}^{\Delta}{}_K(\Omega)$ and $\phi_n \to 0$ in $\mathscr{D}^{\Delta}{}_K(\Omega)$.

$\Delta =$ the empty, $\mathscr{D}'(\Omega)$ distributions,

$\Delta = (s)$, $\mathscr{D}^{(s)'}(\Omega)$ ultradistributions.

Notice that

$$\mathscr{D}^{0'}(\Omega) \subset \mathscr{D}^{1'}(\Omega) \subset \cdots \mathscr{D}'(\Omega) \subset \mathscr{D}^{(s)'}(\Omega)$$

Harmonic representation

For any $f \in \mathscr{D}^{(s)'}(\Omega)$, there is a harmonic function $F(x,t)$, $x \in \mathbb{R}^m$, $t > 0$ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^m} F(x,t)\phi(x)\,dx = < f, \phi >, \quad \phi \in \mathscr{D}^{(s)}(\Omega)$$

If $\tilde{F}(x,t)$ is another such harmonic function, then

$$\lim_{t\to 0}(\tilde{F}(x,t) - F(x,t)) = 0, \quad \text{uniformly for } \; x \in K \text{ and } K \subset\subset \Omega$$

μ_K^Δ: monad of ${}^*\mathscr{D}_K^\Delta(\Omega)$ at 0.

$$\mu_K = \{\ \phi \in {}^*{}_K\ /\ {}^*||\phi||_n \simeq 0\ \ \text{for all standard } n\}$$

$$\mu_K^n = \{\ \phi \in {}^*\mathscr{D}_K^n\ /\ {}^*||\phi||_n \simeq 0\ \ \}$$

$$\mu_K^{(s)} = \{\ \phi \in {}^*\mathscr{D}_K^{(s)}\ /\ {}^*||\phi||_{(s),n} \simeq 0\ \ \text{for all standard } n\}$$

Notice that $\phi \in \mu_K$ iff $D^\alpha\phi \simeq 0$ for all standard α, while for $\phi \in \mu_K^{(s)}$, a necessary condition is that

$$D^\alpha\phi \simeq 0 \quad \text{for all}\ \ \alpha \in {}^*\mathbf{N}^m$$

Define the pairing of u with $\phi \in {}^*\mathscr{D}_K^\Delta$

Two ways:

$$< u, \phi >_d = \sum_{j \in J} u(j)\, \phi(jh) h_1 \cdots h_m, \; jh = (j_1 h_1, \cdots, j_m h_m)$$

$$< u, \phi >_c = \int_{^*\mathbb{R}^m} u(x)\phi(x)\, dx = \sum_{j \in J} u(j) \int_{Q_j} \phi(x)\, dx$$

where $j \in J$,

$$Q_j = \{x = (x_1, \cdots, x_m) \in {}^*\mathbb{R}^m \;\; / \; j_i h_i \le x_i < (j_i + 1) h_i, \; i = 1, 2, \cdots, m\}$$

$u \in G(\Omega)$ as a function defined on $\bigcup_{j \in J} Q_j$ such that

$$u(x) = u(j) \quad \text{for } x \in Q_j$$

Pairing

$< u, \phi >_d$: was commonly adopted by the earlier literature

C.Kessler, Nonstandard methods in the theory of random fields, Doctoral dissertation, Ruhr-University, Bochum, 1984.

M.Kinoshita, Nonstandard representations of distributions I, Osaka J. of Math., 25 (1988) 805-824. II, Osaka J. of Math., 27 (1990) 843-861.

$< u, \phi >_c$ coincides with the hypercontinuous representation of distributions as in

Bang-He Li, On the moiré problem from distributional point of view, J. Sys. Sci. & Math. Scis., 6, (1986) 4, 263-268.

Bang-He Li & Ya-Qing Li, New generalized functions in nonstandard framework, Acta Math. Scientia, 12 (1992) 3, 260-269.

Bang-He Li& Ya-Qing Li, Nonstandard analysis and multiplication of distribution in any dimension, Scientia Sinica, 28, (1985) 9, 923-937.

$\diamondsuit -$ Near Standard Definition: if for any $K \subset\subset \Omega$ and $\phi \in \mu_K^{\Delta}$,

$$< u, \phi >_{\diamondsuit} \simeq 0$$

If u is $\mathscr{D}^{\Delta'}(\Omega) - \diamondsuit$-near standard, then it is easy to prove that for any $\phi \in \mathscr{D}^{\Delta}(\Omega)$, $< u, \phi >_{\diamondsuit}$ is finite, and

$$< U, \phi >= st < u, \phi >_{\diamondsuit}$$

define a $\mathscr{D}^{\Delta}(\Omega)-$distribution $U \in \mathscr{D}^{\Delta'}(\Omega)$. We call such $u \in G(\Omega)$ a nice $\diamondsuit$-representative of U.

$\delta^\alpha u$ and $D^\alpha U$

If $u \in G(\Omega)$ is a nice $\diamondsuit$-representative of $U \in \mathscr{D}^{\Delta'}(\Omega)$,

then $\delta^\alpha u$ is a nice $\diamondsuit$-representative of $D^\alpha U \in \mathscr{D}^{\Delta'}(\Omega)$,

when $\alpha \in \mathbf{N}^m$, and $D^\alpha U$ is the generalized derivative of U of index α.

Lemma 1

If H is positive infinity satisfying

$$H \cdot \max\{h_1, \cdots, h_m\} \simeq 0$$

and $u \in G(\Omega)$ satisfying $|u(j)| < H$ for $j \in J$,

then $\mathscr{D}^{\Delta'}(\Omega) - c-$ near standardness is equvalent to $\mathscr{D}^{\Delta'}(\Omega) - d-$ near standardness, and

$$< u, \phi >_c \simeq < u, \phi >_d \quad \text{for } \phi \in \mathscr{D}^{\Delta}(\Omega)$$

where $\Delta =$ the empty, (s) or $n \in \mathbf{N}$.

Proof of Lemma 1(not for $\Delta = 0$)

First assume that $\Delta \neq 0$. For $\phi \in \mu_K^{\Delta}$ or $\mathscr{D}_K^{\Delta}(\Omega)$, let $J' = \{j \in J \ / \ Q_j \cap Supp\phi \neq \emptyset\}$, then the compactness of K implies $\sum_{j \in J'} \int_{Q_j} dx$ being finite

By the integral mean value theorem $\int_{Q_j} \phi(x)\, dx = \phi(x_j) h_1 \cdots h_m$ for some $x_j \in Q_j$. Since $\Delta \neq 0, \phi \in {}^*\mathscr{D}_K^1$. So
$\phi(x_j) - \phi(jh) = \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(jh)(x_{j,i} - j_i h_i) + \varepsilon \sum_{i=1}^m |x_{j,i} - j_i h_i|$
where ε is an infinitesimal.
$\dfrac{\partial \phi}{\partial x_i}(jh)$ are finite, so
$|\phi(x_j) - \phi(jh)| \leq$ a finite number $\cdot \max\{h_1 \cdots h_m\}$ Thus

$$| < u, \phi >_c - < u, \phi >_d | = |\sum_{j \in J} u(j)(\int_{Q_j} \phi(x) dx - \phi(jh) h_1 \cdots h_m)|$$

$$\leq \text{ a finite number} \cdot \max\{h_1 \cdots h_m\} \cdot H \cdot \sum_{j \in J'} \int_{Q_j} dx$$

$$\simeq 0$$

First to find u, by harmonic representation. Take a harmonic function $\tilde{u}(x,t),\ x \in \mathbb{R}^m, t > 0$ such that for any $\phi \in \mathscr{D}^{(s)}(\Omega)$,

$$\lim_{t \to 0} \int_\Omega \tilde{u}(x,t)\phi(x)\,dx =< U, \phi >$$

Second prove (1): $|\tilde{u}(x,\rho)| < H$, for $x \in Ns(\Omega)$.

Now let $u : J \to {}^*C$ be given by $u(j) = \tilde{u}(jh,\rho)$, then $|u(j)| < H$ for $jh \in Ns(\Omega)$.

Third prove (2): $|\delta^\alpha u(j)| < H \quad$ if $jh \in Ns(\Omega)$.

Thus we have $u \in G_H^\infty(\Omega)$.

Last prove (3): u is a nice representative of $U \in {\mathscr{D}^{(s)}}'(\Omega)$.

Prove (1)

By Theorem 2.13 in (H.Komatsu, Microlocal analysis in Gevrey classes in complex domain), for any compact set K in Ω, there are C and a,
$\sup_{x\in K} |\tilde{u}(x,t)| \leq C \; exp((at)^{-\frac{1}{s-1}}),\; 0 < t < 1$

Let $\rho = (\log H)^b,\; b = -(\log\log H)^{-\frac{1}{2}}$.
Since $\log\rho = b\log\log H = -(\log\log H)^{\frac{1}{2}}$,
thus ρ is a positive infinitesimal.
For any standard positive numbers C, a and s with $s > 1$,

$$\begin{aligned}\frac{1}{a}(\log\frac{H}{C})^{1-s} &< a\log\sqrt{H})^{1-s} = a(\frac{1}{2})^{1-s}(\log H)^{1-s}\\ &< (\log H)^{\frac{s-1}{2}}(\log H)^{1-s} = (\log H)^{\frac{1-s}{2}} \leq \rho\end{aligned}$$

Hence $C\; exp((a\rho)^{-\frac{1}{s-1}}) < H$ i.e. $|\tilde{u}(x,\rho)| < H$, for $x \in {}^*K$

Since for any $x \in Ns(\Omega)$, there is such an K with $x \in {}^*K$, thus (1) hold.

Prove (2)

For simplicity, we assume that $\alpha_1, \cdots, \alpha_k > 0$ and $\alpha_i = 0$ for $k < i \leq m$.

$$\begin{aligned} Q_j^\alpha = & \{t = (t_{1,1}, \cdots, t_{1,\alpha_1}, \cdots, t_{k,1}, \cdots, t_{k,\alpha_k}) \in {}^*\mathbb{R}^{|\alpha|} \ / \\ & j_i h_i \leq t_{i,1} \leq (j_i + 1)h_i,\ 0 \leq t_{i,r} \leq h_i \text{ for } 2 \leq r \leq \alpha_i, \\ & i = 1, 2, \cdots, k\} \end{aligned}$$

we can prove that $\delta^\alpha u(j) = \frac{1}{h^\alpha} \int_{Q_j^\alpha} D_x^\alpha \tilde{u}(t_{1,1} + \cdots + t_{1,\alpha_1}, \cdots, t_{k,1} + \cdots + t_{k,\alpha_k}, j_{k+1} h_{k+1}, \cdots, j_m h_m, \rho) dt$.

If $jh \in Ns(\Omega)$, then $Q_j^\alpha \subset Ns(\Omega)$. Since $D_x^\alpha \tilde{u}(x, \rho)$ represents $D^\alpha U \in \mathscr{D}^{(s)'}(\Omega)$, we have

$$|D_x^\alpha \tilde{u}(x, \rho)| < H \quad \text{for } x \in Q_j^\alpha$$

Hence

$$\boxed{|\delta^\alpha u(j)| < H \quad \text{if } jh \in Ns(\Omega)}$$

(2) holds.

Prove (3)

We may assume that $H \cdot \max\{h_1 \cdots h_m\} \simeq 0$. By Lemma 1, $\mathscr{D}^{\Delta\prime}(\Omega) - c-$ near standardness is equivalent to $\mathscr{D}^{\Delta\prime}(\Omega) - d-$ near standardness, so we only to prove (4), (5).

$(4): \ < U, \phi > \simeq < u, \phi >_c$ for $\phi \in \mathscr{D}^{(s)}(\Omega)$

$(5): \ < u, \phi >_c \simeq 0$ for $\phi \in \mu_K^{(s)}, \ K \subset\subset \Omega$

u is $\mathscr{D}^{(s)\prime} - \diamondsuit$-near standardness, a nice representative of U.

Prove (4)

For $\phi \in \mathscr{D}^{(s)}(\Omega)$, we have

$< u, \phi >_c = \sum_{j\in J} \tilde{u}(jh, \rho) \int_{Q_j} \phi(x)\, dx$,

$\lim_{t\to 0} \int \tilde{u}(x, \rho)\phi(x)\, dx =< U, \phi >$ implies

$< U, \phi > \simeq \int_\Omega \tilde{u}(x, \rho)\phi(x)\, dx = \sum_{j\in J} \int_{Q_j} \tilde{u}(x, \rho)\phi(x) dx$.

So $< U, \phi > - < u, \phi >_c \simeq \sum_{j\in J} \int_{Q_j} (\tilde{u}(x, \rho) - \tilde{u}(jh, \rho))\phi(x) dx$.

Now

$$\tilde{u}(x, \rho) - \tilde{u}(jh, \rho) = \int_0^1 \frac{d}{dt} \tilde{u}(jh + t(x - jh), \rho) dt$$

$$= \sum_{i=1}^m (x_i - j_i h_i) \int_0^1 \frac{\partial \tilde{u}}{\partial x_i}(jh + t(x - jh), \rho) dt$$

Let $Q_j \cap Supp\phi \neq \emptyset$, then $Q_j \subset Ns(\Omega)$, and $x \in Q_j$ together with $0 \leq t \leq 1$ imply

$|x_i - j_i h_i| < h_i$ and $|\frac{\partial \tilde{u}}{\partial x_i}(jh + t(x - jh),\ \rho)| < H$.

Hence $|\tilde{u}(x, \rho) - \tilde{u}(jh, \rho)| \leq H \sum_{i=1}^m h_i \simeq 0$,

$$|\sum_{j\in J} \int_{Q_j} (\tilde{u}(x, \rho) - \tilde{u}(jh, \rho))\phi(x) dx| \leq H \sum_{i=1}^m h_i \int_\Omega |\phi(x)| dx \simeq 0.$$

(4) holds.

Consider $\tilde{u}(\cdot,t),\ 0<t<1$ as a family of continuous functionals on $\mathscr{D}_K^{(s)}$, defined by $<\tilde{u}(\cdot,t),\ \phi>=\int_K \tilde{u}(x,t)\phi(x)dx$ then it is bounded for any ϕ. $\mathscr{D}_K^{(s)}$ is a barrelled space, so it is equi-continuous, i.e. there is a neighborhood V of $0\in\mathscr{D}_K^{(s)}$, such that $|<\tilde{u}(\cdot,t),\ \phi>|<1$ for $\phi\in V$ and $0<t<1$ $\mathscr{D}_K^{(s)}=\bigcap\mathscr{D}_K^{(s),n}$ is a space topologized by countable norms $||\phi||_{(s),1}\leq||\phi||_{(s),2}\leq\cdots$, so there is an $n\in\mathbf{N}$ and a standard positive number ε such that $\{\phi\in\mathscr{D}_K^{(s)}\ /||\phi||_{(s),n}\leq\varepsilon\}\subset V$ Thus for any $\phi\in\mathscr{D}_K^{(s)}$ (and hence for $\phi\in{}^*\mathscr{D}_K^{(s)}$)

$$|<\tilde{u}(\cdot,t),\ \phi>|\leq\frac{1}{\varepsilon}||\phi||_{(s),n},\ 0<t<1$$

Since $\phi\in\mu_K^{(s)}$ implies $||\phi||_{(s),n}\simeq0$, we have $<\tilde{u}(\cdot,\rho),\ \phi>\simeq0$ for $\phi\in\mu_K^{(s)}$. A similar proof as for $\phi\in\mathscr{D}^{(s)}(\Omega)$ yields $<\tilde{u}(\cdot,\rho),\ \phi>\simeq<u,\phi>_c$ for $\phi\in\mu_K^{(s)}$

Theorem 2

$u \in G(\Omega)$ is locally absolutely summable

if and only if

u is $\mathscr{D}^{\Delta'} - \diamondsuit$-near standard and

invariant for any possible Δ and $\diamondsuit$.

$u \in G(\Omega)$ is said to be **locally absolutely summable**, if for any compact $K \subset \Omega$, $\sum_{jh \in {}^*K} |u(j)| h_1 \cdots h_m$ is finite.

An internal bijection B of J is called an **$\Omega-$infinitesimal transformation**, if $jh \in Ns(\Omega)$ implies $B(jh) \simeq jh$

Let u be $\mathscr{D}^{\Delta'}(\Omega) - \diamondsuit$-near standard, we say that u is **invariant** if for any $\Omega-$infinitesimal transformation B,

$$< u \circ B, \phi >_{\diamondsuit} \simeq < u, \phi >_{\diamondsuit}, \quad \text{for any } \phi \in \mathscr{D}^{\Delta}(\Omega)$$

For $u \in G(\Omega)$, the following are equivalent

1) u is locally absolutely summable.

2) u is $\mathscr{D}^{0\prime}(\Omega) - d-$near standard.

3) u is $\mathscr{D}^{0\prime}(\Omega) - c-$near standard.

4) for any internal set A in the monad of any $x_0 \in \Omega$,

$$\sum_{jh \in A} |u(j)| h_1 \cdots h_m \ \text{ is finite.}$$

Property in monad

If $u \in G(\Omega)$ is real and $\mathscr{D}^{\Delta'} - \diamondsuit$-near standard, and for some $x_0 \in \Omega$, there is an internal A in the monad of x_0 such that $u(j) > 0$ if $jh \in A$, and $\sum_{jh\in A} u(j)h_1 \cdots h_m$ is infinite, then there is an internal A' in the monad of x_0 such that $u(j) < 0$ if $jh \in A'$, and

$$\sum_{jh\in A'} u(j)h_1 \cdots h_m \text{ is infinite.}$$

$L^P_{sloc}(\Omega)$: $u \in G(\Omega)$ is locally $L^p -$ summable,

$p \geq 1$ be a standard real number.

If for any compact $K \subset \Omega$, $\sum_{jh \in {}^*K} |u(j)|^p h_1 \cdots h_m$ is finite.

$L^\infty_{sloc}(\Omega)$: $u \in G(\Omega)$ for any compact $K \subset \Omega$,

there is a standard real number $C(K)$ such that

$jh \in {}^*K$ implies $|u(j)| \leq C(K)$.

$l^1_{sloc}(\Omega)$: for any internal set A of J, if $j \in A$

implies $jh \in {}^*K$ for some compact $K \subset \Omega$,

and $(\#A)h_1 \cdots h_m \simeq 0$, then $\sum_{j \in A} |u(j)| h_1 \cdots h_m \simeq 0$.

Kinoshita's definition of the set $E(\Omega)$ of locally $S-$integrable $u \in G(\Omega)$ in the case of dimension 1 can be stated in any dimension as $u \in E(\Omega)$ iff for any compact $K \subset \Omega$ and any positive infinity ω,

$$\sum_{j \in A(u,K,\omega)} |u(j)| h_1 \cdots h_m \simeq 0$$

where

$$A(u, K, \omega) = \{\, j \;/\; jh \in {}^*K,\ |u(j)| \geq \omega\}$$

$$l^1_{sloc}(\Omega) = E(\Omega)$$

$L^p_{loc}(\Omega)$- the set of standard locally L^p-functions on Ω.

When $p = 1$, we have the following:

Theorem 3.
1) $u \in l^1_{sloc}(\Omega)$ is a nice representative of a $\tilde{u} \in L^1_{loc}(\Omega)$.

2) For any $\tilde{u} \in L^1_{loc}(\Omega)$, there is a nice representative $u \in l^1_{sloc}(\Omega)$ of $\tilde{u}$.

Remark: $L^1_{sloc}(\Omega)$ is the nonstandard representation of Radon measure on Ω,

$$l^1_{sloc}(\Omega) \text{ strictly} \subset L^1_{sloc}(\Omega).$$

Theorem 4.
Assume $p > 1$.

1). $u \in L^p_{sloc}(\Omega)$ is a nice representative of a function $\tilde{u} \in L^p_{loc}(\Omega)$.

2). For $\tilde{u} \in L^p_{loc}(\Omega)$, there is $u \in L^p_{sloc}(\Omega)$ representing $\tilde{u}$.

$W^{p,n}_{sloc}(\Omega)$:the $S-$local Sobolev space consisting of $u \in G(\Omega)$ and for any index $\alpha = (\alpha_1, \cdots, \alpha_m)$ with $0 \leq |\alpha| \leq n$ $\delta^\alpha u \in L^p_{sloc}(\Omega)$

Sobolev imbedding theorem:

If $p > 1$, and $u \in W^{p,n}_{sloc}(\Omega)$, then for $\sigma \in \mathbf{N}$ with $\sigma < n - \frac{m}{p}$,

$$u \text{ represents } \tilde{u} \in C^\sigma(\Omega) \cap W^{p,n}_{loc}(\Omega)$$

where $W^{p,n}_{loc}(\Omega)$ is the standard local Sobolev space.

Theorem 5. If $m = 1$ and $u \in W^{p,n}_{sloc}(\Omega)$, where $n \geq 1$ and $1 < p \leq \infty$, then u is a nice representative of $\tilde{u} \in C^{n-1}(\Omega)$ such that $\tilde{u}^{(n-1)}$ is locally Hölder continuous with exponent $1 - \frac{1}{p}$ (for $p = \infty$, $\tilde{u}^{(n-1)}$ is locally Lipschitz continuous).

Remark: For $p = 1$, Theorem 5 not true.

$$u(j) = \begin{cases} 0 & j \leq 0 \\ 1 & j > 0 \end{cases} \qquad \delta u(j) = \begin{cases} \frac{1}{h} & j = 0 \\ 0 & j \neq 0 \end{cases}$$

then $u \in W^{1,1}_{sloc}(\mathbb{R})$, but u is a representative of Heaviside function which not continuous.

Dedekind completion ${}^{\#}\mathbb{R}_{+}$ of ${}^{*}\mathbb{R}_{+} = \{t \in {}^{*}\mathbb{R} | t \geq 0\}$.
Wattenberg and Banghe Li and Jijiang Zhang have studied.
Li,Zhang proved there are lots of elements U in ${}^{\#}\mathbb{R}_{+} \bigcup (\infty)$ with

$$U + U = U, \quad U \cdot U = U$$

$\Longleftrightarrow$

1). $a \in U$ and $a > b \in {}^{*}\mathbb{R}_{+}$ imply $b \in U$ (i.e. U is ${}^{*}\mathbb{R}_{+}$ or a Dedekind cut of ${}^{*}\mathbb{R}_{+}$, hence an element of ${}^{\#}\mathbb{R}_{+} \cup \{\infty\}$).
2). $a \in U$ and $b \in U$ imply $ab \in U$ (i.e. $U \cdot U = U$)
3). $2 \in U$

Let $\mathcal{U} = \{U | U + U = U,\ U \cdot U = U,\ U \subset {}^{\#}\mathbb{R}_+ \cup \{\infty\}\}$
$\mathcal{U}$ has a minimal element

$$U_0 = \{t \in {}^*\mathbb{R}_+ \ / \ t \ \text{ is finite}\}$$

a maximal element ${}^*\mathbb{R}_+$.
For any $U \in \mathcal{U}$,
$\mathbb{C}'_U = \{z \in {}^*\mathbb{C} | |z| \in U\}$ is algebra over $\mathbb{C}$.

$$\mathbb{C}'_{{}^*\mathbb{R}_+} = {}^*\mathbb{C}$$

$$\mathbb{C}'_{U_0} = \{z \in {}^*\mathbb{C} | |z| \text{ is finite}\}$$

For any infinite $H \in {}^*\mathbb{R}_+$

$$\mathcal{U}_H = \{U \in \mathcal{U} | U < H\}$$

Various algebras of NGF

For any $U \in \mathcal{U}$ and $n \in \mathbf{N}$, we call **$u \in G(\Omega)$ a $U-$GNF of order n**, if

$$\delta^\alpha u(j) \in \mathbb{C}'_U, \quad \text{for all } j \text{ with } jh \in Ns(\Omega), \ \text{and all } \alpha \in \mathbf{N}^m \text{ with } |\alpha| \leq n$$

$G^n_U(\Omega)$ is an algebra over $\mathbb{C}$.

U-GNF:

$$G^\infty_U(\Omega) = \bigcap_{n \in \mathbf{N}} G^n_U(\Omega)$$

$$G^\infty_{^*\mathbb{R}_+}(\Omega) = G(\Omega)$$

Quotient algebra of NGF

For $U \in \mathcal{U}$,

$$U^{-1} = \{x \in {}^*\mathbb{R}_+ \mid x = 0 \text{ or } x^{-1} > a \text{ for any } a \in U\}$$

$$\mathbb{C}'_{U^{-1}} = \{z \in {}^*\mathbb{C} \mid |z| \in U^{-1}\}$$

Then $\mathbb{C}_U = \mathbb{C}'_U / \mathbb{C}'_{U^{-1}}$ is a field.

$\mathbb{C}_{{}^*\mathbb{R}_+} = {}^*\mathbb{C}, \qquad \mathbb{C}_{U_0} = \mathbb{C}.$

$\boxed{\widetilde{G}^n_U(\Omega) = G^n_U(\Omega)/G^n_{U^{-1}}(\Omega)}$ algebra over $\mathbb{C}_U$.

$\widetilde{G}^n_U(\Omega)$ share most properties of $G^n_U(\Omega)$;
$\boxed{\widetilde{G}^n_U(\Omega) \text{ similar to Colombeau's NGF.}}$

Thanks!