On pseudo–intersections and condensers

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Basic remarks

- we will consider ultrafilters on $P(\mathbb{N})$ and on Boolean subalgebras of $P(\mathbb{N})$;
- if $\mathfrak{A}$ is a subalgebra of $P(\mathbb{N})$, then every ultrafilter on $\mathfrak{A}$ is generated by a filter on $P(\mathbb{N})$.

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Preliminaries

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Pseudo–intersection

Definition

We say that $P \subseteq \mathbb{N}$ is a *pseudo–intersection* of a filter $\mathcal{F}$ if $P \setminus F$ is finite ($P \subseteq^* F$) for every $F \in \mathcal{F}$.

Definition

The *pseudo–intersection number* $\mathfrak{p}$ is a minimal cardinality of a base of a filter without a pseudo–intersection.

- $\aleph_0 < \mathfrak{p} \leq \mathfrak{c}$;
- $\mathfrak{p} = \mathfrak{c}$ under MA;
- $\mathfrak{p} = \aleph_1 < \mathfrak{c}$ in Sacks model (and many others).

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Asymptotic density

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The *asymptotic density* of a set $A \subseteq \mathbb{N}$ is defined as

$$d(A) = \lim_{n\to\infty} \frac{|A \cap [1, \ldots, n]|}{n},$$

provided this limit exists. The family $\{A\colon d(A) = 1\}$ forms a filter on $\mathbb{N}$.

Definition

For an infinite $B = \{b_1 < b_2 < b_3 < \ldots\} \subseteq \mathbb{N}$ define the *relative density of A in B* by

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Remarks

- Every pseudo–intersection is a condenser;
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Another approach to condensation

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We say that a filter $\mathcal{F}$ on $\mathbb{N}$ is *condensed* if there is a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$d(f[F]) = 1$$

for every $F \in \mathcal{F}$.

Remarks

- if $\mathcal{F}$ has a condenser, then it is condensed;
- if $\mathcal{F}$ is condensed, then it is *feeble*, i.e. there is a finite–to–one function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f[F]$ is co–finite for every $F \in \mathcal{F}$.

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Special Boolean algebras

Remarks

- it is quite easy to construct a subalgebra $\mathfrak{A}$ of $P(\mathbb{N})$ such that each ultrafilter on $\mathfrak{A}$ does not have pseudo–intersection ...
- ... even if this $\mathfrak{A}$ has to be small (i.e. does not contain uncountable independent family).

Loosely speaking

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Problem

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Can we construct a subalgebra $\mathfrak{A}$ of $P(\mathbb{N})$ such that

- no ultrafilter on $\mathfrak{A}$ has a pseudo–intersection;
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Answer - partial and easy

- assume CH;
- suppose no ultrafilter on $\mathfrak{A}$ has a pseudo–intersection;
- then, it has to be $2^{\mathfrak{c}}$ ultrafilters on $\mathfrak{A}$;
- thus, there is no enough bijections to ensure that every ultrafilter is condensed;
- conclusion: under CH there is no such an algebra.

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What about the general result?

Can we construct such an algebra in other models of ZFC?

Definition

A condensation number $\mathfrak{k}$ is a minimal cardinality of a base of filter on $\mathbb{N}$ without a condenser.

Facts

- $\mathfrak{p} \leq \mathfrak{k}$;
- $\mathfrak{k} \leq \mathfrak{b}$ (a consequence of P. Simon's result);
- consistently $\mathfrak{k} < \mathfrak{b}$ (a consequence of M. Hrusak's result).

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Construction of demanded algebra

Result (?)

If $\mathfrak{k} > \mathfrak{h}$, then there is a Boolean algebra $\mathfrak{A}$ such that no ultrafilter on $\mathfrak{A}$ has a pseudo–intersection, but each ultrafilter is condensed.

Sketch of proof

- consider a *base matrix tree* $\mathcal{T}$ (Balcar, Simon, Pelant);
- let $\mathfrak{B}$ be a Boolean algebra generated by $\mathcal{T}$;
- there are two types of ultrafilters on $\mathfrak{B}$: *branches* and *knots*;
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Theorem

In the same manner it can be proved that if $\mathfrak{b} > \mathfrak{h}$ (eg. in Hechler model), then there is a Boolean algebra $\mathfrak{A}$ such that

- no ultrafilter on $\mathfrak{A}$ has a pseudo–intersection;
- every ultrafilter on $\mathfrak{A}$ is feeble.

Dualities

Boolean algebra $\mathfrak{A}$

compact space $K = ult(\mathfrak{A})$

Banach space $X = C(K)$

dual Banach space $M = C^*(K) = M(K)$

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Easy application

Fact

Let p be an ultrafilter on a Boolean algebra $\mathfrak{A} \subseteq P(\mathbb{N})$. The following conditions are equivalent:

- p has a pseudo–intersection $\{n_1, n_2, \ldots\}$;
- $\lim n_k = p$ in $ult(\mathfrak{A})$.

Gelfand–Phillips Property and Mazur property

Definition

A Banach space X has a *Mazur property* if every *weak**–sequentially continuous $x^{**} \in X^{**}$ is continuous.

A bounded subset A of a Banach space X is said to be *limited* if

$$\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0$$

for every weak*–null sequence $x_n^* \in X^*$.

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Mazur Property vs. Gelfand-Phillips Property

Problem

It is known that there are Banach spaces with a Gelfand–Phillips property but without a Mazur property. **Does Mazur property imply Gelfand–Phillips property?**

Fact

If $\mathfrak{A}$ is a Boolean algebra such that no ultrafilter on $\mathfrak{A}$ has a pseudo–intersection but each ultrafilter on $\mathfrak{A}$ has a condenser, then $C(ult(\mathfrak{A}))$ is an example of a Mazur space which does not possess the Gelfand–Phillips property.

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The end

Slides and a preprint concerning the subject will be available on

http://www.math.uni.wroc.pl/~pborod