ULTRAFILTERS, CLOSURE OPERATORS AND THE AXIOM OF CHOICE

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It is well known that, in a topological space, the open sets can be characterized using filter convergence. In ZF (*Zermelo-Fraenkel set theory without the Axiom of Choice*), we cannot replace filters by ultrafilters. It can be proven that the ultrafilter convergence determines the open sets for every topological space if and only if the *Ultrafilter Theorem* holds. More, we can also prove that the Ultrafilter Theorem is equivalent to the fact that $u_X = k_X$ for every topological space X, where k is the usual Kuratowski closure operator and u is the ultrafilter closure, with

$$u_X(A) := \{x \in X : (\exists \mathcal{U} \text{ ultrafilter in } X)[\mathcal{U} \text{ converges to } x \text{ and } A \in \mathcal{U}]\}.$$

These facts arise two different questions that we will try to answer in this talk.

(1) Under which set theoretic conditions the equality $u = k$ is true in some subclasses of topological spaces, such as first countable spaces, second countable metric spaces or $\{\mathbb{R}\}$.

(2) Is there any topological space X for which $u_X \neq k_X$, but the open sets are characterized by the ultrafilter convergence? Making a parallel with sequential convergence case, this is the correspondent to find a sequential space which is not a Fréchet space.

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