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Regularity Problems for Young Differential Equations

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Chapter 1

Introduction

Young Integration firstly introduced by L.C. Young in [You36] had a key role in the development of the modern Rough Paths' theory. Nowadays it is a tool in the study of stochastic differential equations driven by continuous processes other than the Brownian motion such as the fractional Brownian motion with Hölder regularity $> \frac{1}{2}$. in this work we study the regularity of solutions to Young differential equations. In Chapter two we introduce the theory of Young integration for function f, g from a compact interval $[0, T] \subset \mathbb{R}$ with value in some Banach spaces A, B with some Hölder regularity. In Chapter three we focus our attention on the differential problem:

$$y(t) = y_0 + \int_0^t f(y(s))\ dx(s),\ t \in [0, T] \tag{1.1}$$

showing that under suitable condition on f, x there exists a solution under the Young theory. In the last part of this chapter we study the Itō map, defined as the map that associates the initial datum y_0 and the noise x to the solution of the problem (1.1) for a vector field $f \in \mathcal{C}^{1,\gamma}(A, \mathcal{L}(A, B))$. Our aim is to study the differentiability of the Itō map with respect to the initial datum and the regularity of its Fréchet derivative. The fourth chapter contains an approach to study the sharpness of the regularity of the Fréchet derivative of the Itō map.
For the firsts chapters we followed the article [Lej10] and in showing the Fréchet derivative with respect to the initial datum of the Itō map we used the approach of [CL18], via Omega Lemma. The results in Chapter four are an original computation suggested by R. Züst and E. Stepanov.

1.1 Notation

Here is a list of frequently used notations:

- $\mathcal{L}(A, B)$ is the space of bounded linear functional from a Banach space A to a Banach space B,
- $\mathcal{C}^\alpha([0, T], A)$ is the space of α-Hölder functions from $[0, T]$ to a Banach space A,

- $\mathcal{C}^{1,\alpha}([0,T],A)$ is the space of functions from $[0,T]$ to a Banach space that admit α-Hölder Fréchet derivative,

- $\mathcal{C}_b^{1,\alpha}(A,B)$ is the space of functions from a Banach space$A$ to a Banach space B that admit α-Hölder Fréchet derivative which is uniformly bounded.

To improve readibility we use for function $f\colon [0,T] \to X$, where X is a set, the equivalent notation $f(t) = f_t$, whenever the second choice is needed to seek a lighter reading. For the same reasons we decided to use the symbol $|\cdot|$ whenever this is referred to an operator norm. So it will be frequent to read, whenever $f\colon [0,T] \to A$, where A is a Banach space, the symbol $|f(t)|$ instead of $\|f(t)\|_A$.

Chapter 2

Young's Integration

In this chapter we will introduce the theory of Young's Integration. Let us fix a compact interval $[a,b] \subset \mathbb{R}$ and let us consider two continuous function $f, g : [a,b] \to \mathbb{R}$. Our aim is to define the integral

$$\int_a^b f \; dg = \int_a^b f(t) \; dg(t)$$

for function of low regularity. Generally speaking, such an integral can represent many phenomena:

- if g(t) is the cumulative distribution function of a random variable X and f(t)=t for every t, then one can interpret the integral as the expectation of the random variable X;
- if g(t) is the position of a particle at a certain time t and f(t) is the force applied to that particle, then one can interpret the integral as the work made by the force in the interval $[a,b]$.

Note that in the first case, there exists a robust definition of the integral that is the Lebesgue-Stieltjes construction, since F is a monotone positive function. Moreover, if g is a differentiable function, namely $g \in \mathcal{C}^1([a,b],\mathbb{R})$, then the Riemann construction gives the following equality:

$$\int_a^b f(t) \; dg(t) = \int_a^b f(t) g'(t) \; dt.$$

Our goal is to extend the integral to function α-Hölder and β-Hölder continuous, such that $\alpha + \beta > 1$.
The constraint introduced is sharp. In fact, if one consider the trajectories of a Brownian Motion $(B_t)_{t \in [0,T]}$, a typical result of stochastic analysis states that the trajectories are α-Hölder, for every $\alpha < \frac{1}{2}$. Whenever we define a certain integral we expect that it satisfies a "stability property" under different choices of the Riemann sums. Let us consider a sequence of partitions $\Pi_n = \{a = t_0 < t_1 < ... < t_n = b\}$ of the interval $[a,b]$ such that

$$|\Pi_n| = \sup_{\substack{t_i \in \Pi_n \\ a = t_0 < t_1 < \cdots < t_n = b}} |t_{i+1} - t_i| \to 0$$

as $n \to \infty$ and $\Pi_n \subset \Pi_{n+1}$, then we want that the following equality holds true:

$$\int_a^b f(t)\, dg(t) = \lim_{n\to\infty} \sum_{\substack{t_i \in \Pi_n \\ \xi_i \in [t_i, t_{i+1}]}} f(\xi_i)(g(t_{i+1}) - g(t_i))$$

for every family of nested partitions $(\Pi_n)_{n\in\mathbb{N}}$ and for every choices of $\xi_i \in [t_i, t_{i+1}]$. Unfortunately this may not be true for f, g Hölder continuous with $\alpha + \beta < 1$. This fact is shown by the following:

Proposition 1. *Let $(B_t)_{t\in[0,T]}$ be a Brownian motion and let $t \in [0,T]$ then the integral*

$$\int_0^t B_s\, dB_s$$

depends on the choice of the point $\xi_i \in [t_i, t_{i+1}]$. In particular, let $\Pi_n = \{0 = t_0 < t_1 < ... < t_n = t\}$, then:

$$I_t = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} B_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

$$BI_t = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} B_{t_{i+1}} \left(B_{t_{i+1}} - B_{t_i}\right) = \frac{1}{2}B_t^2 + \frac{1}{2}t$$

$$SI_t = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \frac{1}{2}(B_{t_{i+1}} + B_{t_i}) \left(B_{t_{i+1}} - B_{t_i}\right) = \frac{1}{2}B_t^2$$

Proof. It is useful to rewrite I as:

$$I_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \frac{1}{2}\left(B_{t_{i+1}} - B_{t_i}\right)^2\right\}.$$

Now the first term is telescoping, so that:

$$\sum_{i=0}^{n-1} \frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) = \frac{1}{2}B_{t_n}^2 - B_{t_0}^2 = \frac{1}{2}B_t^2 - B_0^2 = \frac{1}{2}B_t^2.$$

The other therm converges to the quadratic variation of a Brownian motion, that is t for every $t \in [0,T]$, yielding:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} B_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \frac{1}{2}\left(B_{t_{i+1}} - B_{t_i}\right)^2\right\} = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

As previously done, let us write:

$$BI_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{t_{i+1}} \left(B_{t_{i+1}} - B_{t_i}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \frac{1}{2}\left(B_{t_{i+1}} - B_{t_i}\right)^2\right\}.$$

Then as we have seen before, the first term converges to $\frac{1}{2}B_t^2$ and the second term to t, yielding:

$$\begin{aligned} BI_t &= \lim_{n\to\infty}\sum_{i=0}^{n-1} B_{t_{i+1}}\left(B_{t_{i+1}} - B_{t_i}\right) \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \frac{1}{2}\left(B_{t_{i+1}} - B_{t_i}\right)^2\right\} \\ &= \frac{1}{2}B_t^2 + \frac{1}{2}t. \end{aligned}$$

Finally note that $SI_t = \frac{1}{2}(I_t + BI_t)$ yelding the thesis. □

In general I is called Itō integral, while BI is called Backward Itō integral and SI is called Stratonovich integral.

2.1 Hölder spaces

We will introduce some basic facts about Hölder space. To focus on the theory of Young's integral some proofs will be left to appendix. Let us start with defining the space of continuous functions. We will denote by $\mathcal{C}^0([a,b],A)$ the space of continuous function from $[a,b]$ to A, where A is a Banach space, endowed with the norm:

$$\|f\|_\infty = \sup_{s\in[a,b]} |f(s)|.$$

Theorem 1. *The space $\mathcal{C}^0([a,b],A)$ endowed with the norm $\|.\|_\infty$ is a Banach space.*

From now on we will work with Hölder continuous functions. Let us introduce a definition:

Definition 1. Let $[a,b] \subset \mathbb{R}$, and A be a Banach space A, and $\alpha \in (0,1)$. A function $f : [a,b] \to A$ is said to be α-Hölder continuous if it satisfies the following estimate:

$$|f(x) - f(y)| \leq C|x-y|^\alpha \quad (x,y) \in [a,b]$$

for some constant $C \geq 0$.
Moreover for such a function it is well defined the Hölder norm as:

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{x,y\in[a,b] \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

The constraint $\alpha \in (0,1)$ is explained by the following:

Lemma 1. *If for some $\alpha > 1$ the following inequality holds true:*

$$|f(t) - f(s)| \leq c|t-s|^\alpha.$$

Then f is constant. Moreover, whenever $|f(t) - f(s)| = o(|t-s|)$ uniformly as $|t-s| \to 0$ then f is constant.

Proof. By continuity of f for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - s| \leq \delta$ implies $|f(t) - f(s)| \leq \varepsilon |t - s|$. Let $[c, d] \subset [a, b]$ be a compact interval and consider a partition $c = t_0 < t_1 < \cdots < t_n = d$ with $t_{i+1} - t_i \leq \delta$, then $|f(t_{i+1}) - f(t_i)| \leq \varepsilon |t_{i+1} - t_i|$ so that:

$$|f(d) - f(c)| \leq \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|\, | \leq \sum_{i=0}^{n-1} \varepsilon |t_{i+1} - t_i| = \varepsilon(d - c).$$

Sending ε to 0, $f(d) = f(c)$ for every compact $[c, d] \subset [a, b]$, i.e. f is constant. □

From now on we will denote by $\mathcal{C}^\alpha([a, b], A)$ the space of α-Hölder functions from $[a, b]$ to A, where A is a Banach space. For our scope it will be useful to have some properties of completeness for the space $\mathcal{C}^{1,\alpha}([a, b], A)$, that is the space of differentiable function from $[a, b]$ to a Banach space A such that the derivative is Hölder continuous.

Theorem 2. *The space $\mathcal{C}^\alpha([a, b], A)$ endowed with the norm $\|.\|_\alpha$ is a Banach space.*

Let us denote by $\mathcal{C}^\infty([a, b], A)$ the space of infinitely many differentiable functions. In particular note that $\mathcal{C}^\infty([a, b], A) \subset \mathcal{C}^\alpha([a, b], A)$ for every $\alpha \in (0, 1)$,but it is not true that $\mathcal{C}^\infty([a, b], A)$ is dense in $\mathcal{C}^\alpha([a, b], A)$.

Theorem 3. *The closure of $\mathcal{C}^\infty([a, b], A)$ in $\mathcal{C}^\alpha([a, b], A)$ is the subset $\mathcal{C}_0^\alpha([a, b], A)$ defined by*

$$\mathcal{C}_0^\alpha([a, b], A) = \{f : |f(t) - f(s)| = o(|t - s|) \mathit{uniformly\ as} |t - s| \to 0\}.$$

However the following inclusions hold true:

$$\mathcal{C}^{\alpha+\varepsilon}([a, b], A) \subset \mathcal{C}_0^\alpha([a, b], A) \subset \mathcal{C}^\alpha([a, b], A),$$

for any $\varepsilon > 0$.

Proposition 2. *Let $\alpha, \beta \in (0, 1)$ with if $\beta > \alpha$. Then $\mathcal{C}^\beta([a, b], A) \subset \mathcal{C}^\alpha([a, b], A)$.*

Proof. Given $f \in \mathcal{C}^\beta([a, b], A)$ we have:

$$\begin{aligned}\sup_{\substack{x,y\in[a,b]\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} &= \sup_{\substack{x,y\in[a,b]\\ x\neq y}} \frac{|f(x) - f(y)|^{\frac{\alpha}{\beta}}}{|x - y|^{\beta\frac{\alpha}{\beta}}} \cdot \sup_{\substack{x,y\in[a,b]\\ x\neq y}} |f(x) - f(y)|^{\frac{\beta-\alpha}{\beta}} \\ &\leq \|f\|_\beta^{\frac{\alpha}{\beta}} \left(2 \|f\|_\infty\right)^{\frac{\beta-\alpha}{\beta}}\end{aligned}$$

□

For our scope it is importan that the space $\mathcal{C}^{1,\alpha}([a, b], A)$ is a Banach space. The following theorem will be proven in the appendix.

Theorem 4. *The space $\mathcal{C}^{1,\alpha}([a, b], A)$ endowed with the norm $\|.\|_\alpha$ is a Bancach space.*

2.2 Construction of Young Integral

In this section we will construct the Young Integral by proving a more general result called Sewing Lemma. Let us recall the following proposition, that will be useful in advance.

Proposition 3. *Let $(x_n)_{n\in\mathbb{N}} \subset A$, where A is a Banach space, be a sequence such that*

$$\sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty$$

then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. Let $\bar{n} \in \mathbb{N}$ such that

$$\sum_{n=\bar{n}}^{\infty} |x_{n+1} - x_n| < \varepsilon.$$

The for every $m \geq n \geq \bar{n}$:

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n| \\ &\leq \sum_{s=n}^{m-1} |x_{s+1} - x_s| \\ &\leq \sum_{s=\bar{n}}^{\infty} |x_{s+1} - x_s| < \varepsilon. \end{aligned}$$

Where in the last inequality we used that the latter is the tail of a convergent series. □

Now we are ready to state the key result on the construction of the Young Integral. For its proof we follow [FdLPM08].

Lemma 2 (Sewing Lemma)**.** *Let $\gamma > 1$, $\mu(a, b)$ be a continuous function defined for $0 \leq a \leq b \leq T$ to A, where A is a Banach space, satisfying the relation:*

$$|\mu(a, b) - \mu(a, c) - \mu(c, b)| \leq K |b - a|^{\gamma}$$

for every $c \in [a, b]$ and for some constant $K \in \mathbb{R}$. Then there exists a function $\varphi(t)$ on $[0, T]$ to A, unique up to an additive constant, such that:

$$|\varphi(b) - \varphi(a) - \mu(a, b)| \leq c(\gamma) K |b - a|^{\gamma}$$

where $c(\gamma) = \frac{2^{\gamma}}{2^{\gamma}-2}$.

Proof. **Existence.** Let us set $\mu'(a, b) = \mu(a, c) + \mu(c, b)$ where $c = \frac{a+b}{2}$ and define $\mu^{(n+1)} = \mu^{(n)'}$, where we set $\mu^{(0)} = \mu$. For $n \geq 0$ we obtain that:

$$\left|\mu^{(n)}(a, b) - \mu^{(n+1)}(a, b)\right| \leq K \frac{|b - a|^{\gamma}}{2^{(\gamma-1)n}}$$

We will prove this fact by induction. For $n = 0$ the thesis holds true, in fact:

$$\left|\mu^{(0)}(a,b) - \mu^{(1)}(a,b)\right| = |\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq K\,|b-a|^\gamma$$

by hypothesis. Then let us suppose that the thesis holds true for $n-1$ and let us prove it for $k = n$. We have that:

$$\begin{aligned}\left|\mu^{(n)}(a,b) - \mu^{(n+1)}(a,b)\right| &= \left|\mu^{(n)}(a,b) - \mu^{(n)}(a,c) - \mu^{(n)}(b,c)\right| \\ &= \left|\mu^{(n-1)}(a,c) + \mu^{(n-1)}(c,b) - \mu^{(n)}(a,c) - \mu^{(n)}(c,b)\right| \\ &\leq \left|\mu^{(n-1)}(a,c) - \mu^{(n)}(a,c)\right| + \left|\mu^{(n-1)}(c,b) - \mu^{(n)}(c,b)\right| \\ &\leq K2^{n-1}\left(\frac{|c-a|}{2^{n-1}}\right)^\gamma + K2^{n-1}\left(\frac{|b-c|}{2^{n-1}}\right)^\gamma \\ &\leq K2^{n-1}\left(\frac{|b-a|}{2^{n}}\right)^\gamma + K2^{n-1}\left(\frac{|b-a|}{2^{n}}\right)^\gamma = K\frac{|b-a|^\gamma}{2^{(\gamma-1)n}}.\end{aligned}$$

Moreover, since $\gamma - 1 > 0$, $(\mu^{(n)}(a,b))_{n\in\mathbb{N}}$ is a Cauchy sequence, in fact:

$$\sum_{n=0}^{\infty}\left|\mu^{(n+1)}(a,b) - \mu^{(n)}(a,b)\right| \leq K\sum_{n=0}^{\infty}\frac{|b-a|^\gamma}{2^{(\gamma-1)n}} < \infty.$$

So by Proposition 3 $(\mu^{(n)}(a,b))_{n\in\mathbb{N}}$ is a Cauchy sequence. We let $u(a,b) = \lim_{n\to\infty}\mu^{(n)}(a,b)$. The function $u(a,b)$ is midpoint additive, in other words $u(a,b) = u(a,c) + u(c,b)$.

Uniqueness. By its definition we have that:

$$|u(a,b) - \mu(a,b)| \leq c(\gamma)K\,|b-a|^\gamma\,.$$

This follows from the inequality:

$$\left|\mu^{(n)}(a,b) - \mu(a,b)\right| \leq \sum_{s=0}^{n-1}\left|\mu^{(s+1)}(a,b) - \mu^{(s)}(a,b)\right|$$

taking n to ∞ we have that:

$$\sum_{s=0}^{\infty}\left|\mu^{(s+1)}(a,b) - \mu^{(s)}(a,b)\right| \leq c(\gamma)K\,|b-a|^\gamma$$

Let $v(a,b)$ be another midpoint additive function, such that

$$|v(a,b) - \mu(a,b)| \leq c(\gamma)K\,|b-a|^\gamma\,.$$

We prove by induction that:

$$|v(a,b) - u(a,b)| \leq C2^n\left(\frac{|b-a|}{2^n}\right)^\gamma$$

for some constant $C \in\geq 0$. Firstly note that:

$$\begin{aligned}|v(a,b) - u(a,b)|_A &\leq \|v(a,b) - \mu(a,b)\| + \|\mu(a,b) - u(a,b)\| \\ &\leq 2c(\gamma)K\,|b-a|^\gamma = C\,|b-a|^\gamma\end{aligned}$$

where we put $C = 2c(\gamma)K$.
For $n = 1$ the thesis is true since, by midpoint additivity, we have that:

$$\begin{aligned} |v(a,b) - u(a,b)| &= |v(a,c) - v(c,b) - u(a,c) - u(c,b)| \\ &\leq |v(a,c) - u(a,c)| + |v(c,b) - u(c,b)| \\ &\leq C\,|c-a|^\gamma + C\,|b-c|^\gamma = 2C\left(\frac{|b-a|}{2}\right)^\gamma . \end{aligned}$$

Then let us suppose the thesis true for $k \leq n-1$ and let us set $k = n$, then we have:

$$\begin{aligned} |v(a,b) - u(a,b)| &= |v(a,c) - v(c,b) - u(a,c) - u(c,b)| \\ &\leq |v(a,c) - u(a,c)| + |v(c,b) - u(c,b)| \\ &\leq 2^{n-1}C\left(\frac{|c-a|}{2^{n-1}}\right)^\gamma + 2^{n-1}C\left(\frac{|b-c|}{2^{n-1}}\right)^\gamma \\ &= C2^n\left(\frac{|b-a|}{2^n}\right)^\gamma . \end{aligned}$$

Then $|v(a,b) - u(a,b)| \leq C2^n\left(\frac{|b-a|}{2^n}\right)^\gamma \to 0$ as $n \to \infty$, so that $v = u$.

Continuity of u. Note that by its definition:

$$\mu^{(n)}(a,b) = \sum_{i=1}^{2^n} \mu\left(a + \frac{(i-1)(b-a)}{2^n}, a + \frac{i(b-a)}{2^n}\right).$$

In fact, by induction the equality holds true for $n = 0$ then, supposing the thesis holds for $k \leq n$, for $k = n+1$ we have:

$$\begin{aligned} \mu^{(n+1)}(a,b) &= \mu^{(n)}\left(a, \frac{b+a}{2}\right) + \mu^{(n)}\left(\frac{b+a}{2}, b\right) \\ &= \sum_{i=1}^{2^n} \mu\left(a + \frac{(i-1)(\frac{b+a}{2} - a)}{2^n}, a + \frac{i(\frac{b+a}{2} - a)}{2^n}\right) \\ &+ \sum_{i=1}^{2^n} \mu\left(\frac{b+a}{2} + \frac{(i-1)(b - \frac{b+a}{2})}{2^n}, \frac{b+a}{2} + \frac{i(b - \frac{b+a}{2})}{2^n}\right) \\ &= \sum_{i=1}^{2^n} \mu\left(a + \frac{(i-1)(b-a)}{2^{n+1}}, a + \frac{i(b-a)}{2^{n+1}}\right) \\ &+ \sum_{i=1}^{2^n} \mu\left(\frac{(b+a)2^n + (i-1)(b-a)}{2^{n+1}}, \frac{(b+a)2^n + i(b-a)}{2^{n+1}}\right). \end{aligned}$$

Let us study the second sum, setting $j = i + 2^n$ we have:

$$\sum_{j=2^n+1}^{2^{n+1}} \mu\left(\frac{(b+a)2^n + (j-2^n-1)(b-a)}{2^{n+1}}, \frac{(b+a)2^n + (j-2^n)(b-a)}{2^{n+1}}\right) =$$

$$= \sum_{j=2^n+1}^{2^{n+1}} \mu\left(\frac{(b+a)2^n + (j-1)(b-a) - 2^n(b-a)}{2^{n+1}}, \frac{(b+a)2^n + j(b-a) - 2^n(b-a)}{2^{n+1}}\right)$$

$$= \sum_{j=2^n}^{2^{n+1}} \mu\left(a + \frac{(j-1)(b-a)}{2^n}, a + \frac{j(b-a)}{2^n}\right).$$

And the equality holds true adding the two pieces. Now let us consider the quantity:

$$\sup_{a,b\in[0,T]} \left|u(a,b) - \mu^{(n)}(a,b)\right| = \sup_{a,b\in[0,T]} \left|\sum_{i=2^n+1}^{\infty} \mu(a + \frac{(i-1)(b-a)}{2^n}, a + \frac{i(b-a)}{2^n})\right|.$$

And the term on the right hand side goes to 0 as $n \to \infty$ since it is the tail of a convergent series. Since the series that defines u converges uniformly and μ is continuous by hypothesis, then u is continuous.

Additivity of u We prove that for every $c \in [a,b]$

$$u(a,b) - u(a,c) - u(c,b) = 0.$$

Let $k \geq 3$ be an integer and let:

$$w(a,b) = \sum_{i=0}^{k-1} u(t_i, t_{i+1})$$

where $t_i = a + i\frac{(b-a)}{k}$. Then also w is midpoint additive, and by induction and using the assumption on μ

$$|\mu(a,b) - w(a,b)| \leq CKc(\gamma)\,|b-a|^\gamma$$

hence $w = u$.

At this point we have proved that u is additive on rational points, moreover let us consider $(c_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $c_n \to c$ as $n \to \infty$, then:

$$u(a,b) = u(a,c_n) + u(c_n,b),$$

and taking the limit, by the continuity of u:

$$u(a,b) = u(a,c) + u(c,b)$$

for every $0 \leq a \leq b \leq c \leq T$.

So u is additive and finally we put $\varphi(t) = u(0,t)$. □

Proposition 4. *Let* $\Pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ *be a finite partition of* $[a,b]$*, let* μ, φ *as in the Sewing Lemma, then*

$$\lim_{|\Pi_n|\to 0} \sum_{t_i \in \Pi_n} \mu(t_i, t_{i+1}) = \varphi(b) - \varphi(a).$$

Proof. We have that:

$$\begin{aligned}
\left|\varphi(b) - \varphi(a) - \sum_{t_i \in \Pi_n} \mu(t_i, t_{i+1})\right| &= \left|\sum_{t_i \in \Pi_n} \varphi(t_{i+1}) - \varphi(t_i) - \mu(t_i, t_{i+1})\right| \\
&\leq \sum_{t_i \in \Pi_n} c(\gamma) K (t_{i+1} - t_i)^\gamma \\
&\leq c(\gamma) K \left|\Pi_n\right|^{\gamma - 1} \sum_{t_i \in \Pi_n} (t_{i+1} - t_i) \\
&\leq c(\gamma) K \left|\Pi_n\right|^{\gamma - 1} (b - a)
\end{aligned}$$

And the quantity on the right hand side vanishes as $n \to \infty$. □

Now we are ready to define the Young's Integral. Let us set:

$$\mu(a,b) = f(a)(g(b) - g(a))$$

for some $f \in \mathcal{C}^\alpha([0,T], \mathcal{L}(A,B)), g \in \mathcal{C}^\beta([0,T], A)$, with $\alpha + \beta > 1$. Note that for such μ we have that:

$$\begin{aligned}
|\mu(a,b) - \mu(a,c) - \mu(c,b)| &= |f(a)(g(b) - g(a)) - f(a)(g(c) - g(a)) - f(c)(g(b) - g(c))| \\
&= |(f(c) - f(a)(g(b) - g(a))| \\
&\leq \|f\|_\alpha \|g\|_\beta |b - a|^{\alpha + \beta}.
\end{aligned}$$

Then let φ be the function defined by the Sewing Lemma. By proposition 4 it is well defined the Young Integral:

$$\int_a^b f(r)\ dg(r) = \lim_{|\Pi_n| \to 0} \sum_{t_i \in \Pi_n} \mu(t_i, t_{i+1}) = \varphi(b) - \varphi(a) \tag{2.1}$$

where $\Pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a finite partition of $[a,b]$.

2.3 Properties of Young Integral

In this section we will prove some basic properties of Young Integral. The characterization in 2.1 assures the linearity of the Young Integral, now let us start with some calculus properties analogous to the ones of the Riemann Integral.

Proposition 5 (Chain Rule)**.** *Let* $F : A \to \mathcal{L}(A,A)$ *be differentiable and with* DF *Lipschitz. Then, if* $f \in \mathcal{C}^\alpha([a,b], A)$ *with* $\alpha > \frac{1}{2}$*, we have that*

$$F(f(t)) = F(f(a)) + \int_a^t DF(f(s))\ df(s), \quad t \in [a,b].$$

Proof. By Taylor expansion, for any $x, y \in [a, b]$ we have that:

$$F(x) - F(y) - DF(y)(x - y) = o(|x - y|^{2\alpha}),$$
$$F(x) - F(y) - DF(y)(x - y) = \int_y^x (F'(s) - F'(x))\ ds.$$

Now set $x = f(t)$ and $y = f(s)$, for $s, t \in [a, b]$, we can write

$$F((f(t)) - F(f(s)) = DF(f(s))(f(t) - f(s)) + o(|t - s|^{2\alpha}),$$

since $|f(t) - f(s)| \le \|f\|_\alpha |t - s|^\alpha$. Let us call $J(t) = F(f(t)) - F(f(a))$, then

$$J(t) - J(s) = DF(f(s))(f(t) - f(s)) + o(|t - s|)$$

so by the uniqueness of Lemma 2 it follows that $J(t)$ is the Young integral of DF against f, whith $\alpha = \beta$. □

Proposition 6 (Integration by parts)**.** *For $f \in \mathcal{C}^\alpha([a, b], \mathcal{L}(A, B)), g \in \mathcal{C}^\beta([a, b], A)$ with $\alpha + \beta > 1$, then we can write*

$$\int_a^t f(s)\ dg(s) = f(t)g(t) - f(a)g(a) - \int_a^t g(s)\ df(s).$$

Proof. Let us set $J(t) = f(t)g(t) - f(a)g(a) - \int_a^t g(s)\ df(s)$, note that:

$$\begin{aligned} J(t) - J(s) &= f(t)g(t) - f(s)g(s) - g(s)(f(t) - f(s)) + o(|t - s|^{\alpha+\beta}) \\ &= f(t)(g(t) - g(s)) + o(|t - s|^{\alpha+\beta}) \\ &= f(t)(g(t) - g(s)) + (f(t) - f(s))(g(t) - g(s)) + o(|t - s|^{\alpha+\beta}) \\ &= f(s)(g(t) - g(s)) + o(|t - s|^{\alpha+\beta}). \end{aligned}$$

Then again by Lemma 2 $J(t) = \int_a^t f(s)\ dg(s)$. □

Proposition 7. *Let $f \in \mathcal{C}^\alpha([0, T], \mathcal{L}(A, B)), g \in \mathcal{C}^\beta([0, T], A)$, with $\alpha + \beta > 1$ then*

$$\int_0^t f(r)\ dg(r) - \int_0^t f(r)\ dh(r) = \int_0^t f(r)\ d(g - h)(r).$$

Proof. By the characterization given by proposition 4 we have:

$$\begin{aligned} \int_0^t f(r)\ d(g - h)(r) &= \lim_{|\Pi_n| \to 0} \sum_{t_i \in \Pi_n} f(t_i)(g(t_{i+1}) - h(t_{i+1}) - g(t_i) + h(t_i)) \\ &= \lim_{|\Pi_n| \to 0} \sum_{t_i \in \Pi_n} f(t_i)(g(t_{i+1}) - g(t_i)) - \\ &\quad - \lim_{|\Pi_n| \to 0} \sum_{t_i \in \Pi_n} f(t_i)(h(t_{i+1}) - h(t_i)) \\ &= \int_0^t f(r)\ dg(r) - \int_0^t f(r)\ dh(r). \end{aligned}$$

□

Proposition 8. *Let $f \in \mathcal{C}^\alpha([0,T],\mathcal{L}(A,B)), g \in \mathcal{C}^\beta([0,T],A), T>0, \alpha,\beta \in (0,1)$ such that $\alpha+\beta>1$, then*

$$\int_0^\cdot f(r)\ dg(r) \in \mathcal{C}^\beta([0,T],B).$$

Moreover:

$$\left\|\int_0^\cdot f(r)\ dg(r)\right\|_\beta \leq (K\|f\|_\alpha T^\alpha + \|f\|_\infty)\|g\|_\beta (T^\beta+1),$$

where $K = \frac{2^{\alpha+\beta}}{2^{\alpha+\beta}-2}$.

Proof. Since by Sewing Lemma $I(t) = \int_0^t f(r)\ dg(r)$ is continuous for $t \in [0,T]$, to prove that $I(t) \in \mathcal{C}^\beta([0,T],\mathbb{R}])$, it sufficies to show that the following inequality holds true:

$$|I(t) - I(s)| \leq C\,|t-s|^\beta$$

for some constant $C \in \mathbb{R}$. We have that:

$$\begin{aligned}
|I(t)-I(s)| &= \left|\int_0^t f(r)\ dg(r) - \int_0^s f(r)\ dg(r)\right| = \left|\int_s^t f(r)\ dg(r)\right| \\
&= \left|\int_s^t f(r)\ dg(r) - f(s)(g(t)-g(s)) + f(s)(g(t)-g(s))\right| \\
&\leq \left|\int_s^t f(r)\ dg(r) - f(s)(g(t)-g(s))\right| + |f(s)(g(t)-g(s))|\,.
\end{aligned}$$

Then using using the inequality we obtained in Sewing Lemma 2 and setting $K = \frac{2^{\alpha+\beta}}{2^{\alpha+\beta}-2}$ we obtain:

$$\begin{aligned}
|I(t)-I(s)| &\leq \left|\int_s^t f(r)\ dg(r) - f(s)(g(t)-g(s))\right| + \|f(s)(g(t)-g(s))\| \\
&\leq K\|f\|_\alpha \|g\|_\beta |t-s|^{\alpha+\beta} + \|f\|_\infty \|g\|_\beta |t-s|^\beta \\
&= (K\|f\|_\alpha \|g\|_\beta |t-s|^\alpha + \|f\|_\infty \|g\|_\beta)\,|t-s|^\beta \\
&\leq (K\|f\|_\alpha T^\alpha + \|f\|_\infty)\|g\|_\beta |t-s|^\beta \\
&= C\,|t-s|^\beta\,.
\end{aligned}$$

Since C does not depend on $t,s \in [0,T]$, then $I \in \mathcal{C}^\beta([0,T],\mathbb{R})$.
For the second part of the proposition, we need to compute $\|I\|_\infty$. Then, by the same argument used above:

$$\begin{aligned}
|I(t)| &= |I(t) - I(0) - f(0)(g(t)-g(0)) + f(0)(g(t)-g(0))| \\
&\leq |I(t) - I(0) - f(0)(g(t)-g(0))| + |f(0)(g(t)-g(0))| \\
&\leq K\|f\|_\alpha \|g\|_\beta T^{\alpha+\beta} + \|f\|_\infty \|g\|_\beta T^\beta.
\end{aligned}$$

Then, putting together the pieces, we obtain:

$$\left\|\int_0^\cdot f(r)\ dg(r)\right\|_\beta \leq (K\|f\|_\alpha T^\alpha + \|f\|_\infty)\|g\|_\beta (T^\beta+1).$$

□

Sometimes it is useful to integrate against an integral function, we show that under certain condition the Young integral well behaves with this kind of integration.

Proposition 9. *Let* $A \in \mathcal{C}^{\gamma}([0,T], \mathcal{L}(A, B)), x \in \mathcal{C}^{\alpha}([0,T], A), B \in \mathcal{C}^{\beta}([0,T], \mathcal{L}(B, C))$ *where* A, B *are Banach spaces and* $\alpha + \beta > 1, \gamma + \beta > 1$ *then if:*

$$y(t) = \int_0^t A(s)\ dx(s)$$

the folllowing identity holds true:

$$\int_0^t B(s)\ dy(s) = \int_0^t B(s)A(s)\ dx(s).$$

Proof. By the definition of the Young Integral we have:

$$\int_s^t B(r)\ dy(r) = B(s)(y(t) - y(s)) + o(|t-s|^{\alpha+\beta}).$$

On the other hand:

$$\int_s^t B(r)A(r)\ dx(r) = B(s)A(s)(x(t) - x(s)) + o(|t-s|^{\alpha+\beta}).$$

We want to show that:

$$B(s)(y(t) - y(s)) + o(|t-s|^{\alpha+\beta}) = B(s)A(s)(x(t) - x(s)) + o(|t-s|^{\alpha+\beta}).$$

Now

$$y(t) - y(s) = \int_s^t A(r)\ dx(r) = A(s)(x(t) - x(s)) + o(|t-s|^{\alpha+\beta}).$$

So that:

$$\begin{aligned} B(s)(y(t) - y(s)) + o(|t-s|^{\alpha+\beta}) &= B(s)(A(s)(x(t) - x(s)) + o(|t-s|^{\alpha+\beta})) \\ &\quad + o(|t-s|^{\alpha+\beta}) \\ &= B(s)A(s)(x(t) - x(s)) + o(|t-s|^{\alpha+\beta}). \end{aligned}$$

□

Chapter 3

Young Differential Equations

In this section we will study differential equations of the following type:

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s))\ dx(s) \tag{3.1}$$

for some $f \in \mathcal{C}_b^{1,\gamma}(B, \mathcal{L}(A,B)), y_0 \in B, t \in [t_0, T]$, where B is a Banach space.

Definition 2. A solution to (3.1) is a function $y : [t_0, T] \to B$ such that (3.1) holds for any $t \in [t_0, T]$.

Note that by Proposition 8 one should expect that a solution to (3.1) inherit the regularity of x, and that is true under suitable condition. Let us assume that a solution to (3.1) exists and let us give an a priori estimate to such a solution.

Proposition 10. *Let $x \in \mathcal{C}^\alpha([t_0,T], A), y \in \mathcal{C}^\alpha([t_0,T], B)$ for some $\alpha \in (\frac{1}{2}, 1]$ and let $f \in Lip(B, \mathcal{L}(A,B))$, with Lipschitz constant L, such that given $y_0 \in \mathbb{R}$ we have that:*

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s))\ dx(s).$$

Then $\|y\|_\beta$ is bounded, uniformly in $\|x\|_\alpha, L, T - t_0, \|f\|_\infty$.

Proof. Using the inequality given by Lemma 2 we have that:

$$\begin{aligned} |y(t) - y(s) - f(y(s))(x(t) - x(s))| &= \left| \int_{s}^{t} f(y(r))\ dx(r) - f(y(s))(x(t) - x(s)) \right| \\ &\leq C \|f \circ y\|_\alpha \|g\|_\alpha |t-s|^{2\alpha}. \end{aligned}$$

Where $C = \frac{2^{2\alpha}}{2^{2\alpha} - 2}$. So that:

$$\begin{aligned} |y(t) - y(s)| &= |y(t) - y(s) - f(y(s))(x(t) - x(s)) + f(y(s))(x(t) - x(s))| \\ &\leq |y(t) - y(s) - f(y(s))(x(t) - x(s))| + |f(y(s))(x(t) - x(s))| \\ &\leq C \|f \circ y\|_\alpha \|x\|_\beta |t-s|^{\alpha+\beta} + \|f\|_\infty \|x\|_\beta |t-s|^\beta. \end{aligned}$$

It remains to study $\|f \circ y\|_\alpha$, we have that:

$$\begin{aligned}\|f \circ y\|_\alpha &= \|f \circ y\|_\infty + \sup_{t_0 \le s < t \le T} \frac{|f(y(t)) - f(y(s))|}{|t-s|^\alpha} \\ &\le \|f\|_\infty + \sup_{t_0 \le s < t \le T} \frac{|f(y(t)) - f(y(s))|}{|t-s|^\alpha}\end{aligned}$$

Moreover since f is Lipschitz, we have:

$$\begin{aligned}\sup_{t_0 \le s < t \le T} \frac{|f(y(t)) - f(y(s))|}{|t-s|^\alpha} &\le \sup_{t_0 \le s < t \le T} \frac{L\,|y(t) - y(s)|}{|t-s|^\alpha} \\ &\le \sup_{t_0 \le s < t \le T} \frac{L\,\|y\|_\alpha |t-s|^\alpha}{|t-s|^\alpha} = L\,\|y\|_\alpha .\end{aligned}$$

So that:

$$\|f\|_\alpha \le \|f\|_\infty + L\,\|f\|_\alpha < \infty.$$

Finally we obtain that

$$\begin{aligned}|y(t) - y(s)| &\le c(\|f\|_\infty + L\,\|y\|_\alpha)\,|t-s|^{\alpha+\beta} + \|f\|_\infty \|x\|_\beta |t-s|^\beta \\ &\le (c(\|f\|_\infty + L\,\|y\|_\alpha)\,|T-t_0|^\alpha + \|f\|_\infty \|x\|_\beta)\,|t-s|^\beta .\end{aligned}$$

So that $y \in \mathcal{C}^\beta$ and $\|y\|_\beta$ is bounded. □

The previous proposition proves that if a solution exists then it is bounded for any finite interval of time.

3.1 An existence and uniqueness result

In this section we will prove that there exists a unique solution to the Young Differential Equation

$$y(t) = y_0 + \int_{t_0}^t f(y(s))\,dx(s)$$

under the assumptions $x \in \mathcal{C}^\alpha([t_0,T],A), \alpha \in (0,1), f \in \mathcal{C}_b^{1,\gamma}(B, \mathcal{L}(A,B)), y_0 \in B$, with $\alpha(1+\gamma) > 1$. For the sake of simplicity we will consider interval of time of the type $[0,T]$ as, up to a translation $y(t-t_0)$, the problem does not change. Our proof will be based on what we would do if we are handling usual ordinary differential equation. Indeed we will define the map:

$$M_{T,y_0}(y) = y_0 + \int_0^t f(y(r))\,dx(r),\ t \in [0,T],$$

then we will prove it is a contraction for balls of a suitable radius and then we will obtain a solution thanks to the Banach Fixed Point Theorem.
Let us start with a definition.

Definition 3. Let $(\mathbb{X}, d)$ be a metric space. A map $T : \mathbb{X} \to \mathbb{X}$ is called a contraction mapping on $\mathbb{X}$ if there exists $L \in [0, 1)$ such that $d(T(x), T(y)) \leq Ld(x, y)$ for all $x, y \in \mathbb{X}$.

The proof of the following theorem is due to [Pal07].

Theorem 5 (Banach Fixed Point Theorem)**.** *Let $(\mathbb{X}, d)$ be a complete non empty metric space. Let $T : \mathbb{X} \to \mathbb{X}$ a contraction mapping on $\mathbb{X}$ then the map T admits a unique fixed point.*

Proof. Firstly let us note that for every $n \in \mathbb{N}, x, y \in \mathbb{X}$ we have $d(T^n(x), T^n(y)) \leq L^n d(x, y)$. By induction the thesis holds for $n = 0$, then supposing the thesis holds true for n let us prove the thesis for $n + 1$:

$$d(T^{n+1}(x), T^{n+1}(y)) \leq Ld(T^n(x), T^n(y)) \leq L^{n+1}d(x, y).$$

Moreover by the triangle inequality we have:

$$\begin{aligned} d(x, y) &\leq d(x, T(x)) + d(T(x), T(y)) + d(T(y), y)) \\ &\leq d(x, T(x)) + Ld(x, y) + d(T(y), y). \end{aligned}$$

So that:

$$(1 - L)d(x, y) \leq d(x, T(x)) + d(T(y), y),$$

then:

$$d(x, y) \leq \frac{d(T(x), x) + d(T(y), y)}{1 - L}. \tag{3.2}$$

Existence. Let us prove that the sequence $(T^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. By 3.2 we have:

$$\begin{aligned} d(T^n(x), T^m(x)) &\leq \frac{d(T(T^n(x)), T^n(x)) + d(T(T^m(x)), T^m(x))}{1 - L} \\ &= \frac{d(T^n(T(X)), T^n(x)) + d(T^m(T(x)), T^m(x))}{1 - L} \\ &\leq \frac{L^n d(T(x), x) + L^m d(T(x), x)}{1 - L} \\ &= \frac{L^n + L^m}{1 - L} d(T(x), x) \to 0 \end{aligned}$$

as $n, m \to \infty$ since $L \in [0, 1)$ proving that $(T^n(x))_{n\in\mathbb{N}}$ is Cauchy. So that there exists a limit $x^\star = \lim_{n\to\infty} T^n(x)$. The point $x^\star$ is a fixed point, in fact since T is a contraction then it is continuous, so that:

$$x^\star = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} T^{n+1}(x) = T(\lim_{n\to\infty} T^n(x)) = T(x^\star).$$

Uniqueness. Let $x, y \in \mathbb{X}$ such that $T(x) = x, T(y) = y$, then by (3.2)

$$d(x, y) \leq \frac{d(T(x), x) + d(T(y), y)}{1 - L} = \frac{d(x, x) + d(y, y)}{1 - L} = 0.$$

So that $x = y$. □

Note that in the proof of theorem 5 we obtained the fixed point by an iterative construction. In particular, if the map M_{T,y_0} is a contraction mapping of some suitable space into itself, then we can construct an approximation to a solution, whenever it exists, of 3.1 by considering the sequence:

$$M^n_{T,y_0} = y_0 + \int_0^t f(M^{n-1}_{T,y_0}(r))\ dx(r).$$

In order to study whether the map $M_{T,y_0}(y)$ is a contraction or not, we need to study its norm. To do this we need to study the behaviour of the composition $f \circ y$. To do this let us introduce a result that will be useful to study the behaviour of the composition.

Proposition 11. *Let $x, y > 0, \alpha \in (0,1)$ then*

$$(x+y)^\alpha \leq x^\alpha + y^\alpha.$$

Proof. We have that:

$$\begin{aligned}(x+y)^\alpha - x^\alpha &= \int_x^{x+y} \alpha t^{\alpha-1}\ dt \leq \int_x^{x+y} \alpha (t-x)^{\alpha-1}\ dt \\ &\leq \alpha \frac{1}{\alpha}(t-x)^\alpha \Big|_x^{x+y} = y^\alpha.\end{aligned}$$

The inequalities follows from the fact that $1-\alpha > 0$, yielding $(x+y)^\alpha \leq x^\alpha + y^\alpha$. □

Now we are ready to give an estimate on the composition $f \circ y$.

Lemma 3. *Given $\alpha \in (0,1]$, let $x, y \in \mathcal{C}^\alpha([0,T], B)$ and $f \in \mathcal{C}^{1,\gamma}(B, B')$, where B, B' are Banach spaces. Then:*

$$\|f \circ x - f \circ y\|_{\alpha\gamma} \leq (\|Df\|_\infty (1 + T^{\alpha(1-\gamma)}) + \|Df\|_\gamma (\|x\|^\gamma_\alpha + \|y\|^\gamma_\alpha))\, \|x-y\|_\alpha.$$

Proof. Let us define

$$\begin{aligned} g :& B \times B \to \mathcal{L}(B, B') \\ &(u,v) \mapsto \int_0^1 Df(tu + (1-t)v)\ dt. \end{aligned}$$

Then the following identity holds

$$f(u) - f(v) = g(u,v)(u-v)$$

by applying the Fundamental Theorem of Calculus to the function $f \circ \tau$ where

$$\begin{aligned} \tau :& [0,1] \to B \\ & t \mapsto tu + (1-t)v. \end{aligned}$$

Now we can observe that:

$$\|g\|_\infty \leq \int_0^1 \|Df\|_\infty \ dt = \|Df\|_\infty .$$

Moreover:

$$\begin{aligned}
|g(u,v) - g(\bar{u},\bar{v})| &\leq \int_0^1 |Df(tu + (1-t)v) - Df(t\bar{u} + (1-t)\bar{v}| \ dt \\
&\leq \|Df\|_\gamma \int_0^1 |t(u-\bar{u}) + (1-t)(v-\bar{v})|^\gamma \ dt \\
&\leq \|Df\|_\gamma \int_0^1 (t\,|u-\bar{u}| + (1-t)\,|v-\bar{v}|)^\gamma \ dt \\
&\leq \|Df\|_\gamma \int_0^1 (t^\gamma\,|u-\bar{u}|^\gamma + (1-t)^\gamma\,|v-\bar{v}|^\gamma) \ dt \\
&\leq \|Df\|_\gamma \int_0^1 (|u-\bar{u}|^\gamma + |v-\bar{v}|^\gamma) \ dt \\
&= \|Df\|_\gamma\,(|u-\bar{u}|^\gamma + |v-\bar{v}|^\gamma).
\end{aligned}$$

Before going on let us introduce the following notation for g from $[0,T]$ to some Banach space B: $g_{s,t} = g(t) - g(s)$. Now by substituting x, y we obtain:

$$\begin{aligned}
|(f\circ x)_{s,t} - (f\circ y)_{s,t}| &= |(f(x(t) - f(y(t))) - (f(x(s)) - f(y(s)))| \\
&= |g(x(t),y(t))(x(t) - y(t)) - g(x(s),y(s))(x(s) - y(s))| \\
&= |g(x(t),y(t))(x_{s,t} - y_{s,t} + \\
&\quad + x(s) - y(s)) - g(x(s),y(s))(x(s) - y(s))| \\
&\leq |g(x(t),y(t))(x_{s,t} - y_{s,t})| + \\
&\quad + |(g(x(t),g(y(t)) - g(x(s),y(s)))(x(s) - y(s)| \\
&\leq \|Df\|_\infty\,|x_{s,t} - y_{s,t}| + \|Df\|_\gamma\,(|x_{s,t}|^\gamma + |y_{s,t}|^\gamma)\,|x(s) - y(s)|\,.
\end{aligned}$$

Then we have:

$$\begin{aligned}
\|f\circ x - f\circ y\|_{\alpha\gamma} &\leq \|f\circ x - f\circ y\|_\infty + \\
&\quad + \|Df\|_\infty\,\|x-y\|_\alpha\,T^{\alpha(1-\gamma)} + \|Df\|_\infty\,(\|x\|_\alpha^\gamma + \|y\|_\alpha^\gamma)\,\|x-y\|_\infty \\
&\leq \|f\|_\infty\,\|x-y\|_\alpha + \\
&\quad + \|Df\|_\infty\,\|x-y\|_\alpha\,T^{\alpha(1-\gamma)} + \|Df\|_\infty\,(\|x\|_\alpha^\gamma + \|y\|_\alpha^\gamma)\,\|x-y\|_\alpha \\
&\leq (\|Df\|_\infty\,(1 + T^{\alpha(1-\gamma)}) + \|Df\|_\gamma\,(\|x\|_\alpha^\gamma + \|y\|_\alpha^\gamma))\,\|x-y\|_\alpha\,.
\end{aligned}$$

That is the thesis. □

Now we are ready to prove the existence and uniqueness theorem for solution to 3.1.

Theorem 6 (Existence and Uniqueness of the solutions to Young Differential Equations)**.** *Given $\alpha \in (\frac{1}{2}, 1]$, A, B two Banach spaces, let $x \in \mathcal{C}^\alpha([0,T], A)$ and*

$f \in \mathcal{C}_b^{1,\gamma}(B, \mathcal{L}(A,B))$ *with* $\alpha)1+\gamma) > 1$*. For every* $y_0 \in B$ *there exists a unique solution* $y \in \mathcal{C}^\alpha([0,T], B)$ *to the differential equation:*

$$y(t) = y_0 + \int_0^t f(y(r))\ dx(r).$$

Proof. Firstly note that the Young's integral should be defined since $f \circ y \in \mathcal{C}^\alpha([0,T], \mathcal{L}(A,B))$ and $2\alpha > 1$ by hypothesis. We fix $\beta \in (\frac{1}{2}, \alpha)$ such that $\beta\gamma + \alpha > 1$ and consider the operator between Banach spaces defined as follows:

$$\begin{aligned} M_{T,y_0} :& \mathcal{C}^\beta([0,T], B) \to \mathcal{C}^\alpha([0,T], B) \\ & y \mapsto y_0 + \int_0^t f(y(r))\ dx(r). \end{aligned}$$

The integral is well defined since $f \circ y$ lies in $\mathcal{C}^\beta([0,T], B)$ and, by the choice of β, $\alpha+\beta > 1$. Moreover M_{T,y_0} maps the subspace $V = \{y \in \mathcal{C}^\beta([0,T], B) : y(0) = y_0\}$ into itself.
As we briefly said at the beginning of the paragraph, we want to find a suitable space where M_{T,y_0} is a contraction. Since V is invariant under M_{T,y_0}, then a good choice should be a subspace of V. Let us consider $W = \{y \in V : \|y\|_\beta \leq 1\}$ then, for a suitable $T > 0$ small enough $M_{T,y_0}(W) \subset W$.

$$\begin{aligned} \|M_{t,y_0}(y)\|_\beta &= \left\| \int_0^t f(y(r))\ dx(r) \right\| \\ &\leq (K \|f \circ y\|_\beta T^\beta + \|f \circ y\|_\infty)(T^\beta + 1) \|x\|_\beta \\ &\leq (K \|Df\|_\infty \|y\|_\beta T^\beta + \|f\|_\infty)(T^\beta + 1) \|x\|_\alpha T^{\alpha-\beta}. \end{aligned}$$

Then choosing $T \leq \min\{1, T_1\}$ where

$$T_1 = (2(K \|Df\|_\infty + \|f\|_\infty) \|x\|_\alpha)^{\frac{1}{\alpha-\beta}}.$$

Now we will show that M_{T,y_0} is a contraction for a suitable $T > 0$ and then by Theorem 5, there exists a solution $\bar{y}$ to $M_{T,y_0}(\bar{y}) = \bar{y}$ in the interval $[0,T]$ meaning that $\bar{y}$ is a solution to the intial problem on $[0,T]$. For $z, y \in W$:

$$\begin{aligned} \|M_{T,y_0}(z) - M_{T,y_0}(y)\|_\beta &= \left\| \int_0^\cdot (f(z(r)) - f(y(r)))\ dx(r) \right\|_\beta \\ &\leq K \|f \circ z - f \circ y\|_{\beta\gamma} T^\beta + \|f \circ z - f \circ y\|_\infty)(T^\beta + 1) \|x\|_\beta \\ &\leq K((\|Df\|_\infty (1 + T^{\beta(1-\gamma)}) + \\ &\quad + 2 \|Df\|_\gamma) \|z - y\|_\beta) T^\beta + \\ &\quad + \|f \circ z - f \circ y\|_\infty)(T^\beta + 1) \|x\|_\alpha T^{\alpha-\beta} \\ &\leq K((\|Df\|_\infty (1 + T^{\beta(1-\gamma)}) + \\ &\quad + 2 \|Df\|_\gamma)) T^\beta + \\ &\quad + \|Df\|_\infty)(T^\beta + 1) \|z - y\|_\beta \|x\|_\alpha T^{\alpha-\beta} \end{aligned}$$

Then by choosing $T \leq \min\{1, T_1, T_2\}$ where:

$$T_2 = \left(\frac{1}{2} K \left(2 \|Df\|_\infty + 2 \|Df\|_\gamma \right) + 2 \|Df\|_\infty) \|x\|_\alpha \right)^{\frac{1}{\alpha - \beta}},$$

we have that:

$$\|M_{T,y_0}(z) - M_{T,y_0}(y)\|_\beta \leq \frac{1}{2} \|z - y\|_\beta .$$

So M_{T,y_0} is a contraction on W for T as above. Finally, since M_{T,y_0} takes also value in $\mathcal{C}^\alpha([0,T], B)$ we have $y \in \mathcal{C}^\alpha([0,T], B)$.
Now we want to show that the results holds true for any $T > 0$. Let $n \in \mathbb{N}$ such that $n \min\{1, T_1, T_2\} \geq T$ then, for what we have already shown, there exists unique $y^{(1)}, \dots, y^{(n)} \in \mathcal{C}^\alpha([0, \frac{T}{N}], B)$ that solve the differential problems:

$$y_t^{(1)} = y_0 + \int_0^t f\left(y^{(1)}(r)\right)\, dx(r)$$

$$y_t^{(i+1)} = y^{(i-1)}\left(i\frac{T}{n}\right) + \int_0^t f\left(y^{(i+1)}(r)\right)\, dx\left(r + i\frac{T}{n}\right)$$

Given $y \in \mathcal{C}^\alpha([0,T], B)$ solution to the differential equation, by the uniqueness of the solution onto $[\frac{(i-1)T}{n}, i\frac{T}{n}]$, the following identity holds:

$$y^{(i)}(t) = y\left(t + (i-1)\frac{T}{n}\right)$$

for every i, then the solution is unique. On the other hand y constructed as above is a solution to the differential equation and that concludes the proof of the theorem. □

Before going on let us remark that the contraction does not depend on y_0. Since we have obtained a solution let us note that it behaves well with composition. The following result is a refining of proposition 5.

Proposition 12 (Itô's Formula)**.** *For some $\alpha > \frac{1}{2}$, let $x \in \mathcal{C}^\alpha([0,T], A), y \in \mathcal{C}^\alpha([0,T], B)$ satisfying 3.1 where $f \in Lip(B, \mathcal{L}(A, B))$. Then if $G : B \to B$ is differentiable, with Lipschitz derivative, then for all $t \in [a,b]$,*

$$G(y(t)) = G(y_0) + \int_a^t G'(y(s))\, dy(s) = G(y_0) + \int_a^t G'(y(s)) f(y(s))\, dx(s).$$

Proof. By Taylor expansion:

$$G(y(t)) - G(y(s)) = G'(y(s))(y(t) - y(s)) + o(|t-s|^{2\alpha}).$$

By the definition of the Young Integral:

$$\int_s^t G'(y(r))\, dy(r) = G'(y(s))(y(t) - y(s)) + o(|t-s|^{2\alpha}).$$

On the other hand:

$$\int_s^t G'(y(r))f(y(r)\ dx(r) = G'(y(s))f(y(s))(x(t)-x(s)) + o(|t-s|^{2\alpha}).$$

So we need to show that:

$$G'(y(s))(y(t)-y(s)) + o(|t-s|^{2\alpha}) = G'(y(s))f(y(s))(x(t)-x(s)) + o(|t-s|^{2\alpha}).$$

Now:

$$y(t)-y(s) = \int_s^t f(y(r))\ dx(r) = f(y(s))(x(t)-x(s)) + o(|t-s|^{2\alpha}).$$

So that:

$$G'(y(s))(y(t)-y(s)) + o(|t-s|^{2\alpha}) = G'(y(s))f(y(s))(x(t)-x(s)) + o(|t-s|^{2\alpha}).$$

Then the thesis holds by Sewing Lemma. □

This is a refinement of proposition 5 and proposition 9.

3.2 Flow properties

Given $f \in \mathcal{C}_b^{1,\gamma}(B, \mathcal{L}(A,B)), x \in \mathcal{C}^\alpha([0,T],A), y_0 \in B$, where A, B are Banach spaces and $\alpha \in (\frac{1}{2}, 1)$, with $\alpha(1+\gamma) > 1$ let us define the following map:

$$\begin{aligned} \mathbf{Y}(y_0,x) :& B \times \mathcal{C}^\alpha([0,T],A) \to \mathcal{C}^\beta([0,T],B), \quad (\beta \geq \alpha) \\ & (y_0,x) \mapsto y_0 + \int_0^t f(\mathbf{Y}_s(y_0,x))\ dx(s). \end{aligned}$$

The map $\mathbf{Y}(y_0,x)$ is usually called Itō Map (or flow in the case of ODEs). Note that the map is well defined since it satisfies the hypothesis of Theorem 6.

Proposition 13. *The map $\mathbf{Y}(.,x)$ is Lipschitz for every fixed x.*

Proof. Firstly we show that there exists a sufficiently small $\tilde{T} > 0$ such that $\mathbf{Y} : B \times \mathcal{C}^\alpha([0,\tilde{T}],A) \to \mathcal{C}^\alpha([0,\tilde{T}],B)$ is Lipschitz.
Let $y_0, \bar{y}_0 \in B, \beta \in (\frac{1}{2}, \alpha)$ then:

$$\begin{aligned} \|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\beta &= \left\| y_0 - \bar{y}_0 + \int_0^\cdot \{f(\mathbf{Y}_r(y_0,x)) - f(\mathbf{Y}_r(\bar{y}_0,x))\}\ dx(r) \right\|_\beta \\ &\leq |y_0 - \bar{y}_0| + \left\| \int_0^\cdot \{f(\mathbf{Y}_r(y_0,x)) - f(\mathbf{Y}_r(\bar{y}_0,x))\}\ dx(r) \right\|_\beta. \end{aligned}$$

By the same arguments used in the proof of Theorem 6 we obtain the following inequality:

$$\begin{aligned} &\left\| \int_0^\cdot \{f(\mathbf{Y}_r(y_0,x)) - f(\mathbf{Y}_r(\bar{y}_0,x))\}\ dx(r) \right\|_\beta \leq \\ &\leq (K\|Df\|_\infty (1+T^{\beta(1-\gamma)}) + \|Df\|_\gamma (\|\mathbf{Y}(y_0,x)\|_\beta^\gamma + \|\mathbf{Y}(\bar{y}_0,x)\|_\beta^\gamma) T^\beta + \\ &+ \|Df\|_\infty)(T^\beta + 1))\ \|x\|_\alpha \|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\beta T^{\alpha-\beta}. \end{aligned}$$

Setting:

$$C(\beta,\gamma) = (K \|Df\|_\infty (1 + T^{\beta(1-\gamma)}) + \|Df\|_\gamma (\|\mathbf{Y}(y_0,x)\|_\beta^\gamma + \\ + \|\mathbf{Y}(\bar{y}_0,x)\|_\beta^\gamma) T^\beta + \|Df\|_\infty)(T^\beta + 1)) \|x\|_\alpha$$

we obtain:

$$\|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\beta \le |y_0 - \bar{y}_0| + C(\beta,\gamma) \|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\beta T^{\alpha-\beta}.$$

Then if $\tilde{T} < \frac{1}{C(\beta,\gamma)}$ we obtain:

$$\|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\beta \le \frac{1}{1 - \tilde{T}C(\beta,\gamma)} |y_0 - \bar{y}_0|$$

As $\|.\|_\beta \ge \|.\|_\infty$ we have:

$$\|\mathbf{Y}(y_0,x) - \mathbf{Y}(\bar{y}_0,x)\|_\infty \le \frac{1}{1 - \tilde{T}C(\beta,\gamma)} |y_0 - \bar{y}_0|$$

Finally, remembering the semigroup law of $\mathbf{Y}(y_0,x)$, meaning that

$$Y_{T-t}(y_0,x) = Y_t(Y_T(y_0,x),x),$$

and by the uniqueness of the solution to (3.1) the result can be extended by the whole interval $[0,T]$. □

Moreover, fixed $T > 0$, one can interpret $\mathbf{Y}_T(.,x)$ as the map $B \mapsto \mathbf{Y}_T(B,x)$, where

$$\mathbf{Y}_T(B,x) = \{\mathbf{Y}_T(y_0,x) \in \mathcal{C}^\alpha([0,T],B) : \mathbf{Y}_T(y_0,x) \text{ satisfies } y_0 + \\ + \int_0^T f(\mathbf{Y}_r(y_0,x)\ dx(r) \text{ for some } y_0 \in B\}.$$

Furthermore $\mathbf{Y}_T(.,x)$ is a homeomorphism between the two spaces. To show this we need a preliminary result regarding time inversion for Young Integral.

Lemma 4. *Given $y \in \mathcal{C}^\alpha([0,T],\mathcal{L}(A,B)), x \in \mathcal{C}^\beta([0,T],A)$ with $\alpha+\beta > 1$, then*

$$\int_0^t y(s)\ dx(s) = -\int_{T-t}^T y(T-s)\ dx(T-s).$$

Proof. As we have seen in proposition 4 the Young is define as the limit of the Riemann sums:

$$J_n(0,t) = \sum_{t_i \in \Pi_n} y(t_i)(x(t_{i+1}) - x(t_i))$$

among a partition $\Pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0,T]$ such that $\lim_{n\to\infty} |\Pi_n| = 0$.

Hence:

$$\begin{aligned} J_n(0,t) &= \sum_{t_i \in \Pi_n} (y(t_i) - (y(t_{i+1}) - y(t_{i+1})))(x(t_{i+1}) - x(t_i)) \\ &= \sum_{t_i \in \Pi_n} y(t_{i+1})(x(t_{i+1}) - x(t_i)) - \sum_{t_i \in \Pi_n} (y(t_{i+1}) - y(t_i))(x(t_{i+1}) - x(t_i)). \end{aligned}$$

On the other hand:

$$\left|\sum_{t_i\in\Pi_n}(y(t_{i+1})-y(t_i))(x(t_{i+1})-x(t_i)\right| \leq \|y\|_\alpha \|x\|_\beta \sum_{t_i\in\Pi_n}|t_{i+1}-t_i|^{\alpha+\beta}$$
$$\leq \|y\|_\alpha \|x\|_\beta |\Pi_n| \sum_{t_i\in\Pi_n}|t_{i+1}-t_i|^{\alpha+\beta-1}$$
$$\leq \|y\|_\alpha \|x\|_\beta |\Pi_n| T^{\alpha+\beta-1} \to 0$$

as $n\to\infty$. Setting $u(t)=y(T-t), v(t)=x(T-t)$ we have:

$$J_n(0,t) = \sum_{t_i\in\Pi_n} u(T-t_{i+1})(v(T-t_{i+1})-v(T-t_i)) + o(|\Pi_n|)$$
$$= -\sum_{t_i\in\Pi_n} u(T-t_{i+1})(v(T-t_i)-v(T-t_{i+1})) + o(|\Pi_n|),$$

which converges to $-\int_{T-t}^{T} u(s)\ dv(s)$ yelding the thesis. □

Now let us consider the map, for $T>0$, $y_T\mapsto \mathbf{Y}_T^{-1}(y_T,\hat{x})$ defined as:

$$\mathbf{Y}_T^{-1}(y_T,x) = y_T + \int_0^T f(\mathbf{Y}_s^{-1}(y_T,\hat{x}))\ d\hat{x}(T-s),$$

where $\hat{x}(s)=x(T-s)$. Using the time inversion property we obtain:

$$\mathbf{Y}_{T-t}(y_T,x) = y_T - \int_{T-t}^{T} f(\mathbf{Y}_{T-s}(y_T,x)\ d\hat{x}(s)$$
$$= \mathbf{Y}_T(y_T,x) + \int_0^t f(\mathbf{Y}_{t-s}(y_T,x))\ d\hat{x}(s).$$

By the uniqueness of the solution to the differential problem

$$\mathbf{Y}_t(y_0,x) = y_0 + \int_0^t f(\mathbf{Y}_r(y_0,x))\ dx(r),\ \ y_0\in B,$$

we obtain that $\mathbf{Y}_{T-t}(y_0,x) = \mathbf{Y}_t^{-1}(\mathbf{Y}_T(y_0,x),x)$ and $\mathbf{Y}_0(y_0,x)=y_0$. Similarly $\mathbf{Y}_{T-t}^{-1}(\mathbf{Y}_T(y_0,x),x) = \mathbf{Y}_{T-t}^{-1}(y_0,x)$ and then $y_0 = \mathbf{Y}_T(\mathbf{Y}_T^{-1}(y_0,x),x) = \mathbf{Y}_T^{-1}(\mathbf{Y}_T(y_0,x),x)$. We can resume all the discussion above in the following result.

Proposition 14. *Let $f\in\mathcal{C}^{1,\gamma}(A,\mathcal{L}(A,B))$ and $x\in\mathcal{C}^\alpha([0,T],A)$ with $\alpha(1+\gamma)>1$. For any $T>0$, the map $y_0\mapsto\mathbf{Y}_T(y_0,x)$ defines a Lipschitz homeomorphism from B to $\mathbf{Y}_T(B,x)$ and its inverse is $y_0\mapsto\mathbf{Y}_T^{-1}(y_0,x)$*

From now on we are interested in studying the differentiability of the map $(y_0,x)\mapsto Y(y_0,x)$ with respect to the initial datum y_0. To do this we use a result due to Da Prato [DP14, Theorem D.2], that we report here as a Lemma.

Lemma 5. *Let A,B two Banach spaces and $F: A\times B\to B$ such that:*

1. *There exists $K < 1$ such that, given $x \in A, X, Y \in B$ the following inequality holds true:* $\|F(x,X) - F(x,Y)\|_B \leq K \|X - Y\|_B$,

2. *There exists $F_x \in \mathcal{L}(A,B)$ and $F_X \in \mathcal{L}(B,B)$ such that, given $x, y \in \mathbb{R}, Y \in B$ we have:*
$$\lim_{h \to 0} \frac{F(x+hy, X+hY) - F(x,X)}{h} = F_x(x,X)y + F_X(x,X)Y,$$

3. *The functions $F_x(.,.) : A \times B \to \mathcal{L}(A,B)$ and $F_X(.,.) : A \times B \to \mathcal{L}(B,B)$ are continuous.*

Then the function $x \mapsto X^x$ where X^x is the only solution to $F(x, X^x) = X^x$ is differentiable and
$$\frac{\partial X^x}{\partial x} = F_x(x, X^x) + F_X(x, X^x)\frac{\partial X^x}{\partial x}.$$

Proof. Note that X^x is unique and well defined since $F(x,.)$ is a contraction, then for every x there exists, by Theorem 5 a unique solution to $F(x, X^x) = X^x$.
Let us start proving that the map $x \mapsto X^x$ is continuous. In fact:
$$X^x - X^y = F(x, X^x) - F(y, X^x) + F(y, X^x) - F(y, X^y),$$
then:
$$\|X^x - X^y\|_B \leq \|F(x, X^x) - F(y, X^x)\|_B + K \|X^x - X^y\|_B.$$
So that:
$$(1 - K) \|X^x - X^y\|_B \leq \|F(x, X^x) - F(y, X^x)\|_B.$$
Since $1 - K > 0$ we can write:
$$\|X^x - X^y\|_B \leq \frac{1}{1-K} \|F(x, X^x) - F(y, X^x)\|_B.$$
Since $F(., X^x)$ is continuous by hypothesis, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - y\|_A < \delta \Rightarrow \|F(x, X^x) - F(y, X^x)\|_B < \varepsilon$. Then choosing $\delta_\varepsilon < (1-K)\varepsilon\delta$, we obtain that for every x, y such that $\|x - y\|_A < \delta_\varepsilon$ we have:
$$\|X^x - X^y\|_B \leq \varepsilon$$
proving that $x \mapsto X^x$ is continuous.
By 2) for every fixed $x \in A, F(x,.)$ is Gateaux differentiable with respect to X. Moreover by 1):
$$\|F_X(x,.)\|_{\mathcal{L}(B,B)} \leq K.$$
From $2), 3)$ the function
$$t \mapsto F((1-t)x + ty, (1-t)X + tY)$$
is differentiable and
$$\frac{d}{dt} F((1-t)x + ty, (1-t)X + tY)) =$$
$$= F_x((1-t)x + ty, (1-t)X + tY)(y - x) + F_X((1-t)x + ty, (1-t)X + tY)(Y - X).$$

So that:

$$F(y,Y)-F(x,X)=\int_0^1 F_x((1-t)x+ty,(1-t)X+tY)(y-x)\;dt+$$
$$+\int_0^1 F_X((1-t)x+ty,(1-t)X+tY)(Y-X)\;dt.$$

In particular

$$\begin{aligned}X^{x+y}-X^x&=F(x+y,X^{x+y})-F(x,X^x)\\&=\int_0^1 F_x(x+ty,(1-t)X^{x+y}+tX^x)y\;dt+\\&+\int_0^1 F_X(x+ty,(1-t)X^{x+y}+tX^x)(X^{x+y}-X^x)\;dt.\end{aligned}$$

Let us define the operator:

$$G_y=\int_0^1 F_X(x+ty,(1-t)X^x+tX^{x+y})\;dt.$$

Then $G_y\in\mathcal{L}(B,B)$, $\|G_y\|_{\mathcal{L}(B,B)}\le K$ and $y\mapsto G_y$ is continuous. Moreover for every $Z\in B$ we have:

$$\lim_{y\to 0}G_yZ=F_X(x,X^x)Z.$$

Since $\|G_y\|_{\mathcal{L}(B,B)}\le K<1$ it is defined $(I-G_y)^{-1}=\sum_{n=0}^{\infty}G_y^n$. Then:

$$\begin{aligned}X^{x+y}-X^x&=G_y(X^{x+y}-X^x)+y\int_0^1 F_x(x+ty,(1-t)X^x+tX^{x+y})\;dt,\\\frac{(I-G_y)(X^{x+y}-X^x)}{y}&=\int_0^1 F_x(x+ty,(1-t)X^x+tX^{x+y})\;dt,\\\frac{X^{x+y}-X^x}{y}&=(I-G_y)^{-1}\int_0^1 F_x(x+ty,(1-t)X^x+tX^{x+y})\;dt.\end{aligned}$$

Hence:

$$\frac{\partial X^x}{\partial x}=\lim_{y\to 0}\frac{X^{x+y}-X^x}{y}=(I-F_X(x,X^x))^{-1}F_x(x,X^x).\tag{3.3}$$

In fact $\lim_{y\to 0}(I-G_y)^{-1}=\lim_{y\to 0}\sum_{n=0}^{\infty}G_y^n=\sum_{n=0}^{\infty}F_X(x,X^x)^n=(I-F_X(x,X^x))^{-1}$. Finally we observe that 3.3 is equivalent to the thesis. □

Now we need some result to assure ourselves that the function $\mathbf{Y}(y_0,x)$ satisfies the hypothesis of Lemma 5. To do this let us start with a result regarding different estimates that we can obtain for Hölder function by evaluating them in four points:

Lemma 6. *Let A be a Banach space, let $g \in \mathcal{C}^\alpha([0,T],A)$ with $\alpha \in (0,1)$. Then for any $k \in [0,1]$:*

$$\|g(z) - g(y) - g(z') - g(y')\|_A \leq \|g\|_\alpha (|y'-y|^{\alpha k} + |z'-z|^{\alpha k})(|z'-y'|^{\alpha\bar{k}} + |z-y|^{\alpha\bar{k}}),$$

for every $y, z, y', z' \in [0,T]$ and $\bar{k} = 1-k$.

Proof. Let us consider $y, z, y', z' \in [0,T]$, then:

$$\begin{aligned}|g(z) - g(y) - g(z') + g(y')| &\leq \|g(y') - g(y)\|_A + \|g(z') - g(z)\|_A \\ &\leq \|g\|_\alpha |y'-y|^\alpha + \|g\|_\alpha |z'-z|^\alpha \\ &\leq \|g\|_\alpha (|y'-y|^\alpha + |z'-z|^\alpha).\end{aligned}$$

Then elevating to k and by proposition 11 we obtain:

$$\begin{aligned}|g(z) - g(y) - g(z') + g(y')|^k &\leq \|g\|_\alpha^k (|y'-y|^\alpha + |z'-z|\,\alpha)^\gamma \\ &\leq \|g\|_\alpha^k (|y'-y|^{\alpha k} + |z'-z|^{\alpha k}).\end{aligned}$$

Moreover:

$$\begin{aligned}|g(z) - g(y) - g(z') + g(y')| &\leq |g(z) - g(y)| + |g(y') - g(z')| \\ &\leq \|g\|_\alpha (|z-y|^\alpha + |y'-z'|^\alpha).\end{aligned}$$

Then elevating to $\bar{k}$ and using proposition 11 we obtain:

$$|g(z) - g(y) - g(z') + g(y')|^{\bar{k}} \leq \|g\|_\alpha^{\bar{k}} (|z-y|^{\alpha\bar{k}} + |y'-z'|^{\alpha\bar{k}}).$$

Then multiplying the two inequalities we finally obtain the thesis. □

We now use a simplified version of the "Omega Lemma" from [CL18].

Lemma 7 (Omega Lemma)**.** *Let $\alpha \in (0,1), \gamma \in (0,1), k \in (0,1)$ and $f \in \mathcal{C}^{1,\gamma}(B, \mathcal{L}(A,B))$. Let us set $F(y) = (f(y_t)_{t\in[0,T]})$ then $F(y) \in \mathcal{C}^{1,(1-k)\gamma}$ from any ball of radius $\rho > 0$ of $\mathcal{C}^\alpha([0,T],B)$ to $\mathcal{C}^\beta([0,T],\mathcal{L}(A,B))$ where $\beta = \alpha k\gamma$. In addition $D_yF(y)\cdot h = (Df(y_t)\cdot h_t)_{t\in[0,T]}$ belongs to $\mathcal{C}^\beta([0,T],B)$.*

Proof. Call $\mathcal{B} = \mathcal{C}^\beta([0,T],\mathcal{L}(A,B)), \mathcal{A} = \mathcal{C}^\alpha([0,T],B)$ Firstly let us show that $D_yF(y)\cdot h \in \mathcal{B}$, for every $h \in \mathcal{A}$. We have:

$$\begin{aligned}&\|Df(y_s)\cdot h_s - Df(y_t)\cdot h_t\|_B = \\ &= \|Df(y_s)\cdot h_s - Df(y_t)\cdot h_s + Df(y_t)\cdot h_s - Df(y_t)\cdot h_t\|_B \\ &\leq \|Df(y_s)\cdot h_s - Df(y_t)\cdot h_s\|_B + \|Df(y_t)\cdot h_s - Df(y_t)\cdot h_t\|_B \\ &\leq \|Df\|_\gamma \|y_t - y_s\|_A^\gamma \|h\|_\alpha + \|Df\|_\infty \|h_t - h_s\|_A \\ &\leq \|Df\|_\gamma \|y\|_\alpha^\gamma \|h\|_\alpha |t-s|^{\alpha\gamma} + \|Df\|_\infty \|h\|_\alpha |t-s|^\alpha \\ &= (\|Df\|_\gamma \|y\|_\alpha^\gamma + |t-s|^{\alpha\gamma(1-k)} + \|Df\|_\infty |t-s|^{\alpha(1-k\gamma)}) \|h\|_\alpha |t-s|^{\alpha k\gamma} \\ &\leq (\|Df\|_\gamma \|y\|_\alpha^\gamma + T^{\alpha\gamma(1-k)} + \|Df\|_\infty T^{\alpha(1-k\gamma)}) \|h\|_\alpha |t-s|^{\alpha k\gamma}.\end{aligned}$$

Since $\alpha k\gamma = \beta$ the thesis holds.
Now let us show that $D_yF(y) \in \mathcal{C}^{(1-k)\gamma}_{loc}(\mathcal{A}, \mathcal{L}(\mathcal{A}, \mathcal{B})$. Let $\rho > 0$ and let us consider y, z such that $\|y\|_\alpha, \|z\| < \rho$ then we want to show that for every $h \in \mathcal{A}$ we have:

$$\|(D_yF(y) - D_yF(z)) \cdot h\|_\beta \leq C \|y - z\|_\alpha^{(1-k)\gamma} \|h\|_\alpha ,$$

where $C > 0$ is a constant that does not depend on h. Then we have:

$$\|(D_yF(y) - D_yF(z)) \cdot h\|_\infty \leq \|Df\|_\gamma \|y - z\|_\infty^\gamma \|h\|_\alpha \leq \|Df\|_\gamma (2\rho)_\alpha^\gamma \|h\|_\alpha .$$

On the other hand:

$$\begin{aligned}
&|(Df(y_t) - Df(z_t)) \cdot h_t - (Df(y_s) - Df(z_s)) \cdot h_s| \leq \\
&\leq |(Df(y_t) - Df(z_t)) \cdot h_t - (Df(y_s) - Df(z_s)) \cdot h_t| + \\
&\quad + |(Df(y_s) - Df(z_s)) \cdot h_t - (Df(y_s) - Df(z_s)) \cdot h_s|
\end{aligned}$$

Now let us estimate the two terms separately.
For the first we have:

$$\begin{aligned}
&|(Df(y_t) - Df(z_t)) \cdot h_t - (Df(y_s) - Df(z_s)) \cdot h_t| \leq \\
&\leq \|Df\|_\gamma (|y_t - y_s|^{k\gamma} + |z_t - z_s|^{k\gamma})(|y_t - z_t|^{(1-k)\gamma} \\
&\leq \|Df\|_\gamma (4(\|y\|_\alpha^{k\gamma} + \|z\|_\alpha^{k\gamma})T^{\alpha k\gamma} \|y - z\|_\alpha^{(1-k)\gamma}) \\
&\leq \|Df\|_\gamma (8\rho^{k\gamma})T^{\alpha k\gamma} \|y - z\|_\alpha^{(1-k)\gamma} .
\end{aligned}$$

Where we used Lemma 6.
For the second term we have:

$$\begin{aligned}
|(Df(y_s) - Df(z_s)) \cdot h_t - (Df(y_s) - Df(z_s)) \cdot h_s| &\leq \|Df\|_\gamma |y_s - z_s|^\gamma |h_t - h_s| \\
&\leq \|Df\|_\gamma (2\rho)^{k\gamma} \|h\|_\alpha T^{\alpha k\gamma} \|y - z\|^{(1-k)\gamma} .
\end{aligned}$$

Then we are able to find a constant $C > 0$, that does not depend on h, such that:

$$\|(D_yF(y) - D_yF(z)) \cdot h\|_\beta \leq C \|y - z\|_\alpha^{(1-k)\gamma} \|h\|_\alpha \tag{3.4}$$

yelding the thesis.
Finally let us how that the derivative exists, in other words let us prove that:

$$\|F(y + h) - F(y) - D_yF(y) \cdot h\|_\beta = o(\|h\|_\alpha).$$

We have that:

$$\begin{aligned}
&\|f(y_t + h_t) - f(y_t) - Df(y_t) \cdot h_t - f(y_s + h_s) + f(y_s) + Df(y_s) \cdot h_s\|_B = \\
&= \left\| \int_0^1 \{(Df(y_t + \tau h_t) - Df(y_t)) \cdot h_t - (Df(y_s + \tau h_s) - Df(y_s)) \cdot h_s\} \, d\tau \right\|_B \\
&\leq \int_0^1 |(Df(y_t + \tau h_t) - Df(y_t))h_t - (Df(y_s + \tau h_s) - Df(y_s))h_s| \, d\tau \\
&\leq \int_0^1 |Df(y_t + \tau h_t) - Df(y_t) + Df(y_s + \tau h_s) - Df(y_s)| \|h\|_\alpha \, d\tau + \\
&\quad + \int_0^1 |Df(y_s + \tau h_s) - Df(y_s)| |h_t - h_s| \, d\tau.
\end{aligned}$$

Now let us study the two terms independently. For the first term, by applying Lemma 6 we have:

$$\int_0^1 |Df(y_t+\tau h_t) - Df(y_t) + Df(y_s+\tau h_s) - Df(y_s)| \, \|h\|_\alpha \; d\tau \leq$$
$$\leq \int_0^1 \|Df\|_\gamma \, (|y_t + \tau h_t - y_s - \tau h_s)| + |y_t - y_s|)^{\gamma k} (|\tau h_t| + |\tau h_s|)^{\gamma(1-k)} \, \|h\|_\alpha \; d\tau$$
$$\leq K_1 \, \|h\|_\alpha^{(1-k)\gamma+1} .$$

For a suitable constant $K_1 > 0$.
For the second term we have

$$\int_0^1 |Df(y_s \tau h_s) - Df(y_s)| \, |h_t - h_s| \; d\tau \leq$$
$$\leq \int_0^1 \|Df\|_\gamma \, |\tau h_s| \, \|h\|_\alpha \, |t-s|^\alpha \; d\tau \leq K_2 \, \|h\|_\alpha^{1+\gamma} .$$

Then, putting together the pieces, we have obtained (3.4) with $o(\|h\|_\alpha^{1+(1-k)\gamma})$. □

Let us consider the map $M_{T,y_0}(y) = M_T(y_0, y)$ define in Theorem 6. This map has a unique fixed point $\mathbf{Y}(y_0, x)$ then if $M_T(y_0, x)$ satisfies the hypothesis of Lemma 5, the Fréchet derivative of $\mathbf{Y}(y_0, x)$ would satisfie:

$$\frac{\partial \mathbf{Y}}{\partial y_0}(y_0, x) = \frac{\partial M_T}{\partial y_0}(y_0, \mathbf{Y}(y_0, x)) + \frac{\partial M_T}{\partial y}(y_0, \mathbf{Y}(y_0, x)\frac{\partial \mathbf{Y}}{\partial y_0}(y_0, x).$$

On the other hand we have:

- $\frac{\partial M_T}{\partial y_0}(y_0, \mathbf{Y}(y_0, x)) = \frac{\partial}{\partial y_0}(y_0 + \int_0^\cdot f(y(s)) \; dx(s))|_{y=\mathbf{Y}(y_0,x)} = I$
- By Lemma 7 and since the Young integral map $z \mapsto \int_0^t z(s) \; dx(s)$ is linear and smooth, the derivative $\frac{\partial}{\partial y} M_T(y_0, \cdot)$ is $(1-k)\gamma$ - Hölder.

It remains to verify that $M_T(y_0, \cdot)$ is a contraction. By theorem 6 $M_T(\cdot, y)$ is a contraction for a suitable $\tilde{T} > 0$. So that, by Lemma 5, $\mathbf{Y}(y_0, x)$ admits Frêchet derivative at least for a suitable ball of $\mathcal{C}^\alpha([0, \tilde{T}], B)$. By the same argument used in theorem 6 one can extend the result by gluing the derivative as done for the solution to (3.1).
Let us now focus on the equation satisfied by the Fréchet derivative of $\mathbf{Y}(y_0, x)$:

$$\frac{\partial \mathbf{Y}}{\partial y_0}(y_0, x) = \frac{\partial M_T}{\partial y_0}(y_0, \mathbf{Y}(y_0, x)) + \frac{\partial M_T}{\partial y}\left(y_0, \mathbf{Y}(y_0, x)\right)\frac{\partial \mathbf{Y}}{\partial y_0}(y_0, x).$$

From what we already noticed, $\frac{\partial M_T}{\partial y}$ is, by Lemma 7, $(1-k)\gamma$ - Hölder, then $\frac{\partial \mathbf{Y}(y_0,x)}{\partial y_0}$ can only be δ - Hölder with $\delta \leq (1-k)\gamma$. In fact this is given by the following proposition:

Proposition 15. *Let $f \in \mathcal{C}^\alpha([0,T], A), g \in \mathcal{C}^\beta([0,T], A)$, then $f+g$ is $\{\delta\}$-Hölder, with $\delta \leq \min\{\alpha, \beta\}$.*

Proof.

$$\begin{aligned}|f(t)+g(t)-f(s)-g(s)| &\le \|f\|_\alpha |t-s|^\alpha + \|g\| |t-s|^\beta \\ &\le (\|f\|_\alpha T^{\alpha-\delta} + \|g\| T^{\beta-\delta}) |t-s|^\delta .\end{aligned}$$

□

Moreover let us now differentiate the equation that defines $\mathbf{Y}(y_0, x)$ then we obtain:

$$\begin{aligned}\frac{\partial}{\partial y_0}\mathbf{Y}(y_0,x) &= \frac{\partial}{\partial y_0}(y_0 + \int_0^t f(\mathbf{Y}_s(y_0,x))\ dx(s) \\ &= I + \int_0^t Df(\mathbf{Y}_s(y_0,x))\frac{\partial}{\partial y_0}\mathbf{Y}_s(y_0,x)\ dx(s).\end{aligned}$$

Then we obtain another formulation for the Fréchet derivative of $\mathbf{Y}(y_0, x)$.
In conclusion we have obtained that $\frac{\partial \mathbf{Y}(y_0,x)}{\partial y_0}$, if exists, satisfies the differential problem given above and has δ- Hölder regularity with $\delta \le (1-k)\gamma$.

Chapter 4

Regularity of Young integrals depending on a Parameter

From the last chapter we know that the Fréchet derivative with respect to the intial datum of the Itō map previously defined is δ-Hölder with $\delta \leq (1-k)\gamma$. In this chapter we partially investigate if this condition is sharp. To do this let us recall the equation satisfied by $\frac{\partial}{\partial y_0}\mathbf{Y}(y_0,x)$:

$$\frac{\partial}{\partial y_0}\mathbf{Y}(y_0,x) = I + \int_0^t Df(\mathbf{Y}_s(y_0,x))\frac{\partial}{\partial y_0}\mathbf{Y}_s(y_0,x)\ dx.$$

If we set $H(y_0,s) = Df(\mathbf{Y}_s(y_0,x))\frac{\partial}{\partial y_0}\mathbf{Y}_s(y_0,x)$, the problems becomes the regularity of the integral:

$$\int_0^t H(y_0,s)\ dx(s),$$

for δ fixed. Indeed if $H \in \mathcal{C}^\delta$ with respect to the two variables and $x \in \mathcal{C}^\alpha([0,T],A)$ we have, for $y,y' \in B$:

$$\left|\int_0^t H(y,s)\ dx(s) - \int_0^t H(y',s)\ dx(s)\right| \leq K \left\|H(y,.) - H(y',.)\right\|_{k\delta} \left\|x\right\|_\alpha,$$

by the definition of the Young Integral.If $k \in (0,1)$ such that $\alpha + k\delta > 1$, by Proposition 6 of the previous chapter we have:

$$|H(y,s) - H(y',s) - H(y,s') + H(y',s')| \leq 2\left\|H\right\|_\delta |y-y'|^{(1-k)\delta}\, 2\,|s-s'|^{k\delta},$$

so that $|H(y,.) - H(y',.)|_{k\delta} \leq 4\left\|H\right\|_\gamma |y-y'|^{(1-k)\gamma}$.
This shows that $y \mapsto \int_0^t H(y,s)\ dx(s)$ is $(1-k)\delta$- Hölder.

In the following section we will introduce the Weierstrass functions that will gave us an interesting example that suggests the sharpness of this condition. However this in a sull result since H will be constructed ad hoc, not of the type $H(y_0,s) = Df(\mathbf{Y}_s(y_0,x))\frac{\partial}{\partial y_0}\mathbf{Y}_s(y_0,x)$.
Before going on we thank Eugene Stepanov and Roger Züst that gave us the idea we developed.

4.1 Hölder-continuity of Weierstrass functions

Definition 4. We say that a function f is a Weierstrass function if f is of the following form:

$$f = \sum_{n=1}^{\infty} a^n \cos(b^n t) \quad t \in (R) \tag{4.1}$$

for some $a, b > 0$.

Now we want to prove that a Weierstrass function is Hölder continuous for a certain $\alpha \in (0,1)$. The following result assures us this condition.

Lemma 8. *Let $0 < a < 1, b > 1$ and $ab > 1$, let f be a Weierstrass function with a, b as above and let $f_N = \sum_{n=1}^{N} a^n \cos(b^n t)$. Then:*

- *f is $-\log_b a$ Hölder continuous*
- $\sup_{t\in\mathbb{R}} |f(t) - f_N(t)| \to 0$ *as* $N \to \infty$
- *Let us set $\gamma = -\log_b a$, then $\|f_N\|_\gamma \leq \|f\|_\gamma$ and for every $\varepsilon \in (0,\gamma)$ $\|f_N - f\|_{\gamma-\varepsilon} \to 0$ as $N \to \infty$*

Proof. Let $t \in \mathbb{R}, h \in (-1,1)$ and let us compute

$$\begin{aligned} f(t+h) - f(h) &= \sum_{n=1}^{\infty} a^n \left(\cos(b^n(t+h)) - \cos(b^n t)\right) \\ &= -2\sum_{n=1}^{\infty} a^n \sin\left(\frac{1}{2}b^n(2t+h)\right) \sin\left(\frac{1}{2}b^n h\right). \end{aligned}$$

Where we used in the second equality the Prosthaphaeresis formulas:

$$\cos(a) - \cos(b) = -2\sin\left(\frac{1}{2}(a+b)\right)\sin\left(\frac{1}{2}(a-b)\right).$$

Since $b > 1, h \in (-1,1)$ there exists $m \in \mathbb{N}$ such that $b^{m-1}|h| \leq 1 < b^m |h|$. So:

$$\begin{aligned} |f(t+h) - f(t)| &\leq 2\sum_{n=1}^{\infty} \left| a^n \sin\left(\frac{1}{2}b^n(2t+h)\right) \sin\left(\frac{1}{2}b^n h\right) \right| \\ &\leq 2\sum_{n=1}^{\infty} \left| a^n \sin\left(\frac{1}{2}b^n h\right) \right| \\ &= 2\sum_{n=1}^{m} \left| a^n \sin\left(\frac{1}{2}b^n h\right) \right| + 2\sum_{n=m+1}^{\infty} \left| a^n \sin\left(\frac{1}{2}b^n h\right) \right|. \end{aligned}$$

Note that the first term on the right hand side is a finite sum while the second, since $0 < a < 1$ can be estimated with the tail of a convergent geometric series, in other words:

$$\begin{aligned}|f(t+h) - f(t)| &\leq 2\sum_{n=1}^{m} \left| a^n \sin\left(\frac{1}{2}b^n h\right)\right| + 2\sum_{n=m+1}^{\infty} a^n \\ &= \frac{ab\,|h|}{ab-1}a^m b^m + \frac{2a}{1-a}a^m.\end{aligned}$$

Since $b^{m-1}|h| \leq 1 < b^m |h|$ and $0 < a < 1$, it follows that $b^m |h| < b$ and $a^m \leq |h|^{-\log_b a}$. In fact

$$a^m < |h|^{-\log_b a} \iff m\log_b a < (-\log_b a)(\log_b |h|)$$

And the last inequality holds true whenever $m > -\log_b |h|$, that is equivalent to $b^m |h| > 1$.
So:

$$\begin{aligned}|f(t+h) - f(t)| &\leq \frac{ab\,|h|}{ab-1}a^m b^m + \frac{2a}{1-a}a^m \\ &\leq \frac{ab^2}{ab-1}a^m + \frac{2a}{1-a}a^m \\ &\leq (\frac{ab^2}{ab-1} + \frac{2a}{1-a})\,|h|^{-\log_b a} \\ &= \frac{ab^2 - a^2b^2 + 2a^2b - 2a}{(ab-1)(1-a)}\,|h|^{-\log_b a}.\end{aligned}$$

Now note that

$$|f(t)| \leq \sum_{n=1}^{\infty} a^n = M_a < \infty.$$

Now take $t, s \in \mathbb{R}, |t-s| > 1$ by the Prostaphaeresis formulas:

$$\begin{aligned}|f(t+h) - f(t)| &\leq \left|2\sum_{n=1}^{\infty} a^n \sin\left(\frac{1}{2}b^n(t+s)\right)\sin\left(\frac{1}{2}b^n(t-s)\right)\right| \\ &\leq 2\left(\sum_{n=1}^{\infty} a^n\right)|t-s|^{-\log_b a} = M_a\,|t-s|^{-\log_b a}.\end{aligned}$$

Where we used that $|t-s| > 1$ implies $|t-s|^{-log_b a} > 1 > \sin(x)$,$\forall x \in \mathbb{R}$ and finally we can conclude that f is Hölder continuos.
For the last two assertions, firstly note that $f(x) - f_N(x)$ is the tail of a convergent series and this implies that

$$\sup_{t\in\mathbb{R}} |f(t) - f_N(t)| \to 0$$

as $N \to \infty$.
Moreover, let γ, ε be as in the statement of the Lemma, by the definition of f_N it follows that $\|f_N\|_\gamma \leq \|f\|_\gamma$. We argue by interpolation that:

$$\begin{aligned}
\|f_N - f\|_{\gamma-\varepsilon} &= \sup_{\substack{t,s\in\mathbb{R}\\ t\neq s}} \frac{|f(t) - f_N(t) - (f(s) - f_N(s))|}{|t-s|^{\gamma-\varepsilon}} \\
&= \sup_{\substack{t,s\in\mathbb{R}\\ t\neq s}} \frac{|f(t) - f_N(t) - (f(s) - f_N(s))|^{\frac{\gamma-\varepsilon}{\gamma}}}{|t-s|^{\frac{\gamma(\gamma-\varepsilon)}{\gamma}}} \cdot \\
&\quad \cdot \sup_{\substack{t,s\in\mathbb{R}\\ t\neq s}} |f(t) - f_N(t) - (f(s) - f_N(s))|^{\frac{\varepsilon}{\gamma}} \\
&\leq (2\|f\|_\gamma)^{\frac{\gamma-\varepsilon}{\gamma}} (2\sup_{t\in\mathbb{R}} |f(t) - f_N(t)|)^{\frac{\varepsilon}{\gamma}} \longrightarrow 0.
\end{aligned}$$

as $N \to \infty$ □

We will know consider modified Weierstrass function as functions of the type:

$$\sum_{n=1}^{\infty} \varepsilon_n a^n cos(b^n t),$$

where a, b are as in the previous theorem and $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence that takes values in $(0, 1)$.

Proposition 16. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \in (0, 1]$ then the series*

$$\sum_{n=1}^{\infty} \varepsilon_n a^n cos(b^n t)$$

with $0 < a < 1, b > 1$ and $ab > 1$ is still $-log_b a$- Hölder continuous.

Proof. Following the proof of Theorem 4.1 we obtain

$$\begin{aligned}
|f(t+h) - f(t)| &\leq 2\sum_{n=1}^{\infty} \left| \varepsilon_n a^n \sin\left(\frac{1}{2} b^n (2t+h)\right) \sin\left(\frac{1}{2} b^n h\right) \right| \\
&\leq 2\sum_{n=1}^{\infty} \left| a^n \sin\left(\frac{1}{2} b^n (2t+h)\right) \sin\left(\frac{1}{2} b^n h\right) \right|,
\end{aligned}$$

and the thesis holds by the same arguments used in the previous proof. □

At this point we can ask ourselves whether a Weierstrass function is differentiable at any point or not. Since Weierstrass introduced this function, there were many results regarding the differentiability of this kind of function, the sharpest was proved by Hardy in 1916 in [Har16].

Theorem 7. *For every $a, b \in \mathbb{R}$ such that $b \geq a > 1$ the function (4.1) is bounded and continuous in $\mathbb{R}$, but has no point of differentiability.*

4.2 Study of two particular functions

Now we introduce the following two functions

$$x(s) = \sum_{n=1}^{\infty} 2^{-n\alpha} \sin(2^n s)$$

$$H(y,s) = \sum_{n=1}^{\infty} \psi(y2^n) 2^{-n\gamma} \cos(2^n s)$$

where $\alpha, \gamma \in (0,1)$, $(y,s) \in \mathbb{R}^+ \times [0, 2\pi]$ and $\psi(x)$ is defined as

$$\psi(x) = \begin{cases} 2x - 2 & 1 \leq x \leq \frac{3}{2}, \\ 4 - 2x & \frac{3}{2} \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

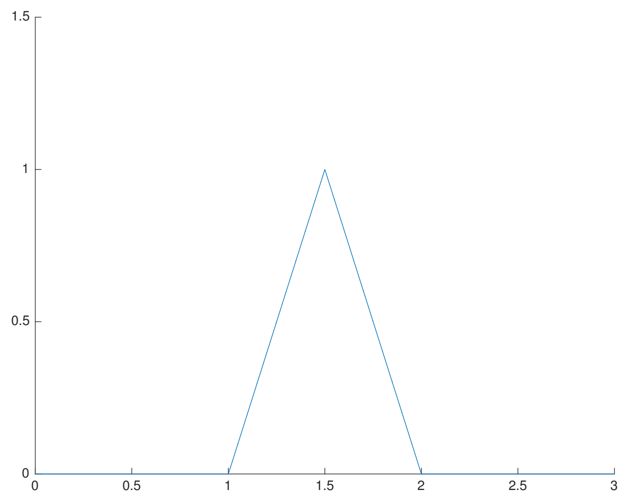

Figure 4.1: Plot of ψ for $x \in [0, 3]$

Firstly we notice that the two series define two functions since they are absolutely convergent for every $(y,s) \in \mathbb{R}^+ \times [0, 2\pi]$, in fact from $\alpha, \gamma \in (0,1)$ it follows that

$$|x(s)| \leq \sum_{n=1}^{\infty} 2^{-n\alpha} \leq \sum_{n=1}^{\infty} (2^{-\alpha})^n < \infty$$

$$|H(y,s)| \leq \sum_{n=1}^{\infty} 2^{-n\gamma} \leq \sum_{n=1}^{\infty} (2^{-\gamma})^n < \infty$$

since the series on the right hand side are two geometric series with ratio less than 1.

From what we already proved $x \in \mathcal{C}^{\alpha}([0, 2\pi], \mathbb{R})$ and $H(\cdot, s) \in \mathcal{C}_s^{\gamma}([0, 2\pi], \mathbb{R})$, since it is a modified Weierstrass function with $\varepsilon_n = \psi(y2^n)$ and it is γ-Hölder by proposition 16.

From now on we will show that $H \in \mathcal{C}_x^\gamma(\mathbb{R}^+, \mathbb{R})$. Primarily we compute:

$$\begin{aligned}
|H(y,s) - H(\bar{y},s)| &= \left| \sum_{n=1}^{\infty} \{\psi(y2^n) - \psi(\bar{y}2^n)\} 2^{-n\gamma} \cos(2^n s) \right| \\
&\leq \sum_{n=1}^{\infty} \left| \{\psi(y2^n) - \psi(\bar{y}2^n)\} 2^{-n\gamma} \cos(2^n s) \right| \\
&\leq \sum_{n=1}^{\infty} \left| \{\psi(y2^n) - \psi(\bar{y}2^n)\} 2^{-n\gamma} \right| \\
&= \sum_{n=1}^{\infty} |\psi(y2^n) - \psi(\bar{y}2^n)| \, 2^{-n\gamma}.
\end{aligned}$$

Now note that $\psi(x)$ is bounded by 1 and Lipschitz with Lipschitz constant 2. Remembering the definition of $H(y,s)$ we observe that there exist only one $n \in \mathbb{N}$ such that $\psi(y2^n) \neq 0$. In fact the condition $\psi(y2^n) \neq 0$ can be translated as $2^{-n} \leq y \leq 2^{-n+1}$ meaning that, for every fixed $y \in \mathbb{R}^+$, there exist a unique $n(y) \in \mathbb{N}$ such that $\psi(y2^{n(y)}) \neq 0$ since the intervals, for $n \neq m$, $[2^{-n}, 2^{-n+1}]$ and $[2^{-m}, 2^{-m+1}]$ are disjoint. From this, let us distinguish between two cases:

1. $n(y) = n(\bar{y}) = n$. Recalling the previous inequality we have, since we are in the case that $n(y) = n(\bar{y}) = n$, that
$$\begin{aligned}
|H(y,s) - H(\bar{y},s)| &\leq |\psi(y2^n) - \psi(\bar{y}2^n)| \, 2^{-n\gamma} \\
&= 2^{-n\gamma} |\psi(y2^n) - \psi(\bar{y}2^n)|^{1-\gamma} |\psi(y2^n) - \psi(\bar{y}2^n)|^{\gamma} \\
&\leq (2 \|\psi\|_\infty)^{1-\gamma} 2^{-n\gamma} 2^\gamma 2^{n\gamma} |y - \bar{y}|^\gamma \\
&= 2 \|\psi\|_\infty^{1-\gamma} |y - \bar{y}|^\gamma = K_1 |y - \bar{y}|^\gamma .
\end{aligned}$$
Where the constant K_1 does not depend on $y, \bar{y}$.

2. $n(y) \neq n(\bar{n})$ in this case notice that $\psi(y2^{n(\bar{y})}) = \psi(\bar{y}2^{n(y)}) = 0$ so that
$$\begin{aligned}
&|H(y,s) - H(\bar{y},s)| \leq \left| \psi(y2^{n(y)}) 2^{-n(y)\gamma} - \psi(\bar{y}2^{n(\bar{y})}) 2^{-n(\bar{y})\gamma} \right| \\
&= \left| \psi(y2^{n(y)}) 2^{-n(y)\gamma} - \psi(\bar{y}2^{n(y)}) 2^{-n(y)\gamma} + \psi(x2^{n(\bar{y})}) 2^{-n(\bar{y})\gamma} - \psi(\bar{y}2^{n(\bar{y})}) 2^{-n(\bar{y})\gamma} \right| \\
&\leq \left| \psi(y2^{n(y)}) - \psi(\bar{y}2^{n(y)}) \right| 2^{-n(y)\gamma} + \left| \psi(y2^{n(\bar{y})}) - \psi(\bar{y}2^{n(\bar{y})}) \right| 2^{-n(\bar{y})\gamma} \\
&= \left| \psi(x2^{n(y)}) - \psi(\bar{y}2^{n(y)}) \right|^{1-\gamma} \left| \psi(y2^{n(y)}) - \psi(\bar{y}2^{n(y)}) \right|^{\gamma} 2^{-n(y)\gamma} + \\
&\quad + \left| \psi(y2^{n(\bar{y})}) - \psi(\bar{y}2^{n(\bar{y})}) \right|^{1-\gamma} \left| \psi(y2^{n(\bar{y})}) - \psi(\bar{y}2^{n(\bar{y})}) \right|^{\gamma} 2^{-n(\bar{y})\gamma} \\
&\leq 4 \|\psi\|_\infty^{1-\gamma} |y - \bar{y}|^\gamma = K_2 |y - \bar{y}|^\gamma .
\end{aligned}$$
Where we obtained the last inequality using the same technique of the previous case. Moreover, the constant K_2 does not depend on $y, \bar{y}$.

Taking $K = \max\{K_1, K_2\}$ we finally obtain:

$$|H(y,s) - H(\bar{y},s)| \leq K |y - \bar{y}|^\gamma$$

Where K does not depend on $y, \bar{y}$, so we can conclude that $H(y,s) \in \mathcal{C}_x^\gamma(\mathbb{R}^+, \mathbb{R})$.

4.3 Integrating $H(y, s)$ against $x(s)$

In this section we integrate $H(y, s)$ against $x(s)$ in the sense of the Young Integral and we study the behaviour of such integral. To do this, by Young Integration theory, we need $\alpha, \gamma \in (0, 1)$ such that $\gamma + \alpha > 1$.

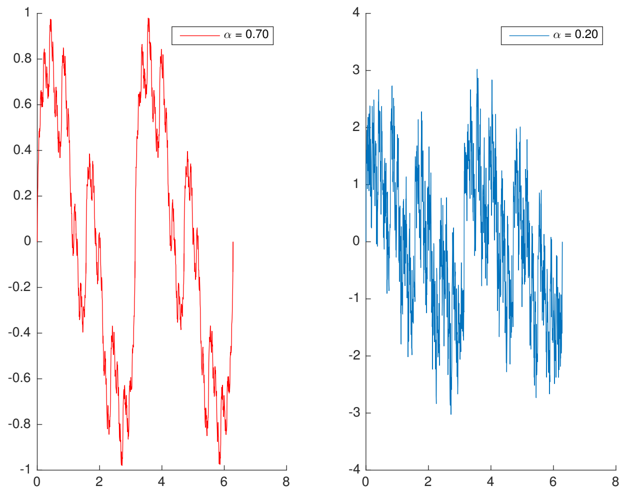

Figure 4.2: Plot of $x(s)$ for $\alpha = 0.20$ and $\alpha = 0.70$

To integrate the two functions let us first recall an integration formula that we can obtain by basic calculus facts.

Proposition 17.

$$\int_0^t \cos(mx)\cos(nx)\ dx = \begin{cases} \frac{\sin((m-n)x)}{2(m-n)} + \frac{\sin((m+n)x)}{2(m+n)}\Big|_0^t & m \neq n \\ \frac{x}{2} + \frac{\sin(2nx)}{4n}\Big|_0^t & m = n \end{cases}$$

Proof. Let us consider the following two cases:

- $m \neq n$

$$\int_0^t \cos(mx)\cos(nx)\ dx = \int_0^t \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]\ dx$$
$$= \frac{\sin((m-n)x)}{2(m-n)} + \frac{\sin((m+n)x)}{2(m+n)}\Big|_0^t .$$

- $m = n$

$$\int_0^t \cos(mx)\cos(nx)\ dx = \int_0^t \cos^2(nx)\ dx$$
$$= \int_0^t \frac{1 + \cos(2nx)}{2}\ dx$$
$$= \frac{x}{2} + \frac{\sin(2nx)}{4n}\Big|_0^t .$$

□

In particular one can observe that for $t = 2\pi$:

$$\int_0^{2\pi} \cos(mx)\cos(nx)\ dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

We are ready to integrate $H(y, s)$ against $x(s)$ from 0 to 2π and study the behaviour, with respect to the variable y, of such integral. Remembering that for every fixed y, there exists a unique $n(y)$ such that $y \in [2^{-n(y)}, 2^{-n(y)+1}]$, we obtain:

$$\int_0^{2\pi} H(y, s)\ dx(s) = \int_0^{2\pi} 2^{-n(y)\gamma} \cos(2^{n(y)} s)\ d(\sum_{n=1}^{\infty} 2^{-n\alpha} sin(2^n s))$$

Let us call

$$S_N(s) = \sum_{n=1}^{N} 2^{-n\alpha} sin(2^n s)$$

Since $\sin(s)$ is a $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ function and $S_N(s)$ is a finite sum of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ functions, then $S_N(s)$ is also a $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ function. For instance consider the integral of $H(x, s)$ against $S_N(s)$, by using the properties of Young's Integral we have:

$$\int_0^{2\pi} H(y, s)\ dS_N(s) = \int_0^{2\pi} 2^{-n(y)\gamma} \cos(2^{n(y)} s) \sum_{n=1}^{m} 2^{-n\alpha} 2^n \cos(2^n s)\ ds$$

Since the sums on the right hand side are finite and by the linearity of the integral we can write:

$$\begin{aligned} \int_0^{2\pi} H(y, s)\ dS_N(s) &= \sum_{n=1}^{m} \int_0^{2\pi} 2^{-n(y)\gamma} \cos(2^{n(y)} s) 2^{-n\alpha} 2^n \cos(2^n s)\ ds \\ &= \begin{cases} \pi 2^{-n(y)\gamma} 2^{n(y)(1-\alpha)} & N \geq n(y) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Where the last equality follows from proposition 17. If $m \geq n(y)$, remembering the choice of $n(y)$, it follows that $2^{-n(y)} \leq y$ and $2^{-n(y)} \geq \frac{y}{2}$ so that:

$$\int_0^{2\pi} H(y, s)\ dS_m(s) \approx y^{\gamma+\alpha-1}$$

for $y \to 0^+$. Now we need to prove that:

$$\lim_{m\to\infty} \int_0^{2\pi} H(y, s)\ dS_m(s) = \int_0^{2\pi} H(y, s)\ dx(s).$$

By proposition 7:

$$\lim_{m\to\infty} \int_0^{2\pi} H(y, s)\ dS_m(s) - \int_0^{2\pi} H(y, s)\ dx(s) = \lim_{m\to\infty} \int_0^{2\pi} H(y, s)\ d(S_m(s) - x(s)).$$

Then by the estimates we obtained in the second chapter, for a every $\varepsilon > 0$ such that $\alpha + \gamma - \varepsilon > 1$, since $\mathcal{C}^{\alpha} \subset \mathcal{C}^{\alpha-\varepsilon}$, we can find a constant $C > 0$ such that:

$$\left\| \int_0^{2\pi} H(y,s)\ d(S_m(s) - x(s)) \right\|_{\alpha-\varepsilon} \leq C \left\| S_m(s) - x(s) \right\|_{\alpha-\varepsilon}.$$

Moreover that quantity goes to zero by Theorem 4.1.
Finally we conclude that for $y \to 0^+$

$$\int_0^{2\pi} H(y,s)\ dg(s) \approx (y^{\gamma+\alpha-1}).$$

Since

$$\int_0^{2\pi} H(0,s)\ dx(s) = 0$$

the function

$$y \mapsto \int_0^{2\pi} H(y,s)\ dx(s)$$

cannot be δ-Hölder for $\delta > \gamma + \alpha - 1$.

Appendix A

Basic Facts on Banach Space

Let A be a real linear space.

Definition 5. A mapping $\|.\| : A \to [0, \infty)$ is called a norm if

1. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in A$,
2. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in A, \lambda \in \mathbb{R}$,
3. $\|x\| = 0$ if and only if $x = 0$.

Definition 6. We say a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converges to x, written $x_n \to x$ if

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

Definition 7. 1. A sequence $(x_n)_{n \in \mathbb{N}} \subset A$ is called a Cauchy sequence provided for each $\varepsilon > 0$ there exists $N > 0$ such that

$$\|x_n - x_m\| < \varepsilon \quad \text{for all } n, m \geq N.$$

2. X is complete if each Cauchy sequence in A converges, that is, whenever $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, there exists $x \in A$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x.
3. A Banach space is a compete, normed linear space.

Let A, B be Banach spaces.

Definition 8. A map $F : A \to B$ is a linear operator provided $F(\lambda x + \mu y) = \lambda F(x) + \mu F(y)$ for all $x, y \in A, \lambda, \mu \in \mathbb{R}$.

Definition 9. A linear operator $F : A \to B$ is bounded if the following norm it is finite:

$$\|F\|_{\mathcal{L}(A,B)} = \inf\{C \geq 0 : \|Ax\|_B \leq C \, \|x\|_A \text{ for all } x \in A\}.$$

Proposition 18. *A bounded linear operator $F : A \to B$ is continuous.*

Proof. Let $\delta < \frac{\varepsilon}{\|F\|_{\mathcal{L}(A,B)}}$ then for every $x, y \in A$ such that $\|x-y\|_A \leq \delta$ we have:

$$\|F(x)-F(y)\|_B \leq \|F(x-y)\|_B \leq \|F\|_{\mathcal{L}(A,B)} \|x-y\|_A < \varepsilon.$$

□

It is known that the viceversa of the proposition above holds thanks to some functional analysis results, that we do not include here.

Definition 10. Let A, B be two Banach spaces. $F : A \to B$ is called Fréchet differentiable at $x \in A$ if there exists a bounded linear operator $DF(x) : A \to B$ such that

$$\lim_{h\to 0} \frac{\|F(x+h)-F(x)-DF(x)\cdot h\|_B}{\|h\|_A} = 0.$$

We call $DF(x)$ the Fréchet derivative of F at x. A function F that is Fréchet differentiable for any points of A is said to be $\mathcal{C}^1(A,B)$ if the function:

$$\begin{aligned} DF :& A \to \mathcal{L}(A,B) \\ & x \mapsto DF(x) \end{aligned}$$

is continuous.

Note that requiring the continuity of the map $x \mapsto DF(x)$ is different from requiring that the map $DF(x)$ is continuous for every $x \in A$.

A.1 Hölder spaces are Banach spaces

Theorem 8. *Let A, B be two Banach spaces. The space $\mathcal{C}^{k,\gamma}(A,B)$ endowed with the norm:*

$$\|f\|_{k,\gamma} = \sum_{i\leq k} \left\|D^i f\right\|_\infty + \left|D^k f\right|_\gamma$$

is a Banach space.

Proof. It is clear that $\|f\|_{k,\gamma}$ ia a norm, and that the space $\mathcal{C}^{k,\gamma}(A,B)$ is lineare. We now prove that it is complete.
Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^{k,\gamma}(A,B)$, we want to show that there exists $u \in \mathcal{C}^{k,\gamma}(A,B)$ such that

$$\lim_{n\to\infty} u_n = u.$$

Note that the Hölder norm is the sum of the $\mathcal{C}^k$ norm and the Hölder seminorm. By the completeness of $\mathcal{C}^k(A,B)$ there exists $u \in \mathcal{C}^k(A,B)$ such that:

$$\lim_{n\to\infty} \|u_n - u\|_{\mathcal{C}^k} = 0.$$

We want to show that the limit u belongs to $\mathcal{C}^{k,\gamma}(A,B)$. To do this let us set:

$$v = D^i u, \ v_n = D^i u_n.$$

Then for $x \neq y$ we have:

$$\begin{aligned}
&\frac{|v(x)-v(y)|}{|x-y|^{\gamma}} = \\
&= \frac{|v(x)-v_n(x)+v_n(x)-v_n(y)+v_n(y)-v(y)|}{|x-y|^{\gamma}} \\
&\leq \frac{|v(x)-v_n(x)|}{|x-y|^{\gamma}} + \frac{|v_n(x)-v_n(y)|}{|x-y|^{\gamma}} + \frac{|v_n(y)-v(y)|}{|x-y|^{\gamma}}.
\end{aligned}$$

Hence the first and third term can be made small enough since $v_n \to v$ uniformly and the middle term can be made small enough since we assumed $(u_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence. Then for every $0 \leq i \leq k$ we can find a constant M_i such that

$$\frac{|v(x)-v(y)|}{|x-y|^{\gamma}} \leq M_i.$$

Then $u \in \mathcal{C}^{k,\gamma}(A,B)$. Now we need to show that the convergence is in $\mathcal{C}^{k,\gamma}(A,B)$. Let us consider the quantity:

$$\frac{|(v-v_m)(x)-(v-v_m)(y)|}{|x-y|^{\gamma}} =$$

And the last term can be made small enough since $(u_n)_{n\in\mathbb{N}}$ ia a Cacuhy sequence in $\mathcal{C}^{k,\gamma}(A,B)$. Then v_n converges to v in the order seminorm, since there are only finitely many indices i to consider, we can conclude that $u_n \to u$ in $\mathcal{C}^{k,\gamma}(A,B)$. □

Proposition 19. *$\mathcal{C}^{1,\gamma}(A,B)$ is a Banach space.*

Here we report a convergence result that links the uniform norm with the Hölder norm.

Lemma 9. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded equicontinuous sequence in $\mathcal{C}^{\beta}([0,T],A)$ with $\beta \in (0,1]$, then $u_n \to u$ in $\mathcal{C}^{\alpha}([0,T],A)$, for every $\alpha < \beta$.*

Proof. Let us assume, without loss of generality $u = 0$. By Ascoli - Arzelá theorem $u_n \to 0$ uniformly, in other words $\|u_n\|_{\infty} \to 0$ as $n \to \infty$. We want to show, assuming that $u_n \in \mathcal{C}^{\beta}([0,T],A)$ there is α - Hölder convergence for $\alpha < \beta$. We have, by interpolation

$$\begin{aligned}
\frac{|u_n(x)-u_n(y)|}{|x-y|^{\alpha}} &\leq \left(\frac{|u_n(x)-u_n(y)|}{|x-y|^{\beta}}\right)^{\frac{\alpha}{\beta}} |u_n(x)-u_n(y)|^{1-\frac{\alpha}{\beta}} \\
&\leq \|u_n\|_{\beta}^{\frac{\alpha}{\beta}} \left(2\|u_n\|_{\infty}\right)^{1-\frac{\alpha}{\beta}} \\
&\leq M^{\frac{\alpha}{\beta}} \left(2\|u_n\|_{\infty}\right)^{1-\frac{\alpha}{\beta}} \to 0
\end{aligned}$$

as $n \to \infty$ since $1-\frac{\alpha}{\beta} > 0$, where $M > 0$ is such that $\|u_n\|_{\beta} < M$. □

For completeness we provide an equivalent Hölder norm that is easier to use in many context, e.g. whenever we handle integral functions, since it coincides with the Hölder seminorm. The result is the following.

Proposition 20. *Let us consider the following two norms:*

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{x,y\in[0,T]\\ x\neq y}} \frac{|f(x)-f(y))|}{|x-y|^\alpha}$$
$$\|f\|_{\alpha\bullet} = |f(0)| + \sup_{\substack{x,y\in[0,T]\\ x\neq y}} \frac{|f(x)-f(y))|}{|x-y|^\alpha}.$$

Then $\|.\|_\alpha, \|.\|_{\alpha\bullet}$ *are equivalent on* $\mathcal{C}^\alpha([0,T],A)$*, where* A *is a Banach space.*

Proof. We need to show the following inequalities $\|f\|_\alpha \geq a\|f\|_{\alpha\bullet}$ and $\|f\|_\alpha \leq b\|f\|_{\alpha\bullet}$ holds true for some constant $a,b>0$. The first inequality is triviale since $|f(0)| \leq \|f\|_\infty$ then setting $a=1$ we have the thesis.
For the second inequality we have:

$$\begin{aligned}|f(x)| &= |f(x)-f(0)+f(0)| \leq |f(x)-f(0)| + |f(0)| \\ &\leq \|f\|_{\alpha\bullet}|x|^\alpha + |f(0)| \\ &\leq \|f\|_{\alpha\bullet}T^\alpha + |f(0)|.\end{aligned}$$

Then taking the sup over $[0,T]$ we have that $\|f\|_\infty \leq |f(0)| + \|f\|_\alpha T^\alpha$. Finally we obtain:

$$\begin{aligned}\|f\|_\alpha &\leq \|f\|_{\alpha\bullet}T^\alpha + |f(0)| + \sup_{\substack{x,y\in[0,T]\\ x\neq y}} \frac{|f(x)-f(y))|}{|x-y|^\alpha} \\ &= \|f\|_{\alpha\bullet}T^\alpha + \|f\|_{\alpha\bullet} = (1+T^\alpha)\|f\|_{\alpha\bullet} = b\|f\|_{\alpha\bullet}.\end{aligned}$$

□

Appendix B

Sewing Lemma: an abstract approach

In this section we show an abstract version of the Sewing Lemma that can be found in [FdLPM08]. Let us start with a definition.

Definition 11. We will say that a non decreasing function $V : [0, T] \to \mathbb{R}$ is a control function if $V(0) = 0$ and satisfies the Dini condition, namely

$$\bar{V}(t) = \sum_{n \geq 0} 2^n V\left(\frac{T}{2^n}\right) < \infty$$

for every $t \in [0, T]$.

Later we will use a particular control function to assure ourselves that the integral we want to define is finite. Let us start with some properties of this class of functions.

Proposition 21. *Le* $V : [0, T] \to \mathbb{R}$ *non decreasing such that* $V(0) = 0$. *The followings are equivalent:*

1. V *is a control function*
2. $\sum_{n \geq 1} V\left(\frac{T}{n}\right) < \infty$
3. $\sum_{n \geq 0} 2^n V\left(\frac{T}{2^n}\right) < \infty$
4. $\int_0^1 \frac{V(Ts)}{s^2}\, ds = \int_1^\infty V\left(\frac{T}{s}\right)\, ds < \infty$

Proof.

1) $\Rightarrow$ 3) By the definition of control function, $\sum_{n \geq 0} 2^n V\left(\frac{T}{2^n}\right) < \infty$ for every $t \in [0, T]$.

3) $\Rightarrow$ 1) Since V si non decreasing, for every $t \in [0, T]$ the following inequality holds true:

$$\sum_{n \geq 0} 2^n V\left(\frac{t}{2^n}\right) < \sum_{n \geq 0} 2^n V\left(\frac{T}{2^n}\right)$$

2) $\Leftrightarrow$ 3) Let us recall the following result:

Proposition 22 (Cauchy condensation test). *Given a non negative and non decreasing sequence$(a_n)_{n\in\mathbb{N}}$ the sum*

$$\sum_{n\in\mathbb{N}} a_n$$

converges if and only if converges the sum

$$\sum_{n\in\mathbb{N}} 2^n a_{2^n}.$$

Applying the test to

$$\sum_{n \geq 0} V\left(\frac{T}{n}\right) \quad \sum_{n \geq 0} 2^n V\left(\frac{T}{2^n}\right) < \infty$$

the thesis holds true.

3) $\Leftrightarrow$ 4) We have that:

$$\begin{aligned}
\int_1^\infty V\left(\frac{T}{s}\right)\, ds &= \sum_{n\geq 0} \int_{2^n}^{2^{n+1}} V\left(\frac{T}{s}\right)\, ds \\
&\leq \sum_{n\geq 0} \int_{2^n}^{2^{n+1}} V\left(\frac{T}{2^n}\right)\, ds \\
&= \sum_{n\geq 0} 2^n V\left(\frac{T}{2^n}\right) \\
&\leq V(T) + 2\left(\sum_{n\geq 0} \int_{2^n}^{2^{n+1}} V\left(\frac{T}{2^n}\right)\, ds\right) \\
&= V(T) + 2\int_0^\infty V\left(\frac{T}{s}\right)\, ds.
\end{aligned}$$

□

Proposition 23. *Given a control function V it holds:*

$$\lim_{t\to 0} \frac{\bar{V}(t)}{t} = 0.$$

Proof. Let $\varepsilon > 0$ and let $(a_n)_{n\in\mathbb{N}} \subset (0, T]$, $a_n \to 0$ such that:

$$\frac{V(a_n)}{a_n} \geq \varepsilon$$

We will also suppose, with no loss of generalities, that for every $n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \leq \frac{1}{2}$. By the monotony of V:

$$V(s) \geq V(a_{n+1}) \geq \varepsilon a_{n+1}$$

for $s \in [a_{n+1}, a_n]$. So:

$$\begin{aligned}
\int_0^1 \frac{V(Ts)}{s^2}\, ds &= \frac{1}{T}\int_0^T \frac{V(s)}{s^2}\, ds \\
&\geq \frac{1}{T}\sum_{n\geq 0}\int_{a_{n+1}}^{a_n} \frac{V(s)}{s^2}\, ds \\
&\geq \frac{\varepsilon}{T}\sum_{n\geq 0} a_{n+1}\int_{a_{n+1}}^{a_n} \frac{ds}{s^2} \\
&\geq \frac{\varepsilon}{T}\sum_{n\geq 0}\left(1\frac{a_n}{a_{n+1}}\right) \\
&\geq \frac{\varepsilon}{T}\sum_{n\geq 0}\frac{1}{2} > \infty.
\end{aligned}$$

And this is not possible by the last point of the previous proposition. □

Theorem 9 (Sewing Lemma 2)**.** *Let $\mu(a, b)$ be a continuous function defined for $0 \leq a \leq b < T$ satisfying the relation:*

$$|\mu(a, b) - \mu(a, c) - \mu(c, b)| \leq V(b - a)$$

for every $c \in [a, b]$, where V is a control function. Then there exists a unique function $\varphi(t)$ on $[0, T)$, up to an additive constant, such that:

$$|\varphi(b) - \varphi(a) - \mu(a, b)| \leq \bar{V}(b - a)$$

Proof. **Existence.** Let us put $\mu'(a, b) = \mu(a, c) + \mu(c, b)$ where $c = \frac{a+b}{2}$ and so we define $\mu^{(n+1)} = \mu^{(n)'}$. For $n \geq 0$ we obtain that:

$$\left|\mu^{(n)}(a, b) - \mu^{(n+1)}(a, b)\right| \leq 2^n V(\frac{|b - a|}{2^n}).$$

We will prove this fact by induction. For $n = 0$ the thesis holds true, in fact:

$$\left|\mu^{(0)}(a, b) - \mu^{(1)}(a, b)\right| = |\mu(a, b) - \mu(a, c) - \mu(c, b)| \leq V(|b - a|)$$

by hypothesis. Then let us suppose that the thesis holds true for $k \leq n-1$ and let us set $k = n$. We have that:

$$\begin{aligned}
\left|\mu^{(n)}(a,b) - \mu^{(n+1)}(a,b)\right| &= \left|\mu^{(n)}(a,b) - \mu^{(n)}(a,c) - \mu^{(n)}(b,c)\right| \\
&= \left|\mu^{(n-1)}(a,c) + \mu^{(n-1)}(c,b) - \mu^{(n)}(a,c) - \mu^{(n)}(c,b)\right| \\
&\leq \left|\mu^{(n-1)}(a,c) - \mu^{(n)}(a,c)\right| + \left|\mu^{(n-1)}(c,b) - \mu^{(n)}(c,b)\right| \\
&\leq 2^{n-1}\bar{V}\left(\frac{|c-a|}{2^n}\right) + 2^{n-1}\bar{V}\left(\frac{|b-c|}{2^n}\right) \\
&\leq 2^{n-1}\bar{V}\left(\frac{|b-a|}{2^n}\right) + 2^{n-1}\bar{V}\left(\frac{|b-a|}{2^n}\right) = 2^n\bar{V}\left(\frac{|b-a|}{2^n}\right).
\end{aligned}$$

Moreover by proposition 23 the sequence $(\mu^{(n)}(a,b))_{n\in\mathbb{N}}$ is a Cauchy sequence, in fact:

$$\left|\mu^{(n+1)}(a,b) - \mu^{(n)}(a,b)\right| \leq 2^n V\left(\frac{|b-a|}{2^n}\right) \to 0$$

as $n \to \infty$. So that $\mu^{(n)}(a,b) \to u(a,b)$ and the function $u(a,b)$ is midpoint additive, in other words $u(a,b) = u(a,c) + u(c,b)$, whenever $c = \frac{b+a}{2}$.

Uniqueness. By its definition we have that:

$$|u(a,b) - \mu(a,b)| \leq c\bar{V}(b-a).$$

This follows from the inequality:

$$\left|\mu^{(n)}(a,b) - \mu(a,b)\right| \leq \sum_{s=0}^{n-1}\left|\mu^{(s+1)}(a,b) - \mu^{(s)}(a,b)\right|$$

taking n to ∞ we have that:

$$\sum_{s=0}^{\infty}\left|\mu^{(s+1)}(a,b) - \mu^{(s)}(a,b)\right| \leq 2^n\bar{V}(\frac{|b-a|}{2^n}).$$

Let $v(a,b)$ be another midpoint additive function, then

$$|v(a,b) - \mu(a,b)| \leq \tilde{c}\bar{V}(b-a).$$

Moreover

$$|v(a,b) - u(a,b)| \leq 2^n V\left(\frac{|b-a|}{2^n}\right).$$

By induction:

$$|v(a,b) - u(a,b)| \leq 2^n K\bar{V}\left(\frac{b-a}{2^n}\right) \longrightarrow 0.$$

Continuity. Note that by its definition:

$$\mu^{(n)}(a,b) = \sum_{i=1}^{2^n}\mu\left(a + \frac{(i-1)(b-a)}{2^n}, a + \frac{i(b-a)}{2^n}\right).$$

In fact, by induction the equality holds true for $n = 0$ then, supposing the thesis holds for $k \leq n$, for $k = n+1$ we have:

$$\begin{aligned}
\mu^{(n+1)}(a,b) &= \mu^{(n)}\left(a, \frac{b+a}{2}\right) + \mu^{(n)}\left(\frac{b+a}{2}, b\right) \\
&= \sum_{i=1}^{2^n} \mu\left(a + \frac{(i-1)(\frac{b+a}{2} - a)}{2^n}, a + \frac{i(\frac{b+a}{2} - a)}{2^n}\right) \\
&+ \sum_{i=1}^{2^n} \mu\left(\frac{b+a}{2} + \frac{(i-1)(b - \frac{b+a}{2})}{2^n}, \frac{b+a}{2} + \frac{i(b - \frac{b+a}{2})}{2^n}\right) \\
&= \sum_{i=1}^{2^n} \mu\left(a + \frac{(i-1)(b-a)}{2^{n+1}}, a + \frac{i(b-a)}{2^{n+1}}\right) \\
&+ \sum_{i=1}^{2^n} \mu\left(\frac{(b+a)2^n + (i-1)(b-a)}{2^{n+1}}, \frac{(b+a)2^n + i(b-a)}{2^{n+1}}\right).
\end{aligned}$$

Let us study the second sum, setting $j = i + 2^n$ we have:

$$\begin{aligned}
&\sum_{j=2^n+1}^{2^{n+1}} \mu\left(\frac{(b+a)2^n + (j-2^n-1)(b-a)}{2^{n+1}}, \frac{(b+a)2^n + (j-2^n)(b-a)}{2^{n+1}}\right) = \\
&= \sum_{j=2^n+1}^{2^{n+1}} \mu\left(\frac{(b+a)2^n + (j-1)(b-a) - 2^n(b-a)}{2^{n+1}}, \frac{(b+a)2^n + j(b-a) - 2^n(b-a)}{2^{n+1}}\right) \\
&= \sum_{j=2^n}^{2^{n+1}} \mu\left(a + \frac{(j-1)(b-a)}{2^n}, a + \frac{j(b-a)}{2^n}\right).
\end{aligned}$$

And the equality holds true adding the two pieces. Now let us consider the quantity:

$$\sup_{a,b\in[0,T]} \left|u(a,b) - \mu^{(n)}(a,b)\right| = \sup_{a,b\in[0,T]} \left|\sum_{i=2^n+1}^{\infty} \mu\left(a + \frac{(i-1)(b-a)}{2^n}, a + \frac{i(b-a)}{2^n}\right)\right|.$$

And the term on the right hand side goes to 0 as $n \to \infty$ since it is the tail of a convergent series. Since the series that defines u converges uniformly and μ is continuous by hypothesis, then u is continuous.
Additivity of u We prove that for every $c \in [a,b]$

$$u(a,b) - u(a,c) - u(c,b) = 0.$$

Let $k \geq 3$ be an integer and let:

$$w(a,b) = \sum_{i=0}^{k-1} u(t_i, t_{i+1})$$

where $t_i = a + i\frac{(b-a)}{k}$. Then also w is midpoint additive, and by induction and using the assumption on μ

$$|\mu(a,b) - w(a,b)| \leq 2^n \bar{V}\left(\frac{|b-a|}{2^n}\right)$$

hence $w = u$.
At this point we have proved that u is additive on rational points, moreover let us consider $(c_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $c_n \to c$ as $n \to \infty$, then:

$$u(a,b) = u(a,c_n) + u(c_n,b),$$

and taking the limit, by the continuity of u:

$$u(a,b) = u(a,c) + u(c,b)$$

for every $0 \leq a \leq b \leq c \leq T$.
So u is additive and finally we put $\varphi(t) = u(0,t)$. □

Note that the function $V(t) = |t|^\gamma$, used in constructing the Young Integral, is a control function whenever $\gamma > 1$ in fact $V(0) = 0$ and

$$\bar{V}(T) = \sum_{n\geq 0} 2^n \frac{T^\gamma}{2^{\gamma n}} < \infty$$

for every $T > 0$, since $\gamma > 1$.

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