Mathematical Aspects of Quantum Information Theory:

Lecture 3

Dario Trevisan

Università di Pisa
dario.trevisan@unipi.it

Plan

1. Quantum Channels (conclusion)
    - CP maps on C^*-algebras
    - Quantum Markov semigroups
2. Inequalities
    - Uncertainty inequalities
    - Monotonicity inequalities
    - Lieb's concavity theorem
    - Exercises

Quantum channels

A completely positive, trace preserving (CPTP) map $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ is also called a quantum channel from the system H to K.

Theorem (Kraus representation of quantum channels)

Let H, K be finite dimensional quantum systems. Any quantum channel $\Phi^\dagger$ from H to K can be represented via a family of Kraus operators:

$$\Phi^\dagger(A) = \sum_{x \in \mathcal{X}} B_x A B_x^* \quad \text{for every } A \in \mathcal{L}(H).$$

One has $|\mathcal{X}| \leq \mathrm{dim}(H)\mathrm{dim}(K)$.

Strategy: factorize the dual map $\Phi : \mathcal{L}(K) \to \mathcal{L}(H)$ via an auxiliary system $\tilde{H}$:

$$\Phi(A) = U^* \pi(A) U,$$

where U is an isometry and π is a $*$-homomorphism.

Proof (sketch)

Stinespring representation

Up to isomorphism, one can let

1. $\tilde{H} = K \otimes \mathbb{C}^{\mathcal{X}}$,
2. $\pi(A) = A \otimes \mathbb{1}_{\mathbb{C}^{\mathcal{X}}}$,
3. $U |\psi\rangle = |\psi\rangle \otimes |0\rangle$.

This yields the Stinespring representation of the quantum channel:

$$\Phi^\dagger(\rho) = \mathrm{tr}_{\mathbb{C}^{\mathcal{X}}}[V(\rho \otimes |0\rangle\langle 0|) V^*],$$

where $V : H \otimes \mathbb{C}^{\mathcal{X}} \to K \otimes \mathbb{C}^{\mathcal{X}}$ is unitary.

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Complete positivity on C^*-algebra

- A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is CP if

$$\sum_{i,j=1}^{d} b_i^* \Phi(a_i^* a_j) b_j$$

  is positive for every $(a_i)_{i=1}^d \subseteq \mathcal{A}$, $(b_i)_{i=1}^d \subseteq \mathcal{B}$, $d \geq 1$.

- It coincides with the previous notion when $\mathcal{A} = \mathcal{L}(H)$, $\mathcal{B} = \mathcal{L}(K)$.
- Φ is unital if $\Phi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$.
- A state $\eta : \mathcal{A} \to \mathbb{C}$ defines a CP unital map:

Repeating the proof of the Kraus representation theorem (with some technicalities because of infinite dimensions!) yields the following:

Theorem (Stinespring dilation)

Let $\mathcal{A}$ be a C^*-algebra and H be a Hilbert space. Given any CP unital map $\Phi : \mathcal{A} \to \mathcal{B}(H)$, there exist

1. *a Hilbert space $\tilde{H}$,*
2. *an isometry $U : H \to \tilde{H}$,*
3. *a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\tilde{H})$,*

such that, for every $a \in \mathcal{A}$,

$$\Phi(a) = U^* \pi(a) U,$$

and

$$\{\pi(a)U\psi \ : \ a \in \mathcal{A}, \psi \in H\} \subseteq \tilde{H} \quad \text{is dense.}$$

Such a triple $(\tilde{H}, U, \pi)$ is unique up to isomorphisms.

The GNS construction

When applied to $\Phi(a) = \eta(a)$ for a state $\eta : \mathcal{A} \to \mathbb{C}$ it yields the following

Theorem (Gelfand-Naimark-Segal)

Let $\mathcal{A}$ be a C^*-algebra and let $\eta : \mathcal{A} \to \mathbb{C}$ be a state. Then, there exists

1. *a Hilbert space H,*
2. *a unit norm vector $|\psi\rangle \in H$,*
3. *and a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$*

such that, for every $a \in \mathcal{A}$,

$$\eta(a) = \langle\psi|\pi(a)\psi\rangle \,,$$

and $\{\pi(a)\,|\psi\rangle\}_{a\in\mathcal{A}} \subseteq H$ is dense.
Such a triple is unique up to isomorphisms.

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- By composing a quantum channel Φ from H into itself, one obtains the analogue of a Markov chain evolution

$$\rho_0, \Phi(\rho_0), \Phi^2(\rho), \Phi^3(\rho), \ldots.$$

- For $\Phi(\rho) = U\rho U^*$ with U unitary, it is the analogue of a discrete-time dynamical system.

- Continuous-time dynamics are described by quantum Markov semigroups $(\Phi^t)_{t\geq 0}$:
  1. for every $t \geq 0$, Φ^t is a quantum channel from H into itself,
  2. (semigroup law) for every $s, t \geq 0$, $\Phi^t\Phi^s = \Phi^{s+t}$,
  3. (strong continuity) for every $A \in \mathcal{L}(H)$, $t \mapsto \Phi^t(A)$ is continuous.
- The generator L is defined as:

- If H is finite-dimensional, L is a bounded operator with an explicit representation (the Lindblad form).
- Stone's theorem describes the generators of semigroups induced by unitary maps:

$$L(A) = -i[H, A]$$

for a suitable Hamiltonian H (self-adjoint but possibly unbounded).

Plan

- Recall that that two compatible measurements can be performed in any order and yield observed outcomes with well-defined joint probabilities.

- Uncertainty inequalities quantify how much this cannot be done for incompatible measurements.

- Usually they are expressed as a lower bound on the product of the standard deviations for uncompatible observables:

$$\sigma_\rho(X)\sigma_\rho(Y) \geq \ell_\rho.$$

Commutator and anti-commutator

- Given $X, Y \in \mathcal{L}(H)$, define the commutator and anti-commutator as

$$[X, Y] = XY - YX \qquad \{X, Y\} = XY + YX,$$

- Notice that

$$XY = \frac{1}{2}\{X, Y\} + \frac{1}{2}[X, Y].$$

- Both are bilinear expressions with respect to X and Y, and satisfy

$$[X, Y]^* = [Y^*, X^*] = -[X^*, Y^*], \quad \{X, Y\}^* = \{X^*, Y^*\}.$$

- Hence, if $X, Y \in \mathcal{O}(H)$ are observables, then $\{X, Y\}, i[X, Y] \in \mathcal{O}(H)$.

Covariance and commutation

- Given $\rho \in \mathcal{S}(H)$, with the notation $(X)_\rho = \mathrm{tr}[X\rho]$, write

$$\tilde{X} = X - (X)_\rho \mathbb{1}_H, \quad \tilde{Y} = Y - (Y)_\rho \mathbb{1}_H.$$

- Define the covariance between X and Y as

$$\mathrm{Cov}_\rho(X, Y) = \frac{1}{2}(\left\{\tilde{X}, \tilde{Y}\right\})_\rho.$$

- Define the commutation

$$\mathrm{Com}_\rho(X, Y) = \frac{1}{2}(i\left[\tilde{X}, \tilde{Y}\right])_\rho = \frac{1}{2}(i\,[X, Y])_\rho,$$

- Notice the identity

$$\mathrm{Cov}_\rho(X, Y) - i\,\mathrm{Com}_\rho(X, Y) = \mathrm{tr}[\tilde{X}\tilde{Y}\rho]. \tag{1}$$

Uncertainty inequality in matrix form

- Given $(X_i)_{i=1,\ldots n} \subseteq \mathcal{O}(H)$, introduce the two matrices

$$\mathsf{Cov}_\rho, \mathsf{Com}_\rho \in \mathbb{R}^{n\times n},$$

given by

$$\mathsf{Cov}_{\rho,ij} = \mathsf{Cov}_\rho(X_i, X_j), \quad \mathsf{Com}_{\rho,ij} = \mathsf{Com}_\rho(X_i, X_j).$$

- Notice that Cov_ρ is symmetric, Com_ρ is anti-symmetric.
- The following inequalities hold:

$$\mathsf{Cov}_\rho \geq \pm i \, \mathsf{Com}_\rho \, .$$

- Proof:

Schrödinger-Robertson uncertainty relation

- Specialize to $n = 2$, i.e.,

$$\begin{pmatrix} \mathrm{Cov}_\rho(X,X) & \mathrm{Cov}_\rho(X,Y) \\ \mathrm{Cov}_\rho(X,Y) & \mathrm{Cov}_\rho(Y,Y) \end{pmatrix} \geq i \begin{pmatrix} 0 & \mathrm{Com}_\rho(X,Y) \\ -\mathrm{Com}_\rho(X,Y) & 0 \end{pmatrix}.$$

- Equivalently

$$\begin{pmatrix} \sigma_\rho^2(X) & b \\ \bar{b} & \sigma_\rho^2(Y) \end{pmatrix} \geq 0,$$

  where $b = \mathrm{Cov}_\rho(X,Y) - i\,\mathrm{Com}_\rho(X,Y)$.

- Taking the determinant, we obtain

$$\sigma_\rho^2(X)\sigma_\rho^2(Y) \geq |b|^2 = |\mathrm{Cov}_\rho(X,Y)|^2 + |\mathrm{Com}_\rho(X,Y)|^2 .$$

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- Recall that quantum channels correspond to Markov kernels $(N(\omega,\cdot))_{x\in\mathcal{X}}$: we investigate analogues of standard functional inequalities.
- Example: since $N(\omega,\cdot)$ is a probability density, given $f:\mathcal{X}\to\mathbb{C}$, Cauchy-Schwarz inequality yields

$$|(Nf)(\omega)|^2 \leq N(|f|^2)(\omega).$$

- Question: does it hold

$$\Phi(A)^*\Phi(A) \leq \Phi(A^*A)$$

for CP unital maps $\Phi:\mathcal{L}(H)\to\mathcal{L}(\tilde{H})$ and $A\in\mathcal{L}(H)$?
- The above indeed holds and is called Kadison-Schwarz inequality.
- To prove it, we use that Φ is CP and apply $\Phi\otimes\mathbb{1}_{\mathcal{L}(\mathbb{C}^2)}$ to

$$M = \begin{pmatrix} A^*A & A^* \\ A & \mathbb{1}_H \end{pmatrix}.$$

But why it is positive?

A general criterion for positivity

Lemma

Let H be a finite dimensional Hilbert space, $X, Y \in \mathcal{O}(H)$, $K \in \mathcal{L}(H)$ with Y positive and invertible. Then,

$$M = \begin{pmatrix} X & K \\ K^* & Y \end{pmatrix}$$

is positive if and only if its Schur complement

$$X - KY^{-1}K^* \in \mathcal{O}(H)$$

is positive.

Proof

- M is positive if and only if, for every pair $|\psi_0\rangle$, $|\psi_1\rangle \in H$,

$$\langle\psi_0|X\psi_0\rangle + \langle\psi_1|K^*\psi_0\rangle + \langle\psi_0|K\psi_1\rangle + \langle\psi_1|Y\psi_1\rangle \geq 0. \tag{2}$$

- Assume that M is positive:

- Viceversa, assume that the Schur complement is positive:

Proof of Kadison-Schwarz inequality

1. The operator
$$M = \begin{pmatrix} A^*A & A^* \\ A & \mathbb{1}_H \end{pmatrix}.$$
is positive since $A^*A - A^*\mathbb{1}_H^{-1}A = 0$.
2. By complete positivity,
$$\begin{pmatrix} \Phi(A^*A) & \Phi(A^*) \\ \Phi(A) & \mathbb{1}_{\tilde{H}} \end{pmatrix} \geq 0.$$
(also using $\Phi(\mathbb{1}_H) = \mathbb{1}_{\tilde{H}}$)
3. Use again the criterion to conclude.

- Beware: many "natural" inequalities valid for functions do not extend to operators, e.g.

$$0 \leq A \leq B \quad \Rightarrow \quad A^2 \leq B^2.$$

- It holds however

$$0 < A \leq B \quad \Rightarrow \quad A^{-1} \geq B^{-1}.$$

  Proof:

- Moreover,

$$0 \leq A \leq B \quad \Rightarrow \quad \sqrt{A} \leq \sqrt{B}.$$

  (the proof is elementary but not very illuminating, please check the lecture notes)

Operator geometric means

- For $t \in [0, 1]$, and positive operators A, $B \in \mathcal{O}_{\geq}(H)$, define

$$A \sharp_t B = A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

- If A and B commute, then

$$A \sharp_t B = A^{1-t} B^t.$$

(use that they can be simultaneously diagonalized)

- For s, $t \in [0, 1]$, one has

and

Proposition (Monotonicity of operator means)

Let H, K be finite dimensional Hilbert spaces,

- let $A, A', B, B' \in \mathcal{O}_{\geq}(H)$,
- and $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ be CP.

Then, for any $t \in [0, 1]$,

$$A' \geq A, B' \geq B \quad \Rightarrow \quad A' \sharp_t B' \geq A \sharp_t B,$$

and

$$\Phi(A) \sharp_t \Phi(B) \geq \Phi(A \sharp_t B).$$

Proof (case $t = 1/2$)

Proof (general case)

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Theorem (Lieb's concavity theorem, monotonicity version)

Let H, $\tilde{H}$ be finite dimensional quantum systems,

- *let $\Phi : \mathcal{L}(\tilde{H}) \to \mathcal{L}(H))$ be CP and unital*
- *so that $\Phi^\dagger$ is a quantum channel from H to $\tilde{H}$,*
- *let $X, Y \in \mathcal{O}_{\geq}(H)$ be positive, $K \in \mathcal{L}(\tilde{H})$.*

Then, for every $t \in [0, 1]$,

$$\mathrm{tr}[\Phi(K)^* X^{1-t} \Phi(K) Y^t] \leq \mathrm{tr}[K^* \Phi^\dagger(X)^{1-t} K \Phi^\dagger(Y)^t].$$

In the case $K = \mathbb{1}_H$, we have $\Phi(\mathbb{1}_H) = \mathbb{1}_{\tilde{H}}$, hence the inequality becomes

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Then, for every $t \in [0, 1]$,

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Remarks

- The inequality
$$\mathrm{tr}[\Phi^\dagger(X^{1-t}Y^t)] \leq \mathrm{tr}[\Phi^\dagger(X)^{1-t}\Phi^\dagger(Y)^t]$$
seems a Hölder inequality $p = 1/(1-t)$ and $p' = 1/t$.
- By monotonicity of operator means, we already have
$$\Phi^\dagger(X\sharp_t Y) \leq \Phi^\dagger(X)\sharp_t\Phi^\dagger(Y),$$
- But X and Y do not necessarily commute. The main idea is to move to a "higher" level to partially restore commutativity.
- The concavity version of Lieb's theorem states that
$$(A, B) \mapsto \mathrm{tr}[K^* A^{1-t} K B^t]$$
is concave on $\mathcal{O}_{\geq 0}(H) \times \mathcal{O}_{\geq 0}(H)$. To deduce it from the monotonicity version, use $\Phi^\dagger = \mathrm{tr}_{\mathbb{C}^2}$ on $H \otimes \mathbb{C}^2$ with
$$X = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad Y = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}, \quad K = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}.$$

Proof

Plan

Uncertainty inequality for Pauli operators

Consider a pure density operator $\rho \in \mathcal{S}(\mathbb{C}^2)$ on a single qubit system.

- Write explicitly the matrix uncertainty inequality for the Pauli operators, in terms of the vector $b = b(\rho)$ of the Bloch parametrization..
- Investigate when equality may occur.
- What about equality cases in the Schrödinger-Robertson inequality for a pair of Pauli operators?

Lieb-Ruskai monotonicity theorem

Let H, $\tilde{H}$ be finite-dimensional quantum systems,

1. let $\Phi : \mathcal{L}(H) \to \mathcal{L}(\tilde{H})$ be CP,
2. $K \in \mathcal{L}(H)$
3. $X \in \mathcal{O}_{>0}(H)$ be positive and invertible such that $\Phi(X)$ is also invertible.

Then,

$$\Phi(K)^* \Phi(X)^{-1} \Phi(K) \leq \Phi(K^* X^{-1} K).$$

A variant of the Schur complement lemma

Let H be a finite dimensional Hilbert space, $X, Y \in \mathcal{O}_{>}(H)$, $K \in \mathcal{L}(H)$. Then, the operator $M \in \mathcal{O}(H \otimes \mathbb{C}^2)$ represented by the block matrix

$$M = \begin{pmatrix} X & K \\ K^* & Y \end{pmatrix}$$

is positive if and only if there exists $Z \in \mathcal{L}(H)$ with operator norm

$$\|Z\| \leq 1, \quad \text{i.e. } Z^* Z \leq \mathbb{1}_H$$

such that

$$K = \sqrt{X} Z \sqrt{Y}.$$

Operator means with negative t

One can extend the definition of $A\sharp_t B$ for any $t \in \mathbb{R}$, provided that A, $B \in \mathcal{O}_>(H)$ are positive and invertible. In turns out that monotonicity inequalities hold true also in the range $t \in [-1, 0] \cup [1, 2]$ (with a reverse inequality than the case $t \in [0, 1]$.

1. Show that, for every $A, B, C \in \mathcal{O}_>(H)$, s, $t \in \mathbb{R}$,
$$C\sharp_s A \le C\sharp_t B \quad \Leftrightarrow \quad C\sharp_{-s} A \ge C\sharp_{-t} A$$
2. Show that, for $t \in [0, 1]$, given A, B, $T \in \mathcal{O}_>(H)$, the inequality
$$A\sharp_{-t} B \le T$$
is equivalent to the following condition:
$$\text{there exists } W \in \mathcal{O}(H) \text{ such that } A\sharp_t B \ge W \text{ and } M = \begin{pmatrix} T & A \\ A & W \end{pmatrix} \ge 0.$$
(Hint: write $A\sharp_t B = A\sharp_{-1}(A\sharp_t B)$ and notice that $A\sharp_t B = AB^{-1}A$.)
3. Deduce that, for $t \in [-1, 0] \cup [0, 1]$ and for every CP map $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ and A, $B \in \mathcal{O}_>(H)$, the inequality
$$\Phi(A\sharp_t B) \ge \Phi(A)\sharp_t \Phi(B)$$

Lieb's theorem for negative exponents

Let H, $\tilde{H}$ be finite dimensional quantum systems, let

- $\Phi : \mathcal{L}(H) \to \mathcal{L}(\tilde{H}))$ be a quantum channel from H to $\tilde{H}$
- X, $Y \in \mathcal{O}_>(H)$ be positive,
- $K \in \mathcal{L}(H)$
- and $t \in [0, 1]$.

Then,

$$\mathrm{tr}[\Phi(K)^* \Phi(X)^{t-1} \Phi(K) \Phi(Y)^{-t}] \leq \mathrm{tr}[K^* X^{1-t} K Y^t],$$

provided that $\Phi(X)$, $\Phi(Y)$ are invertible.