Quantum optimal transport: an invitation

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Non-commutative Optimal Transport
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Outline

Plan

Monge's transport problem

Monge (1781): *sur la théorie des déblais et des remblais*.

How to transport soil during a construction with minimal expenses?

The assignment problem

A discrete formulation: given a

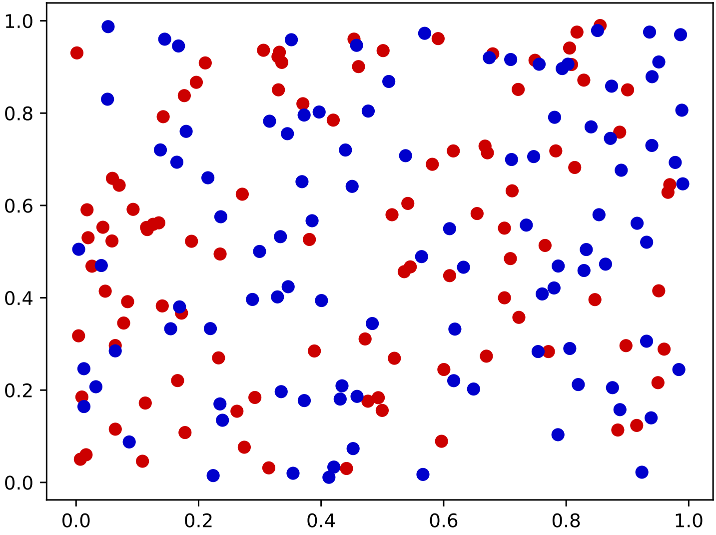
- cost $c(x, y)$ of moving unit of soil from position x to position y, e.g.

$$c(x, y) = |x - y|,$$

- Source distribution of soil $\sigma = (\sigma(x_i))_i$
- Target distribution (dump) $\rho = (\rho(y_j))_j$

Find $T : \{x_i\} \to \{y_j\}$ that moves σ into ρ with minimal transport cost

$$\sum_i c(x_i, T(x_i))\sigma(x_i).$$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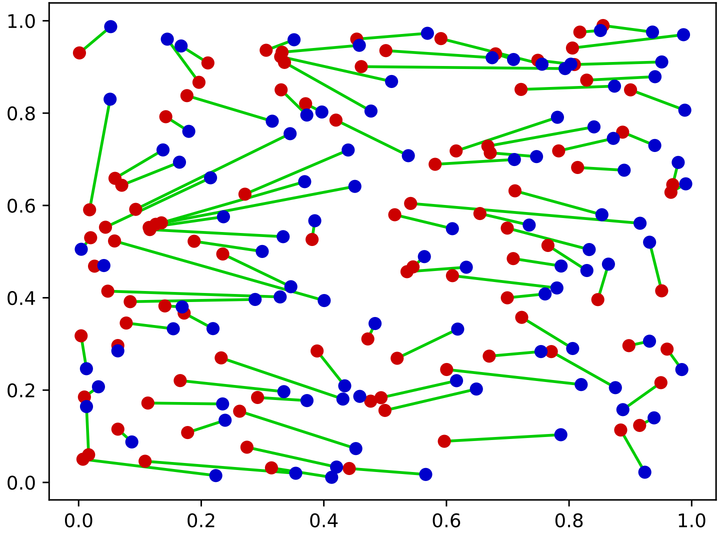
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Some applications of optimal transport:

1. comparison between point clouds
2. geometric interpolation between distributions
3. discriminator in generative AI models (WGANs)
4. functional inequalities (isoperimetric, concentration of measure)
5. PDE's as gradient flows
6. geometry (synthetic Ricci curvature bounds)

Plan

Classical vs Quantum: a dictionary

Classical

Quantum

Classical vs Quantum: a dictionary

E (finite set)	H Hilbert space ($\mathbb{C}^d$)
$e \in E$	$\|\psi\rangle \in H$
$A \subseteq E$	$V < H$
$f : E \to \mathbb{C}$	$A : H \to H$ linear
real-valued	self-adjoint (observable)
non-negative	non-negative
$\|f\|^2$	$\|A\|^2 = A^\dagger A$
$\sum_{x \in E} f(x)$	$\mathrm{Tr}[A]$
probabilities $p \in \mathcal{P}(E)$	quantum states $\rho \in \mathcal{S}(H)$

Classical vs Quantum: a dictionary

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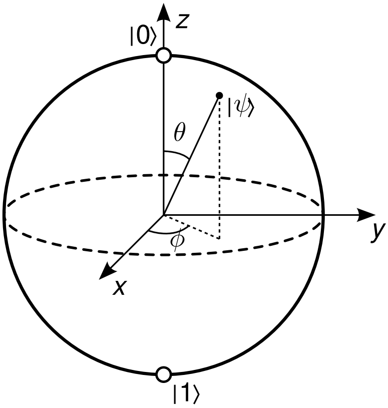
$f : E \to \mathbb{C}$ real-valued non-negative $\|f\|^2$	$A : H \to H$ linear self-adjoint (observable) non-negative $\|A\|^2 = A^\dagger A$
$\sum_{x \in E} f(x)$ probabilities $p \in \mathcal{P}(E)$	$\mathrm{Tr}[A]$ quantum states $\rho \in \mathcal{S}(H)$

Single qubit system

A quantum analogue of $\{0, 1\}$. $H = \mathbb{C}^2$.
Pauli operators σ_x, σ_y, σ_z. Parametrization of states:

$$\rho = \frac{1}{2} \left(\mathbb{I}_{\mathbb{C}^2} + b_x \sigma_x + b_y \sigma_y + b_z \sigma_z \right),$$



Plan

Why Quantum Optimal Transport, why?

Classical distances between probabilities have quantum analogues:

- Total variation $\to$ Trace distance
- Hellinger distance $\to$ Fidelity
- KL divergence $\to$ Relative entropy

Like their classical counterparts:

+ Quite general, easy to compute or approximate
- Not adapted to specific geometry

What about Quantum Optimal Transport?

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What about Quantum Optimal Transport?

A timeline

1992 - Connes/Lott:
spectral distance in non-commutative geometry

1997 - Zyczkowski/Slomczynski:
Wasserstein distance of Husimi distributions

2012 - Maas/Carlen:
quantum analogue of Benamou-Brenier formula

2013 - Agredo:
1-Wasserstein extending any distance on basis vectors

2016 - Golse/Mouhot/Paul:
quantum Kantorovich problem

2019 - De Palma/T.:
Quantum optimal transport using channels

2020 - De Palma/Marvian/T./Lloyd:
Earth mover's distance on qubits

See you at Tutorials!