Probability Master Class 2020

Exercise Sheet on Random Euclidean Bipartite Matching Problems

Please refer to the lecture notes if some definitions are not clear.

Let $d \geq 1$ and consider i.i.d. uniform random variables $(X_i)_{i=1}^{\infty}$, $(Y_i)_{i=1}^{\infty}$ with values in $[0,1]^d$ (if not otherwise stated). For $n \geq 1$, $p > 0$ write

$$B_{p,n} := \min_{\sigma \in \mathcal{S}^n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p$$

$$M_{p,n} = \min_{\sigma \in \mathcal{S}^{2n}} \sum_{i=1}^{n} |X_{\sigma(i)} - X_{\sigma(n+i)}|^p,$$

for the minimum bipartite and monopartite matching costs ($\mathcal{S}^k$ denotes the set of permutations over $\{1, \ldots, k\}$).

Problem 1

Show that $\mathbb{E}[M_{p,n}] \leq \mathbb{E}[B_{p,n}]$.

Problem 2

Show that there exists $c(d,p) > 0$ such that, for every $m \geq 1$, $x \in [0,1]^d$.

$$\mathbb{E}\left[\min_{i=1,\ldots,m} |X_i - x|^p\right] \leq \frac{c(d,p)}{m^{p/d}}.$$

(Hint: follow the proof of the converse inequality and try to bound from above instead of below: where should be located $x \in [0,1]^d$ so that $\left|[0,1]^d \setminus \overline{B(x, t^{1/p})}\right|$ is maximized?)

Problem 3

Show that for some $c(d,p) > 0$ one has $\mathbb{E}[M_{p,n}] \geq c(d,p) n^{1-p/d}$.

Problem 4

Let Z be a Gaussian random variable with mean m and variance σ^2. Then

$$P(|Z - m| > r) \leq \frac{2\sigma}{\sqrt{2\pi} r} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{for every } r > 0.$$

(Hint: reduce to a standard case first, $m = 0$, $\sigma^2 = 1$, and then compute by a change of variables

$$\int_r^{\infty} e^{-x^2/2} \mathsf{d}x = e^{-r^2/2} \int_0^{\infty} e^{-x^2/2} e^{-xr} \mathsf{d}x \leq e^{-r^2/2} \int_0^{\infty} e^{-xr} \mathsf{d}x.$$

Problem 5

Let E be a set, $m \geq 1$ and say that $f : E^m \to \mathbb{R}$ has bounded differences if there exists $(d_i)_{i=1}^m$ such that

$$\text{for every } i \in \{1, \ldots, m\} \text{ there exists } d_i \geq 0 \text{ such that}$$
$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_m)| \leq d_i$$
$$\text{for every } x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m, x_i' \in E.$$

Let $(f_u)_{u \in U}$ be a family of non-negative functions on E^m having bounded differences (and $(d_i)_{i=1}^d$ does not depend on $u \in U$). Then, $\inf_{u \in U} f_u$ also has bounded differences.

Problem 6

Recall the definition of order statistics for a set of n different points $(z_i)_{i=1}^n \subseteq \mathbb{R}$:

$$z_{(1)} = \min\{z_i\}, \quad z_{(k+1)} = \min\left\{z_i : z_i > z_{(k)}\right\}.$$

Recall that we introduce the matching via order statistics between $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$ as the permutation such that

$$\left\{(x_i, y_{\sigma^\dagger(i)})\right\}_{i=1}^n = \left\{(x_{(i)}, y_{(i)})\right\}_{i=1}^n.$$

Give an explicit example showing that $\sigma^\dagger$ is not necessarily an optimal matching if the distance is raised to power p with $0 < p < 1$.

Problem 7

Find an expression for $\mathbb{E}[B_{4,n}]$ and show that the following limit exists:

$$\lim_{n\to\infty} n\mathbb{E}[B_{4,n}].$$

Problem 8

Show that a Poisson random variable Z with parameter λ has mean λ and variance λ and that, for $\alpha \in (0,1]$, $\lambda \geq 1$,

$$\mathbb{E}[Z^\alpha] \geq c(\alpha)\lambda^\alpha,$$

where $c(\alpha) > 0$ depends on α only. *(Hint: use the inequality $Z^\alpha \geq \lambda^\alpha - |Z-\lambda|^\alpha$)*